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Mean value iterations for nonlinear elliptic Cauchy problems

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Summary. We investigate the Cauchy problem for a class of nonlinear elliptic operators with C^∞–coefficients at a regular set $\Omega \subset \mathbb{R}^n$. The Cauchy data are given at a manifold $\Gamma \subset \partial\Omega$ and our goal is to reconstruct the trace of the $H^1(\Omega)$ solution of a nonlinear elliptic equation at $\partial\Omega / \Gamma$. We propose two iterative methods based on the segmenting Mann iteration applied to fixed point equations, which are closely related to the original problem. The first approach consists in obtaining a corresponding linear Cauchy problem and analyzing a linear fixed point equation; a convergence proof is given and convergence rates are obtained. On the second approach a nonlinear fixed point equation is considered and a fully nonlinear iterative method is investigated; some preliminary convergence results are proven and a numerical analysis is provided.

1 Introduction

The main results discussed in this paper are: the solution of a nonlinear elliptic Cauchy problem is written as the solution of fixed point equations; mean value iterations are used to approximate the solution of these fixed point equations. We follow two different approaches: in the first one we use a nonlinear

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transformation in order to obtain a linear Cauchy problem and a corresponding linear fixed point equation. We give a convergence proof and also prove some convergence rates. In the second approach we deduce a nonlinear fixed point equation for the solution of the Cauchy problem and a fully nonlinear iterative method is considered. We prove preliminary convergence results and analyze several numerical examples.

The fixed point equations corresponding to the nonlinear Cauchy problem are obtained in Section 4. In order to construct the fixed point operators, two main steps are required: in Section 2 we obtain a particular version of the Cauchy–Kowalewskaia theorem for the nonlinear elliptic Cauchy problem of interest; in Section 3 we prove existence and uniqueness of solutions for mixed boundary value problems associated with the same nonlinear differential operator. It is worth mentioning that, in the linear case, a fixed point equation for the Cauchy problem is analyzed in [EnLe].

The formulation of the Mann iteration is discussed in Section 5 (see [Ma]). We also analyze some extensions of the original result obtained by W. Mann, among these, a variant introduced by C. Groetsch, called segmenting Mann iteration (see [Gr1], [EnSc]). All these iterative methods aim to approximate the solutions of fixed point equations.

In Section 6 we formulate the mean value iterations for the fixed point equations obtained in Section 4. Some analytical results (convergence, rates, ...) are discussed. In Section 7 we analyze numerically the fully nonlinear iteration.

2 Elliptic Cauchy problems

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $\Gamma \subset \partial\Omega$ a given manifold. Given the function $q : \mathbb{R} \to \mathbb{R}+$, we denote by P a second order elliptic operator of the form

$$P(u) \;=\; -\nabla \cdot (q(u)\nabla u) \tag{1}$$

defined in Ω. We denote by *nonlinear elliptic Cauchy problem* the following (time independent) initial value problem for the operator P

$$(CP) \qquad \left\{ \begin{aligned} Pu &= h, \text{ in } \Omega \\ u &= f, \text{ at } \Gamma \\ q(u)u_\nu &= g, \text{ at } \Gamma \end{aligned} \right.$$

where the given functions $f, g : \Gamma \to \mathbb{R}$ are called *Cauchy data* and the right hand side of the differential equation is a function $h : \Omega \to \mathbb{R}$. The problem we want to solve is to evaluate the trace of the solution of such an initial value problem at the part of the boundary where no data is prescribed, i.e. at $\partial\Omega\backslash\Gamma$. As a solution of the Cauchy problem (CP) we consider a $H^1(\Omega)$–distribution, which solves the weak formulation of the elliptic equation in Ω and also satisfies the Cauchy data at Γ in the sense of the trace operator.

It is well known that linear elliptic Cauchy problems are not well posed in the sense of Hadamard. [1] A famous example given by Hadamard himself (see [Le] and the references therein) shows that we cannot have continuous dependence on the data. Also existence of solutions for arbitrary Cauchy data (f, g) cannot be assured, [2] as shows a simple argumentation with the Schwarz reflection principle (see [Tr]).

In Section 2.1 we extend the Holmgren theorem to the H^1–context, proving uniqueness of solutions in weak sense for linear elliptic Cauchy problems. The next step, described in Section 2.2, is to extend this weak uniqueness result to the nonlinear elliptic Cauchy problem (CP).

2.1 Uniqueness: the linear case

In this section we briefly recall an uniqueness result for elliptic operators. This result extends, in the elliptic case, the well known Holmgren theorem (see, e.g. [Fo]) for operators with analytic coefficients and Cauchy data given at a non characteristic manifold.

