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MEAN VALUE METHODS FOR SOLVING THE HEAT EQUATION BACKWARDS IN TIME

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Abstract

We investigate an iterative mean value method for the inverse (and highly ill-posed) problem of solving the heat equation backwards in time. Semi-group theory is used to rewrite the solution of the inverse problem as the solution of a fixed point equation for an affine operator, with linear part satisfying special functional analytical properties. We give a convergence proof for the method and obtain convergence rates for the residual. Convergence rates for the iterates are also obtained under the so called source conditions.

1 Introduction

The problem of solving the heat equation backwards in time is a classical example of ill-posed problem. Using semi-group theory to represent the solution of parabolic problems, one can verify that this particular problem is modeled by a linear positive operator, whose eigenvalues converge exponentially to zero (see Section 2).

In this paper we introduce a mean value method, based on the Mann iteration, in order to find the solution of a fixed point equation associated to the inverse problem. The development presented in [6] for elliptic Cauchy problems is extended to this inverse parabolic problem.

Our algorithm generalizes the iteration introduced in [1], where this problem is also considered. In that paper the authors obtain the same fixed point equation treated here. However they propose a method based on solving successive well-posed initial value problems, which can be interpreted as a particular case of our mean value method.

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The paper is organized as follows. In Section 2 we introduce some necessary notation and formulate the inverse problem. We also reinterpret the inverse problem in the form of a fixed point equation. In Section 3 we present an overview of the iterative method introduced by W. Mann. In Section 4 we formulate our mean value method and give a convergence proof for the case of exact data. In Section 5 we consider problems with noisy data. The generalized discrepancy principle is used to obtain convergence rates for the iteration residual. Using appropriate *source conditions*, we also prove convergence rates for the iterates. Finally, we discuss an example, which shows that for this particular inverse problem, the source conditions can be interpreted in terms of regularity of the solution in Sobolev spaces.

2 Formulation of the inverse problem

We start this section introducing some notation. Given a normed linear space H, we call an operator $T : H \to H$ *non expansive* if $\|T\| \leq 1$. An arbitrary operator $T : H \to H$ is called *regular asymptotic* in H if

$$\lim_{k\to\infty} \|T^{k+1}(x) - T^k(x)\| \to 0, \ \forall x \in H.$$

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary $\partial\Omega$ and Λ be a positive self-adjoint unbounded operator (with discrete spectrum) densely defined on $L^2(\Omega)$. The resolution of the identity associated to Λ is represented by E_λ, $\lambda \in \mathbb{R}$. The family of Hilbert spaces $\mathcal{H}^s(\Omega)$, $s \geq 0$ (or simply $\mathcal{H}^s$) is now defined by

$$\mathcal{H}^s(\Omega) \ := \ \{\varphi \in L^2(\Omega);\ \|\varphi\|_s^2 := \int_0^\infty (1+\lambda^2)^s d\langle E_\lambda \varphi, \varphi\rangle < \infty\}. \tag{1}$$

Obviously $\mathcal{H}^0 = L^2(\Omega)$. The Hilbert spaces $\mathcal{H}^{-s}(\Omega)$ (with $s > 0$) are defined by duality: $\mathcal{H}^{-s} := (\mathcal{H}^s)'$. One should notice that the embedding $\mathcal{H}^r \hookrightarrow \mathcal{H}^s$, $r > s$ is dense and compact. It is worth mentioning that in the special case $\Lambda = (-\Delta)^{1/2}$, where Δ is the Laplace–Beltrami operator on Ω, we have $\mathcal{H}^s(\Omega) = H_0^{2s}(\Omega)$, where $H_0^s(\Omega)$ are the Sobolev spaces according to Lions and Magenes.