A well known theorem by A.L. Cauchy and S. Kowalewskaia yields uniqueness of a locally analytic solution to a Cauchy problem, where the differential operator has analytic coefficients and the data are analytic on a analytic non-characteristic manifold. The Holmgren theorem guarantees, that for linear differential equations no other non-analytic solution exists, even if one renounces the analyticity of the Cauchy data.

In order to treat weak solutions of Cauchy problems, we take advantage of some regularity results. Essentially one needs to know that, given a strongly elliptic linear operator of second order L with $C^\infty(\Omega)$–coefficients and a distribution $h \in H^k_{\text{loc}}(\Omega)$, then a solution of $Lu = h$ has to satisfy $u \in C^j(\Omega)$ for $j < k + 2 - (n/2)$. In particular, if $h \in C^\infty(\Omega)$, then u also belongs to $C^\infty(\Omega)$.

Next we state the desired uniqueness result for weak solutions of the linear Cauchy problem. For the convenience of the reader a sketch of the proof is given.

Theorem 1 *Let Ω be an open bounded simply connected subdomain of $\mathbb{R}^n$ with analytic boundary, and let Γ be an open, connected part of $\partial\Omega$. Let the*

[1] For a formal definition of *well posed* problems, see e.g. [EHN].

[2] The Cauchy data (f, g) is called *consistent* if the corresponding problem (CP) has a solution. Otherwise (f, g) is called *inconsistent* Cauchy data.

operator L be defined as above. Then, the Cauchy problem

$$\begin{cases} Lu = h, \ in\ \Omega \\ u = f, \ at\ \Gamma \\ u_\nu = g, \ at\ \Gamma \end{cases}$$

for $h \in L^2(\Omega)$, $f \in H^{1/2}(\Gamma)$ *and* $g \in H_{00}^{1/2}(\Gamma)'$ *has at most one solution in* $H^1(\Omega)$. [3]

Sketch of the Proof. If u_1 and u_2 are H^1 –solutions of the Cauchy problem, then $u = u_1 - u_2$ solves the following problem:

$$\begin{cases} Lu = 0, & \text{in } \Omega \\ u = u_\nu = 0, & \text{at } \Gamma \end{cases}$$

Because of $Lu = 0$, our regularity result (with $h = 0$) yields that $u \in C^\infty(\Omega)$ holds. Hence, u is a classical solution to the Cauchy problem with homogeneous data on the manifold Γ.

The operator L is strongly elliptic and both Γ and $\partial\Omega\backslash\Gamma$ are non–characteristic manifolds. Hence, it is possible to find a family of analytic manifolds Γ_λ, $0 \le \lambda \le 1$ such that $\Gamma_0 = \Gamma$, $\Gamma_1 = \partial\Omega\backslash\Gamma$ and all Γ_λ share the same endpoints. If we use the family Γ_λ from the proof of Holmgren theorem, we can conclude that the function u vanishes identically on Ω. □

2.2 Uniqueness: the nonlinear case

In this Section we prove for the nonlinear elliptic Cauchy problem (CP) a result analogous to the one stated in Theorem 1. The argumentation in the next theorem follows the lines of [Kü, Theorem 2.3.2].

Theorem 2 *Let* Ω *be an open bounded simply connected subset of* $\mathbb{R}^n$ *with analytical boundary,* $\Gamma \subset \partial\Omega$ *an open connected manifold, the operator P defined as in* (1)*, where* $q : \mathbb{R} \to [q_{min}, q_{max}] \subset (0, \infty)$ *is a* C^∞*–function. Then the nonlinear Cauchy problem (CP) has for each pair of data* $(f, g) \in H^{1/2}(\Gamma) \times H_{00}^{1/2}(\Gamma)'$ *and* $h \in L^2(\Omega)$ *at most one* $H^1(\Omega)$ *solution.*

Proof. Notice that $Pu = F[u]$, where

$$F[u] \ = \ F\Big(u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}\Big) \ := \ q'(u)(\nabla u)^2 \ + \ q(u)\Delta u.$$

The function $F = F(s, p, R)$, with $s \in \mathbb{R}$, $p = (p_1, \dots, p_n)$, $R = (r_{ij})_{i,j=1}^n$ is continuously differentiable because of the assumption on q. Furhtermore,

[3] The Sobolev spaces are defined as in [DaLi]; see also [Ad].

the operator P is elliptic due to the strict positivity of q. Now let u_1, u_2 be two H^1–solutions of (CP). Notice that

$$F\Big(u_1, \frac{\partial u_1}{\partial x_i}, \frac{\partial^2 u_1}{\partial x_i \partial x_j}\Big) - F\Big(u_2, \frac{\partial u_2}{\partial x_i}, \frac{\partial^2 u_2}{\partial x_i \partial x_j}\Big) = 0. \tag{2}$$