Next we formulate the inverse problem considered in this paper. Given a function $f \in \mathcal{H}^0$ let $u \in (V, \|\cdot\|_V)$, where

$$V := L_2(0,T;\mathcal{H}^1(\Omega)), \quad \|u\|_V^2 := \int_0^T (\|u(t)\|_1^2 + \|\partial_t u(t)\|_{-1}^2)\, dt, \tag{2}$$

be the solution of

$$(P) \qquad \begin{cases} (\partial_t + \Lambda^2)u \;=\; 0, \text{ in } (0,T)\times\Omega \\ u(T) \;=\; f\,. \end{cases}$$

Our goal is to reconstruct $u(t)$ at $t = 0$. In other words, given a temperature profile at the time $t = T$ and a heat transport equation, find the corresponding temperature profile at the initial time $t = 0$. Problem (P) corresponds to the inverse heat transport problem, which is known to be exponentially ill-posed. This assertion follows immediately from the explicit representation of the solution u of problem (P), which is given by

$$u(t) \;=\; \exp(\Lambda^2(T-t))\, f. \tag{3}$$

Indeed, using the eigenfunctions of Λ, one can construct a sequence of data f_k converging (uniformly) to zero in H, while the V-norm of the corresponding solutions u_k do not.

Notice that if $u \in V$, then $u(0),\ u(T) \in \mathcal{H}^0$. Another important remark concerning the inverse problem: given $f \in \mathcal{H}^0$, problem (P) has exactly one solution in V. Thus the problem of determining $u(0)$ (solution) from $u(T)$ (data) has always a solution, this solution is unique, but it does not depend in a stable way on the problem data.

In the sequel we characterize the solution of our inverse problem as the solution of a fixed point equation. We start by considering problem (P) with data $f \in \mathcal{H}^0$ and denote by $\bar{x}$ the solution of the inverse problem, i.e. $\bar{x} = u(0)$. Let the positive constant $\bar{\lambda}$ be defined by

$$\bar{\lambda} := \inf\{\lambda;\ \lambda \in \sigma(\Lambda)\}. \tag{4}$$

Next we chose a parameter

$$\gamma \in (0, 2\exp(\bar{\lambda}^2 T)). \tag{5}$$

Given $\varphi \in \mathcal{H}^0$, let us consider the following initial value problem of parabolic type

$$(Q) \qquad \begin{cases} (\partial_t + \Lambda^2)w = 0, \text{ in } (0,T)\times\Omega \\ w(0) = \varphi \end{cases}$$

We define the affine operator $T : \mathcal{H}^0 \to \mathcal{H}^0$ by

$$T\varphi := \varphi - \gamma(w(T) - f). \tag{6}$$

A straightforward calculation shows that $T\bar{x} = \bar{x}$, i.e. $\bar{x}$ is a fixed point of T. Further, since the solution of problem (Q) can be written as

$$w(t) = \exp(-\Lambda^2 t)\,\varphi, \tag{7}$$

we obtain for the operator T the representation

$$T\varphi = \big(I - \gamma\exp(-\Lambda^2 T)\big)\,\varphi + \gamma f. \tag{8}$$

Remark 1 *In [1] several properties of the linear part of the operator T, namely $T_l = I - \gamma\exp(-\Lambda^2 T)$, are investigated. The most relevant ones are self-adjointness, nonexpansivity, asymptotic regularity and the fact that 1 is not an eigenvalue of T_l (although 1 belongs to the continuous spectrum of T_l). Further, under the stronger assumption $\gamma < 2\exp(\tilde{\lambda}^2 T)$, where $\tilde{\lambda} := (\bar{\lambda}^2 - T^{-1}\ln 2)^{1/2}$, injectivity of T_l can also be proved.*

Remark 2 *If we write the inverse problem in the form $S\bar{x} = f$, with $S := \exp(-\Lambda^2 T)$, equation (6) resembles very much the fixed point equation*

$$\bar{x} = \bar{x} + S^*(f - S\bar{x}), \tag{9}$$

which is based on a transformation of the normal equation. If $\|S\|^2 < 2$, the related fixed point operator $I - S^*S$ is nonexpansive and one may apply the method of successive approximation (this corresponds to the so called Landweber iteration: $\varphi_{k+1} = \varphi_k + S^*(f - S\varphi_k)$). However, this condition is not satisfied is our case. The idea in [1] is to choose a relaxation parameter $\gamma > 0$ such that $I - \gamma S$ is nonexpansive and then apply the successive approximation method for the fixed point equation

$$\bar{x} = \bar{x} + \gamma\,(f - S\bar{x}). \tag{10}$$

3 The Mann iteration

We present a brief overview of the method introduced by W. Mann in 1953 in [9]. Given a Banach space X and $E \subset X$, Mann considered the problem of approximating a solution of the fixed point equation for a continuous operator $\mathcal{T} : E \to E$.

In order to avoid the problem of existence of fixed points, the subset E was assumed to be convex and compact (the existence question is than promptly answered by the Schauder fixed point theorem). Strongly influenced by the works of Cesàro and Topelitz, who used mean value methods in the summation of divergent series, Mann proposed a mean value iterative method based on the Picard iteration ($x_{k+1} := \mathcal{T}(x_k)$), which we shall present next.

Let A be the (infinite) lower triangular matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots \\ a_{21} & a_{22} & 0 & \cdots & 0 & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix},$$

with coefficients a_{ij} satisfying

i) $a_{ij} \ \geq \ 0\,,\ i,j = 1,2,\ldots;$

ii) $a_{ij} \ = \ 0\,,\ j > i;$

iii) $\sum_{j=1}^{i} a_{ij} \ = \ 1\,,\ i = 1,2,\ldots.$

Starting with an arbitrary element $x_1 \in E$, the *Mann iteration* is defined in the following way

1. `Choose` $x_1 \in E$;
2. `For` $k = 1, 2, \ldots$ `do`

   $v_k := \sum_{j=1}^{k} a_{kj}\, x_j\,;$

   $x_{k+1} := \mathcal{T}(v_k)\,;$

In the sequel we denote this iteration briefly by $M(x_1, A, \mathcal{T})$. Notice that with the particular choice $A = I$, this method corresponds to the usual successive approximation method. Next we state the main result in [9].

Lemma 1 *Let X be a Banach space, $E \subset X$ a convex compact subset, $\mathcal{T} : E \to E$ continuous. Further, let $\{x_k\}$, $\{v_k\}$ be the sequences generated by the iteration $M(x_1, A, \mathcal{T})$. If either of the above sequences converges, then the other also converges to the same point, and their common limit is a fixed point of $\mathcal{T}$.*

Proof. See Theorem 1 in [9].

□

In that paper, the case where neither of the sequences $\{x_k\}$, $\{v_k\}$ converges is also considered. Under additional requirements on the coefficients a_{ij}, a relation between the sets of limit points of $\{x_k\}$ and $\{v_k\}$ is proven.

Many authors considered the Mann iteration in other frameworks. Among others we mention [4], [7], [8] and [10]. In the sequel we discuss a result obtained by C. Groetsch, which will be useful in the further discussion. In [7], Groetsch considers a variant of the Mann iteration. The Matrix A is assumed to be *segmenting*, i.e. additionally to properties *i)*, *ii)* and *iii)*, the coefficients a_{ij} must also satisfy

iv) $a_{i+1,j} \; = \; (1 - a_{i+1,i+1})\, a_{ij}, \;\; j \le i.$

One can easily check that, under assumptions *i)*, ..., *iv)*, the element v_{k+1} of $M(x_1, A, \mathcal{T})$ can be written in the form of the convex linear combination

$$v_{k+1} \; = \; (1 - d_k) v_k \; + \; d_k \mathcal{T}(v_k), \tag{11}$$

where $d_k := a_{k+1,k+1}$. In other words, v_{k+1} lies on the line segment joining v_k and $x_{k+1} = \mathcal{T}(v_k)$, what justifies the denomination of *segmenting matrix*. Notice that in this case the choice of the diagonal elements d_k determines completely the matrix A. The following lemma corresponds to the main result in [7].

Lemma 2 *Let X be an* uniformly convex *Banach space, $E \subset X$ a convex subset, $\mathcal{T} : E \to E$ a nonexpansive operator with at least one fixed point in E. If $\sum_{k=1}^{\infty} d_k(1 - d_k)$ diverges, then the sequence $\{(I - \mathcal{T}) v_k\}$ converges strongly to zero, for every $x_1 \in E$.*

Proof. See Theorem 2 in [7].

□

In order to prove strong convergence of the sequence $\{x_k\}$, one needs stronger assumptions on both the set E and the operator $\mathcal{T}$ (e.g. E is also closed and $\mathcal{T}(E)$ is relatively compact in X).