Defining $v := u_1 - u_2$, it follows from (2) together with the mean-value theorem (for functions of several variables) that

$$\sum_{i,j=1}^{n} \Big(\frac{\partial F}{\partial r_{ij}}\Big)_\theta \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \Big(\frac{\partial F}{\partial p_i}\Big)_\theta \frac{\partial v}{\partial x_i} + \Big(\frac{\partial F}{\partial s}\Big)_\theta v = 0, \tag{3}$$

for (different) $\theta \in (0, 1)$ with

$$\begin{aligned}
\Big(\frac{\partial F}{\partial r_{ij}}\Big)_\theta &= q(\theta u_1 + (1-\theta)u_2)\delta_{ij}, \\
\Big(\frac{\partial F}{\partial p_i}\Big)_\theta &= 2q'(\theta u_1 + (1-\theta)u_2)\,(\theta \frac{\partial u_1}{\partial x_i} + (1-\theta)\frac{\partial u_2}{\partial x_i}), \\
\Big(\frac{\partial F}{\partial s}\Big)_\theta &= q''(\theta u_1 + (1-\theta)u_2)\,(\theta \nabla u_1 + (1-\theta)\nabla u_2)^2 \\
&\quad + q'(\theta u_1 + (1-\theta)u_2)\,(\theta \Delta u_1 + (1-\theta)\Delta u_2)
\end{aligned}$$

(note that $\Delta u_i \in L^2$, since $F[u_i] = h$). Thus, v is a $H^1(\Omega)$–solution of the linear elliptic differential equation (3) and further v satisfies $v|_\Gamma = 0$. Now it follows from the identity $q(u_1)u_1 - q(u_2)u_2 = 0$ (again argumenting with the mean-value theorem)

$$\begin{aligned}
&q'(\theta u_1 + (1-\theta)u_2)(\theta(u_1)_\nu + (1-\theta)(u_2)_\nu)v \\
&\quad + q(\theta u_1 + (1-\theta)u_2)v_\nu = 0 \ \text{ on } \ \Gamma
\end{aligned}$$

for another $\theta \in (0, 1)$. Finally, the positivity of q yields $v_\nu|_\Gamma = 0$. From Theorem 1 follows now $v \equiv 0$, proving the assertion. □

Remark 3 There is an alternative way to obtain the uniqueness result described in Theorem 2. Namely, using the nonlinear transformation Q in Section 4 and applying Theorem 1 to the corresponding linear Cauchy problem. The choice of the approach presented in the proof of Theorem 2 is motivated by the nonlinear nature of the operators associated to problem (CP).

3 Elliptic mixed boundary value problems

In this section we prove existence and uniqueness of solutions for mixed boundary value problems modelled by elliptic differential operators. We start presenting some results concerning the linear case in Section 3.1. In Section 3.2 we extend this results for the nonlinear operator defined in (1).

3.1 The linear case

Initially we introduce some useful Sobolev spaces. Let $\Omega \in \mathbb{R}^n$ be an open bounded regular set [4] with C^∞–boundary $\partial\Omega$, which is split in $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$, the subsets Γ_j being open connected and satisfying $\Gamma_1 \cap \Gamma_2 = \emptyset$. We define

$$H_0^s(\Omega \cup \Gamma) \;:=\; \overline{C_0^\infty(\Omega \cup \Gamma)}^{\|\cdot\|_{s;\Omega}},$$

where $\|\cdot\|_{s;\Omega}$ denotes the usual Sobolev s–norm on Ω. [5]

Let L be the linear elliptic operator defined in section 2.1 and denote by $a(\cdot,\cdot)$ the corresponding bilinear form. For the analysis of mixed problems we need a *Poincaré type* inequality on the space $H_0^1(\Omega \cup \Gamma_j)$, i.e.

$$\|u\|_{L^2(\Omega)} \;\leq\; c\,\|a(u,u)\|, \quad u \in H_0^1(\Omega \cup \Gamma_j) \tag{4}$$

(see, e.g. [Tr]). We consider the following mixed problem problem at Ω: given the functions $f \in H^{1/2}(\Gamma_1)$ and $g \in H_{00}^{1/2}(\Gamma_2)'$, find a H^1–solution of

$$(LMP) \qquad \begin{cases} Lu = 0, \text{ in } \Omega \\ u = f, \text{ at } \Gamma_1 \\ u_\nu = g, \text{ at } \Gamma_2 \end{cases}.$$

Existence, uniqueness and continuous dependency of the data for (LMP) are given by a lemma of Lax–Milgram type, which states that for every pair of Cauchy data, problem (LMP) has a unique solution $u \in H^1(\Omega)$ and, further,

$$\|u\|_{H^1(\Omega)} \;\leq\; c\left(\|f\|_{H^{1/2}(\Gamma_1)} \;+\; \|g\|_{H_{00}^{1/2}(\Gamma_2)'}\right). \tag{5}$$

This is a standard result and we omit the proof.