Notice that Lemma 2 gives on $\{x_k\}$ a condition analogous to the *asymptotic regularity*, which is used in both [10] and [2]. This condition is also used in [1] and [6] for the analysis of linear Cauchy problems.

The last result we analyze in this section is due to H.W. Engl and A. Leitão and is the analog of Lemma 2 for affine operators with nonexpansive linear part, defined on Hilbert spaces.

Lemma 3 *Let H be a Hilbert space and $\mathcal{T} : H \to H$ an affine operator with nonexpansive linear part. Further, let A be a segmenting matrix such that $\sum_{k=1}^{\infty} d_k(1 - d_k)$ diverges. The iteration $M(x_1, A, \mathcal{T})$ generates a sequence v_k such that $\{(I - \mathcal{T})v_k\}$ converges strongly to zero, for every $x_1 \in H$.*

Proof. See Theorem 6 in [6].

$\square$

4 A mean value method for the inverse heat transport problem

The iterative method proposed in this paper corresponds to the Mann iteration applied to the fixed point equation $T\varphi = \varphi$, where the operator T is defined in (8). Initially we address the question of convergence for exact data. Given the exact data $f \in \mathcal{H}^0$ we have the following convergence result:

Theorem 1 *Let T be the operator defined in* (8) *and A a segmenting matrix with $\sum_{k=1}^{\infty} d_k(1 - d_k) = \infty$. For every $x_1 \in \mathcal{H}^0$ the iteration $M(x_1, A, T)$ generates sequences x_k and v_k, which converge strongly to $\bar{x}$, the uniquely determined fixed point of T.*

Proof. Existence and uniqueness of the fixed point $\bar{x}$ was already justified on Section 2. Let T_l be the linear part of the operator T. Since $(I - T_l)(v_k - \bar{x}) = (I - T)v_k$ and $\mathrm{Ker}(I - T_l) = \{0\}$ (see Remark 1 or [1, Lemma 8] for details), it is enough to prove that $\lim_k (I - T)v_k = 0$. This however follows from Lemma 3.

$\square$

5 Convergence rates

In this section we consider the case of inexact data. This is a particularly interesting question, what concerns the application of this method to *real live problems*. Let us assume we are given noisy Cauchy data $f_\varepsilon \in \mathcal{H}^0$, such that

$$\|f - f_\varepsilon\| \le \varepsilon, \tag{12}$$

where $\varepsilon > 0$ is the noise level and $f \in \mathcal{H}^0$ represents the exact problem data. This assumption is relevant in the case where f_ε is obtained by means of measurements, since measured data always contain errors. In this case we consider the iteration residual and use the generalized *discrepancy principle* to provide a stopping rule for the algorithm.

In the sequel, we consider for simplicity the particular case where the segmenting matrix is given by $A = I$, the identity matrix. We start by defining the iteration residual. Given $x_1 \in \mathcal{H}^0$, let us consider the sequences

$$\begin{aligned} x_{k+1} &= T_l\, x_k + \gamma f, && (13)\\ x^\varepsilon_{k+1} &= T_l\, x^\varepsilon_k + \gamma f_\varepsilon, && (14)\end{aligned}$$

generated by the iterative method (note that $x^\varepsilon_1 = x_1$). The corresponding residuals (exact and real, i.e. using noisy data) are defined by

$$\begin{aligned} r_k &:= \gamma f - (I - T_l) x_k, && (15)\\ r^\varepsilon_k &:= \gamma f_\varepsilon - (I - T_l) x^\varepsilon_k. && (16)\end{aligned}$$

Notice that the residual sequences $\{\|r^\varepsilon_k\|\}$, $\{\|r_k\|\}$ are nonincreasing. Indeed, this follows from the nonexpansivity of the operator T_l.

Now let $\mu > 1$ be fixed. According to the discrepancy principle, the iteration should be stopped at the step $k(\varepsilon, f_\varepsilon)$, when for the first time $\|r^\varepsilon_{k(\varepsilon,f_\varepsilon)}\| \le \mu\varepsilon$, i.e.

$$k(\varepsilon, f_\varepsilon) \;:=\; \min\{k \in \mathbb{N} \mid \|\gamma f_\varepsilon - (I - T_l)\varphi^\varepsilon_k\| \le \mu\varepsilon\}. \qquad (17)$$

If we do not make any further (regularity) assumption on the solution $\bar{x}$, we cannot prove convergence rates for the iterates $\|x_k - \bar{x}\|$. However, it is possible to obtain rates of convergence for the residuals.

Next we obtain an estimate for $k(\varepsilon, f_\varepsilon)$ in (17). The proof of this result is similar to the one known for the *Landweber iteration* (see, e.g. [5, Section 6.1]). For convenience of the reader we include here the proof.

Proposition 1 *If $\mu > 1$ is fixed, the stopping rule defined by the* discrepancy principle *in* (17) *satisfies* $k(\varepsilon, f_\varepsilon) = O(\varepsilon^{-2})$.

Proof. We start from the identity

$$\begin{aligned} \|\bar{x} - x_{j+1}\|^2 \;=\; & \|\bar{x} - x_j\|^2 - \|\gamma f - (I - T_l) x_j\|^2 \\ & - 2\langle (I - T_l)(\bar{x} - x_j), T_l(\bar{x} - x_j)\rangle \qquad (18)\end{aligned}$$

to obtain the estimate

$$\|\bar{x} - x_j\|^2 - \|\bar{x} - x_{j+1}\|^2 \ge \|\gamma f - (I - T_l) x_j\|^2. \qquad (19)$$

Adding up this inequalities for $j = 1, \ldots, k$ we can conclude

$$\|\gamma f - (I - T_l)\varphi_k\|^2 \leq k^{-1}\|\bar{x} - x_1\|^2. \tag{20}$$

Since the real residual r_k^ε satisfy

$$\|\gamma f_\varepsilon - (I - T_l)x_{k+1}^\varepsilon\| \leq \varepsilon + \|\gamma f - (I - T_l)x_k\|, \tag{21}$$

we obtain from (20)

$$\|\gamma f_\varepsilon - (I - T_l)x_{k+1}^\varepsilon\| \leq \varepsilon \, + \, k^{-\frac{1}{2}} \, \|\bar{x} - x_1\|. \tag{22}$$

Since the right hand side of (22) is lower than $\mu\varepsilon$ for $k > (\mu - 1)^{-2}\|\bar{x} - x_1\|^2\varepsilon^{-2}$, we have $k(\varepsilon, z_\varepsilon) \leq c\,\varepsilon^{-2}$, where the constant $c > 0$ depends only on μ and x_1.

□

From Proposition 1 it is possible to obtain rates of convergence for the residuals.

Corollary 1 *Let f be the exact data, $\tau > 1$ and $\varepsilon > 0$. Given noisy data f_ε, with $\|f_\varepsilon - f\| \leq \varepsilon$, the stopping rule $k(\varepsilon, f_\varepsilon)$ determined by the discrepancy principle satisfies*

i) $\|\gamma f_\varepsilon - (I - T_l)x_{k(\varepsilon, f_\varepsilon)}^\varepsilon\| \leq \mu\varepsilon$;

ii) $k(\varepsilon, z_\varepsilon) = O(\varepsilon^{-2})$.

If appropriate regularity assumptions are made on the fixed point $\bar{x}$, it is possible to obtain convergence rates also for the approximate solutions. This additional assumptions are stated here in the form of the so-called *source conditions* (see, e.g., [5]). Since our inverse problem is exponentially ill-posed, the source condition take the form

$$\bar{x} - x_1 = F(I - T_l)\, y, \tag{23}$$

where y is some function in $\mathcal{H}^0$ and F is defined by

$$F(\lambda) := \begin{cases} (\ln(e/\lambda))^{-p}, & \lambda > 0 \\ 0, & \lambda = 0 \end{cases}$$

with $p > 0$ fixed. This choice of F corresponds to the *logarithmic-type source conditions.* Under these assumptions we can prove the following rates:

Proposition 2 *Let f be given data and assume that the fixed point $\bar{x}$ of T satisfies the source condition*

$$\bar{x} - x_1 \;=\; F(I - T_l)\, y, \ \textit{for some } y \in \mathcal{H}^0, \tag{24}$$

where $x_1 \in \mathcal{H}^0$ is some initial guess and F is defined as above for some $p \geq 1$. Let $\mu > 2$, f_ε some given noisy data with $\|f_\varepsilon - f\| \leq \varepsilon$, $\varepsilon > 0$ and $k(\varepsilon, f_\varepsilon)$ the stopping rule determined by the discrepancy principle. Then there exists a constant C, depending on p and $\|y\|$ only, such that

i) $\|\bar{x} - x_k^\varepsilon\| \leq C(\ln k)^{-p}$;

ii) $\|\gamma f_\varepsilon - (I - T_l)x_k^\varepsilon\| \leq C k^{-1}(\ln k)^{-p}$;

for all iteration index k satisfying $1 \leq k \leq k(\varepsilon, f_\varepsilon)$.