3.2 The nonlinear case

Now, we extend the results of section 3.1 to the following nonlinear mixed boundary value problem: find a H^1–solution of

$$(MP) \qquad \begin{cases} P(u) = h, \text{ in } \Omega \\ u = f, \text{ at } \Gamma_1 \\ q(u)u_\nu = g, \text{ at } \Gamma_2 \end{cases}$$

[4] We mean Ω is locally at one side of $\partial\Omega$.

[5] Analogously one can define

$$H^s(\Omega) := \overline{C^\infty(\overline{\Omega})}^{\|\cdot\|_{s;\Omega}}, \;\; H_0^s(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{s;\Omega}}.$$

where the coefficient $q \in H^1(\mathbb{R})$, the right hand side $h \in L^2(\Omega)$, and the boundary conditions $f \in H^{1/2}(\Gamma_1)$, $g \in H_{00}^{1/2}(\Gamma_2)'$ are given functions.

Before proving the main result of this section, we present an existence (and uniqueness) theorem for abstract operator equations concerning quasi-monotone operators.

Lemma 4 *Let the space $\mathcal{V}$ satisfy $H_0^1(\Omega) \subset \mathcal{V} \subset H^1(\Omega)$ and assume that $\mathcal{V} \hookrightarrow L^2(\Omega)$ is compact. Let the operator $\bar{A} : \mathcal{V} \to \mathcal{V}'$ be given by $\bar{A}u = A(u,u)$, with $A : \mathcal{V} \times \mathcal{V} \to \mathcal{V}'$ defined by*

$$\langle A(u,v), w\rangle_{\mathcal{V}} := \langle A_1(u,v), w\rangle_{\mathcal{V}} + \langle A_0 u, w\rangle_{\mathcal{V}} \tag{6}$$

where

$$\langle A_1(u,v), w\rangle_{\mathcal{V}} := \int_\Omega \Big\{ \sum_{j=1}^n a_j(x, u(x), \nabla v(x)) \frac{\partial}{\partial x_j} w(x) \Big\}\, dx, \tag{7}$$

$$\langle A_0 u, w\rangle_{\mathcal{V}} := \int_\Omega a_0(x, u(x), \nabla u(x))\, w(x)\, dx \tag{8}$$

for all $u, v, w \in \mathcal{V}$. Here, the coefficient functions $a_j : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $0 \le j \le n$ are supposed to satisfy

1. *$a_j(x,\eta,\xi)$ is measurable in $x \in \Omega$ and continuous in $(\eta,\xi) \in \mathbb{R} \times \mathbb{R}^n$;*
2. *$|a_j(x,\eta,\xi)| \le c\,(k(x) + |\eta| + \|\xi\|)$ for almost all $x \in \Omega$, $\eta \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, where $k \in L^2(\Omega)$;*
3. *$\sum_{j=1}^n \big(a_j(x,\eta,\xi) - a_j(x,\eta,\tilde{\xi})\big)\big(\xi_j - \tilde{\xi}_j\big) > 0$ for almost all $x \in \Omega$, $\eta \in \mathbb{R}$ and $\xi \in \mathbb{R}^n / \{\tilde{\xi}\}$;*
4. *$\frac{\sum_{j=1}^n a_j(x,\eta,\xi)\xi_j}{\|\xi\| + \|\xi\|} \to \infty$ as $\|\xi\| \to \infty$, uniformly for η bounded at almost all $x \in \Omega$.*

Furthermore, let $\mathcal{F}$ belong to $\mathcal{V}'$. Then, if $\bar{A}$ is $\mathcal{V}$-coercive and bounded, the equation

$$\bar{A}u = \mathcal{F} \tag{9}$$

has a unique solution in $\mathcal{V}$.

Sketch of the Proof. This lemma follows from an analog result concerning monotone operators by Browder–Vishik (see [Sh, Proposition 5.1], page 60). One has just to relax the monotony assumptions to the class of quasi-monotone operators, as shown in [Sh, Section II.6], on page 74 and subsequent. □

The next result guarantees existence and uniqueness of solution for problem (MP) and is the nonlinear analogous to the result stated in section 3.1.

Theorem 5 *Let the function q be such that $0 < q_{min} \le q \le q_{max} < \infty$ and $h \in L^2(\Omega)$ be given. Then for every pair of data $(f, g) \in H^{1/2}(\Gamma_1) \times H^{1/2}_{00}(\Gamma_2)'$ the problem (MP) has a unique solution $u \in H^1(\Omega)$.*

Proof. We consider first the homogeneous case $f = 0$. Setting $\mathcal{V} = H^1_0(\Omega \cup \Gamma_2)$ we derive by integration by parts the following variational formulation for (MP):

Find $u_0 \in \mathcal{V}$ such that

$$\int_\Omega q(u_0)\nabla u_0 \cdot \nabla v \, dx \; = \; \int_\Omega h(x) v \, dx + \int_{\Gamma_2} g v \, d\Gamma_2 \tag{10}$$

holds for all $v \in \mathcal{V}$.