Proof. Using the estimates in the appendices of [3] and [6] and the discrepancy principle one obtains

$$\|\bar{x} - x_k^\varepsilon\| \leq c\,\|y\|\,\big(1 + \tfrac{1}{\mu-2}\big)\,\big(\ln(k+1)\big)^{-p}. \tag{25}$$

Since $\tilde{c} := \sup_{k \in \mathbb{N}}\{(\ln(k+1)/\ln(k))^{-p}\} < \infty$, assertion *i)* follows from (25) with $C = c\|y\|(1 + \frac{1}{\mu-2})\,\tilde{c}$. To prove *ii)*, we again use the estimates in the appendices of [3] and [6] to obtain

$$\|\gamma f_\varepsilon - (I - T_l)x_k^\varepsilon\| \leq c\,\|y\|\big(1 + \tfrac{2}{\mu-2}\big)\,(k-1)^{-1}\big(\ln(k+1)\big)^{-p}. \tag{26}$$

Since $c^\star := \sup_{k \in \mathbb{N}}\{(k+1)/k)\} < \infty$, assertion *ii)* follows from (26) with $C = c\|y\|(1 + \frac{1}{\mu-2})\, c^\star\, \tilde{c}$.

□

Proposition 3 *Set $k_\varepsilon := k(\varepsilon, f_\varepsilon)$. Under the assumptions of the previous Theorem we have*

i) $k_\varepsilon\big(\ln(k_\varepsilon)\big)^p \;=\; O(\varepsilon^{-1})$;

ii) $\|\bar{x} - x_{k_\varepsilon}^\varepsilon\| \leq O\big((-\ln\sqrt{\varepsilon})^{-p}\big)$.

Proof. Argumenting as in the proof of Proposition 2 we obtain

$$(\mu - 2)\varepsilon \;\leq\; c_1(k_\varepsilon - 2)^{-1}(\ln\, k_\varepsilon)^{-p}, \tag{27}$$

from what follows

$$\varepsilon^{-1} \;\geq\; c_2\,(k_\varepsilon - 2)\,(\ln\, k_\varepsilon)^p \geq c_3\, k_\varepsilon\,(\ln\, k_\varepsilon)^p, \tag{28}$$

proving the first assertion. To prove *ii)*, we first obtain from the iteration rule the estimate

$$\|\bar{x} - x^\varepsilon_{k_\varepsilon}\| \leq \|F(P)v_{k_\varepsilon}\| + \varepsilon k_\varepsilon, \tag{29}$$

where $P = I - T_l$. Using the estimates in the appendices of [3] and [6] we obtain for the first term on the right hand side of (29) the estimate.

$$\|F(P)v_{k_\varepsilon}\| \leq O\big((-\ln(\varepsilon^{\frac{2}{3}}))^{-p}\big). \tag{30}$$

Again using the estimates in the appendices of [3] and [6] we obtain for the second term on the right hand side of (29) the estimate

$$k_\varepsilon \;=\; O\big(\varepsilon^{-1}\,(-\ln\sqrt{\varepsilon})^{-p}\big). \tag{31}$$

Now, substituting the last two estimates in (29), assertion *ii)* follows.

□

The interest in the source conditions of logarithmic-type is motivated by the fact that it can be interpreted in the sense of H^s regularity of $\bar{x} - x_1$. In order to illustrate this fact, let us consider the special case $\Omega = (-\pi, \pi)$, $T = 1$, $\Lambda = (-\Delta)^{1/2}$. We define the Sobolev spaces of periodic functions:

$$H^s_{per}(-\pi,\pi) \;:=\; \{\varphi(t) = \sum_{j\in\mathbb{Z}} \varphi_j\, e^{ijt};\; \sum_{j\in\mathbb{Z}} (1+j^2)^s \varphi_j^2 < \infty\},\; s \in \mathbb{R}. \tag{32}$$

Clearly, the operator T is well defined at $\mathcal{H} = \overline{\text{span}\{\sin(jt);\; j \in \mathbb{N}\}}^{\|\cdot\|_{L^2}}$. The problem data and the initial guess can be represented in the form

$$f(t) = \sum_j f_j \sin(jt); \qquad x_1(t) = \sum_j \Phi_j \sin(jt). \tag{33}$$

Observe also that in $\mathcal{H}$ the operator T can be explicitely represented by

$$(Tx_1)(t) = \sum_j \Big((1 - \gamma e^{-j^2})\Phi_j + \gamma f_j\Big) \sin(jt). \tag{34}$$

The next step corresponds to the choice of the parameter γ. Notice that $\bar{\lambda}$ in (4) satisfies

$$\bar{\lambda} = \inf\{\lambda;\; \lambda \in \sigma(\Lambda)\} = 1,$$

while the parameter $\tilde{\lambda}$ in Remark 1 is given by

$$\tilde{\lambda} = (\bar{\lambda}^2 - T^{-1} \ln 2)^{1/2} = (1 - \ln(2))^{1/2}.$$

Thus, in this particular case, the condition (5) for the choice of γ is given by

$$0 < \gamma < 2 \exp(\tilde{\lambda}^2 T) = e \tag{35}$$

and the choice $\gamma = 1$ is allowed.

From the logarithmic source condition (23), with $y(t) = \sum_j y_j \sin(jt)$ and $q = 2p$, follows

$$\|\bar{x} - x_1\|_q^2 \;=\; \textstyle\sum_{j=1}^{\infty} (1 + j^2)^{2p} \ln \left(\dfrac{\exp(1)}{1 - (1 - e^{-j^2})} \right)^{-2p} y_j^2 \;\le\; \displaystyle\sum_{j=1}^{\infty} y_j^2 \;\le\; \infty.$$

The reciprocal holds, i.e. if $\bar{x} - x_1 \in H^{2p}_{per}$, then exists $y \in \mathcal{H}$ with $\bar{x} - x_1 = f(I - \mathcal{T}_l)\, y$. Thus, the logarithmic source condition can indeed be interpreted as H^p regularity.

References

[1] Baumeister, J.; Leitão, A., *On iterative methods for solving ill-posed problems modeled by partial differential equations,* J. Inverse Ill-Posed Probl. **9** (2001), 13 – 30

[2] Browder, F.; Petryshyn, W., *Construction of fixed points of nonlinear mappings in Hilbert space,* J. Math. Anal. Appl. **20** (1967), 197 – 228

[3] Deuflhard, P.; Engl, H.W.; Scherzer, O., *A convergence analysis of iterative methods for the solution of nonlinear ill-posed problems under affinely invariant conditions,* Inverse Problems, **14** (1998), 1081 – 1106

[4] Dotson, W.G., Jr., *On the Mann iterative process,* Trans. Am. Math. Soc. **149** (1970), 65 – 73

[5] Engl, H.W.; Hanke, M.; Neubauer, A. *Regularization of Inverse Problems,* Kluwer Academic Publishers, Dordrecht, 1996 (Paperback: 2000)

[6] Engl, H.W.; Leitão, A., *A Mann iterative regularization method for elliptic Cauchy problems,* Numer. Funct. Anal. Optim. **22** (2001), 861 – 884

[**7**] Groetsch, C., *A note on segmenting Mann iterates,* J. Math. Anal. Appl. **40** (1972), 369 – 372

[**8**] Kirk, W.A., *On successive approximations for nonexpansive mappings in Banach spaces,* Glasgow Math. J. **12** (1971), 6 – 9

[**9**] Mann, W., *Mean value methods in iteration,* Proc. Amer. Math. Soc. **4** (1953), 506 – 510

[**10**] Opial, Z., *Weak convergence of the sequence of successive approximations for nonexpansive mappings,* Bull. Amer. Math. Soc. **73** (1967), 591 – 597

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