This is equivalent to solve the operator equation

$$\bar{A} u_0 = \mathcal{F}, \tag{11}$$

where $\bar{A}$ is given by $a_0(x, \eta, \xi) = 0$ and $a_j(x, \eta, \xi) = q(\eta)\xi_j$ (see Lemma 4) and $\mathcal{F} \in \mathcal{V}'$ is defined by

$$\langle \mathcal{F}, v \rangle_{\mathcal{V}} \; = \; \int_\Omega h(x) v \, dx + \int_{\Gamma_2} g v \, d\Gamma_2 \;\text{ for all }\; v, w \in \mathcal{V}. \tag{12}$$

We easily see that the conditions 1–4 of Lemma 4 are satisfied. Indeed,

1. $q(\eta)\xi_j$ is continuous in $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$;
2. $|q(\eta)\xi_j| \le q_{\max} \|\xi\|$;
3. $\sum_{j=1}^n \left(q(\eta)\xi_j - q(\eta)\tilde{\xi}_j \right) \left(\xi_j - \tilde{\xi}_j \right) \ge q_{\min} \sum_{j=1}^n \left(\xi_j - \tilde{\xi}_j \right)^2 > 0$ for all $\eta \in \mathbb{R}$, $\xi \ne \tilde{\xi} \in \mathbb{R}^n$;
4. $\frac{\sum_{j=1}^n q(\eta)\xi_j\xi_j}{\|\xi\| + \|\xi\|} \ge \frac{q_{\min}\|\xi\|^2}{2\|\xi\|} \to \infty$ as $\|\xi\| \to \infty$.

Since the coercitivity of $\bar{A}$ follows from the Poincaré inequality and the boundedness is given by the bound $q_{\max}$, the existence of a unique $u_0 \in \mathcal{V}$ can be guaranteed.

Next we consider the non-homogeneous case. By the trace theorem, we obtain a function $\tilde{f} \in H^1(\Omega)$ such that $\tilde{f}|_{\Gamma_1} = f$. Defining the closed convex non-empty affine subspace $K := \tilde{f} + H^1_0(\Omega \cup \Gamma_2)$, it follows from a result by Brezis (see e.g. [Sh, Theorem 2.3], on page 42) that equation (11) has a unique solution in K. This solution clearly satisfies both the differential equation $P(u) = h$ and the mixed boundary conditions. □

4 Cauchy problems and fixed point equations

In this section we characterize the solution of (CP) as solution of certain fixed point equations. We define $\Gamma_1 := \Gamma$, $\Gamma_2 := \partial\Omega \backslash \Gamma$, such that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\overline{\Gamma_1 \cup \Gamma_2} = \partial\Omega$. Further, we assume that the coefficient q of the the second order elliptic operator P defined in (1) satisfies

A1) $q \in C^\infty(\mathbb{R})$;
A2) $q(t) \in [q_{min}, q_{max}]$ for all $t \in \mathbb{R}$, where $0 < q_{min} < q_{max} < \infty$.

Given the Cauchy data $(f, g) \in H^{1/2}(\Gamma_1) \times H^{1/2}_{00}(\Gamma_1)'$ and $h \in L^2(\Omega)$, we assume that there exists an H^1–solution of problem

$$(CP) \qquad \begin{cases} Pu = h, \text{ in } \Omega \\ u = f, \text{ at } \Gamma_1 \\ q(u)u_\nu = g, \text{ at } \Gamma_1 \end{cases}$$

and we are mainly interested in the determination of the Neumann trace

$$\bar{\varphi} := q(u)u_{\nu|_{\Gamma_2}} \in H^{1/2}_{00}(\Gamma_2)'. \tag{13}$$

Notice that, once $\bar{\varphi}$ is known, the solution of (CP) can be determined as the solution of a well posed mixed boundary value problem, namely

$$\begin{cases} Pu = h, \text{ in } \Omega \\ u = f, \text{ at } \Gamma_1 \\ q(u)u_\nu = \bar{\varphi}, \text{ at } \Gamma_2. \end{cases}$$

In order to obtain the fixed point equations for $\bar{\varphi}$, we follow two separate approaches, which we describe next:

First approach: Transformation of (CP) into a linear Cauchy problem
The first step is to introduce the real function

$$Q(t) := \int_0^t q(s)\, ds.$$

Given $u \in H^1(\Omega)$, the function $U := Q(u)$ is also in $H^1(\Omega)$ and satisfies

$$\Delta U = \nabla \cdot (\nabla Q(u)) = \nabla \cdot (q(u)\nabla u) = P(u).$$

Further, $U_\nu = q(u)u_\nu$ holds at $\partial\Omega$. Thus, if u is the solution of problem (CP), the function U solves the linear Cauchy problem

$$(LCP) \qquad \begin{cases} \Delta U = h, & \text{in } \Omega \\ U = Q(f), & \text{at } \Gamma_1 \\ U_\nu = g, & \text{at } \Gamma_1 \end{cases}$$

Reciprocally, if problem (LCP) is consistent for some data $(Q(f), g)$ with solution U, it follows from $Q' = q > 0$ (we use assumption *A2)*), that $u := Q^{-1}(U)$ is well defined in $H^1(\Omega)$ and solves problem (CP).

Thus, it is enough to find the solution of the (consistent) Cauchy problem (LCP). Since this is a linear problem, a fixed point equation based on the composition of adequate mixed boundary value problems is known from the literature (see, e.g., [Le] or [KMF]). This fixed point equation is obtained as follows:

1) Let $\bar{\varphi}$ be given by (13). Notice that U, the solution of (LCP), solves the mixed boundary value problem:

$$\Delta U = h \ \text{ in } \Omega, \quad U = f \ \text{ at } \Gamma_1, \quad U_\nu = \bar{\varphi} \ \text{ at } \Gamma_2.$$

Thus, we define the operator $\mathcal{L}_n : H_{00}^{1/2}(\Gamma_2)' \ni \varphi \mapsto w_{|\Gamma_2} \in H^{1/2}(\Gamma_2)$, where w solves

$$\Delta w = h \ \text{ in } \Omega, \quad w = f \ \text{ at } \Gamma_1, \quad w_\nu = \varphi \ \text{ at } \Gamma_2.$$

2) Notice that U also solves the mixed boundary value problem:

$$\Delta U = h \ \text{ in } \Omega, \quad U_\nu = g \ \text{ at } \Gamma_1, \quad U = \mathcal{L}_n(\bar{\varphi}) \ \text{ at } \Gamma_2.$$

This motivates the definition of the operator $\mathcal{L}_d : H^{1/2}(\Gamma_2) \ni \psi \mapsto v_{\nu|\Gamma_2} \in H_{00}^{1/2}(\Gamma_2)'$, where v solves

$$\Delta v = h \ \text{ in } \Omega, \quad v_\nu = g \ \text{ at } \Gamma_1, \quad v = \psi \ \text{ at } \Gamma_2.$$

3) Finally, we define the operator

$$\mathcal{T} : H_{00}^{1/2}(\Gamma_2)' \ni \varphi \longmapsto \mathcal{L}_d(\mathcal{L}_n(\varphi)) \in H_{00}^{1/2}(\Gamma_2)' \tag{14}$$

and observe that $\mathcal{T}\bar{\varphi} = \bar{\varphi}$. Reciprocally, if φ is a fixed point of $\mathcal{T}$, it follows from an uniqueness argument for problem (LCP) that $\varphi = \bar{\varphi}$ (see, e.g., proof of Proposition 6).

Second approach: Nonlinear mixed problems

In this approach we consider mixed boundary value problems similar to those introduced previously. However, no transformation is performed. We start by defining the operators

$$L_n : H_{00}^{1/2}(\Gamma_2)' \to H^{1/2}(\Gamma_2)\,, \quad L_d : H^{1/2}(\Gamma_2) \to H_{00}^{1/2}(\Gamma_2)' \tag{15}$$

by $L_n(\varphi) := w_{|\Gamma_2}$ and $L_d(\psi) := q(v)v_{\nu|\Gamma_2}$, where the $H^1(\Omega)$–functions w, v solve the nonlinear mixed boundary value problems

$$Pw = h \ \text{ in } \Omega, \quad w = f \ \text{ at } \Gamma_1, \quad q(w)w_\nu = \varphi \ \text{ at } \Gamma_2$$

and

$$Pv = h \ \text{in } \Omega, \quad q(v)v_\nu = g \ \text{at } \Gamma_1, \quad v = \psi \ \text{at } \Gamma_2$$

respectively. Now, defining the operator

$$T : H_{00}^{1/2}(\Gamma_2)' \ni \varphi \longmapsto L_d(L_n(\varphi)) \in H_{00}^{1/2}(\Gamma_2)' \tag{16}$$

and observing that $L_n(\bar{\varphi}) = u_{|\Gamma_2}$ and $L_d(u_{|\Gamma_2}) = \bar{\varphi}$, we obtain the desired characterization of $\bar{\varphi}$ as a solution of the fixed point equation for the operator T. In the next proposition we formalize this result as well as its converse.

Proposition 6 *If problem (CP) admits a solution (say u), then $\bar{\varphi} := q(u)u_{\nu|\Gamma_2}$ is a fixed point of the operator T defined in* (16). *Conversely, if $\bar{\varphi}$ is a fixed point of T, the Cauchy problem (CP) is solvable and it's solution (say u) satisfies $q(u)u_{\nu|\Gamma_2} = \bar{\varphi}$.*

Proof. The first part follows directly from the definition of T. Now let $\bar{\varphi}$ be a fixed point of T. The H^1–functions w and v satisfy

$$w_{|\Gamma_2} = v_{|\Gamma_2} \quad \text{and} \quad q(w)w_{\nu|\Gamma_2} = q(v)v_{\nu|\Gamma_2}.$$

Argumenting with the uniqueness result in Theorem 2 we conclude that $w \equiv v$. Thus, the function w satisfies

$$Pw = h \ \text{in } \Omega, \quad w = f \ \text{at } \Gamma_1, \quad q(w)w_\nu = g \ \text{at } \Gamma_1, \tag{17}$$

i.e. w is a solution of problem (CP). Since we have uniqueness of solutions for (CP) (again by Theorem 2), it follows $q(u)u_{\nu|\Gamma_2} = q(w)w_{\nu|\Gamma_2} = \bar{\varphi}$. □

Using Theorems 2 and 6 we can reinterpret the uniqueness result in Section 3.2 in terms of the fixed point equation $T(\varphi) = \varphi$:

Corollary 7 *Given a pair of consistent Cauchy data (f, g), the nonlinear operator T defined in* (16) *has exactly one fixed point in the Hilbert space $H_{00}^{1/2}(\Gamma_2)'$.*

Remark 8 In the first approach $\mathcal{T}$ is an affine operator, since $\mathcal{L}_n$ and $\mathcal{L}_d$ are both affine as well. The affine part of $\mathcal{T}$ depends only on the Cauchy data (f, g) and on the right hand side h. Denoting this affine term by $z_{f,g} \in H_{00}^{1/2}(\Gamma_2)'$ and the linear part of $\mathcal{T}$ by $\mathcal{T}_l$, it follows that $\bar{\varphi}$ is a solution of the linear Cauchy problem if and only if it is a solution of

$$(I - \mathcal{T}_l)\,\varphi = z_{f,g}. \tag{18}$$

5 The Mann iteration

In this section we present a brief overview of the iterative method introduced by W. Mann in 1953 (see [Ma]). Given a Banach space X and $E \subset X$, Mann considered the problem of approximating the solution of the fixed point equation for a continuous operator $T : E \to E$.

In order to avoid the problem of existence of fixed points, Mann assumed the subset E to be convex and compact (the existence question is then promptly answered by the Schauder fixed point theorem; see [Sc]). Strongly influenced by the works of Cesàro and Toeplitz, who used mean value methods in the summation of divergent series, Mann proposed a mean value iterative method based on the Picard iteration ($x_{k+1} := T(x_k)$), which we shall present next.

Let A be the (infinite) lower triangular matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots \\ a_{21} & a_{22} & 0 & \cdots & 0 & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix},$$

with coefficients a_{ij} satisfying

i) $a_{ij} \ge 0\,,\ i, j = 1, 2, \ldots;$
ii) $a_{ij} = 0\,,\ j > i;$
iii) $\sum_{j=1}^{i} a_{ij} = 1\,,\ i = 1, 2, \ldots.$

Starting with an arbitrary element $x_1 \in E$, the *Mann iteration* is defined by

1. Choose $x_1 \in E$;
2. For $k = 1, 2, \ldots$ do
   $v_k := \sum_{j=1}^{k} a_{kj}\, x_j$;
   $x_{k+1} := T(v_k)$;

and is briefly denoted by $M(x_1, A, T)$. Notice that with the particular choice $A = I$, this method corresponds to the usual Picard iterative process. Next we state the main theorem in [Ma].

Lemma 9 *Let X be a Banach space, $E \subset X$ a convex compact subset, $T : E \to E$ continuous. Further, let $\{x_k\}$, $\{v_k\}$ be the sequences generated by the iteration $M(x_1, A, T)$. If either of the above sequences converges, then the other also converges to the same point, and their common limit is a fixed point of T.*

In that paper, the case where neither of the sequences $\{x_k\}$, $\{v_k\}$ converges is also considered. Under additional requirements on the coefficients a_{ij}, a relation between the sets of limit points of $\{x_k\}$ and $\{v_k\}$ is proven.

Many authors considered the Mann iteration in other frameworks. In the sequel we present some of the main results related, which will be useful for our further discussion.

In [Op], Z. Opial considers the very special case where X is a Hilbert space, $E \subset X$ a closed convex subset, $T : E \to E$ a nonexpansive regular asymptotic application with non empty fixed point set, $A = I$. In the main theorem it is proven that $\{x_k\}$ converges weakly to some fixed point of T (see [EnLe] for a generalization).

In [Do], W. Dotson extends the proof in [Ma] to the case in which X is a locally convex Hausdorff vector space and $E \subset X$ is a convex closed subset. This is achieved by using the regularity of the matrix A and some properties of the continuous semi-norms which generate the topology of E.

In [SeDo], H. Senter and W. Dotson assume X to be an uniformly convex Banach space, $E \subset X$ a closed bounded convex subset, $T : E \to E$ non-expansive. Under special assumptions on the fixed point set of T, they prove that $\{x_k\}$ converge to some fixed point of T.

In [En], H.W. Engl assumes X to be an Opial normed space, $E \subset X$ weakly compact, $T : E \to X$ nonexpansive, $\sum a_{kk}$ divergent. In the main theorem it is proven that if $\{x_k\}$ is well defined for some x_1 (i.e. $x_k \in E, \forall k$) then a (unique) fixed point $\bar{x}$ exists and x_k converges weakly to $\bar{x}$. Further, $\{\bar{x}\}$ is characterized as the Chebyshef-center of the set $\{x_k\}$.

In [Gr1], C. Groetsch considers a variant of the Mann iteration. The Matrix A is assumed to be *segmenting*, i.e. additionally to properties *i)*, *ii)* and *iii)*, the coefficients a_{ij} have also to satisfy

iv) $a_{i+1,j} = (1 - a_{i+1,i+1})\, a_{ij}, \ j \le i.$

One can easily check that, under assumptions *i)*, ... , *iv)*, v_{k+1} can be written as the convex linear combination

$$v_{k+1} = (1 - d_k)v_k + d_k T(v_k), \tag{19}$$

where $d_k := a_{k+1,k+1}$. In other words, v_{k+1} lies on the line segment joining v_k and $x_{k+1} = T(v_k)$, what justifies the denomination of *segmenting matrix*. Notice that the choice of the diagonal elements d_k determines completely the matrix A. Next we enunciate the main theorem in [Gr1]:

Lemma 10 *Let X be an* uniformly convex *Banach space, $E \subset X$ a convex subset, $T : E \to E$ a nonexpansive operator with at least one fixed point in E. If $\sum_{k=1}^{\infty} d_k(1 - d_k)$ diverges, then the sequence $\{(I - T)v_k\}$ converges strongly to zero, for every $x_1 \in E$.*

In order to prove strong convergence of the sequence $\{x_k\}$, one needs stronger assumption on both the set E and the operator T (e.g. E is also closed and $T(E)$ is relatively compact in X).

Notice that Lemma 10 gives on $\{x_k\}$ a condition analogous to the *asymptotic regularity*, which is used in [Op] (this condition is also used in [Le] and [EnLe] for the analysis of linear Cauchy problems).

6 Iterative methods for nonlinear Cauchy problems

6.1 Analysis of the first approach

The first iterative method proposed in this paper corresponds to the Mann iteration applied to the fixed point equation $\mathcal{T}\varphi = \varphi$ with $\mathcal{T}$ defined in (14). Initially we address the question of convergence for exact data. Given a pair of consistent Cauchy data (f, g) we have the following result:

Theorem 11 *Let $\mathcal{T}$ be the operator defined in* (14) *and A a segmenting matrix with $\sum_{k=1}^{\infty} d_k(1-d_k) = \infty$. For every $\varphi_1 \in H_{00}^{1/2}(\Gamma_2)'$ the iteration $(\varphi_1, A, \mathcal{T})$ generates sequences φ_k and ϕ_k, which converge strongly to $\bar{\varphi}$, the uniquely determined fixed point of $\mathcal{T}$.*

Sketch of the Proof. Existence and uniqueness of the fixed point $\bar{\varphi}$ follow from the assumption that (f, g) are consistent data. Since $(I-\mathcal{T}_l)(\phi_k-\bar{\varphi}) = (I-\mathcal{T})\phi_k$ and $\mathrm{Ker}(I-\mathcal{T}_l) = \{0\}$ (see [Le]), it is enough to prove that $\lim_k (I-\mathcal{T})\phi_k = 0$.

This however follows from [EnLe, Theorem 6], which is the analog of Lemma 10 for affine operators, with nonexpansive linear part, defined on Hilbert spaces. □

Now let us consider the case of inexact data. We assume we are given noisy Cauchy data $(f_\varepsilon, g_\varepsilon)$, or alternatively z_ε, such that

$$\|z_{f,g} - z_\varepsilon\| \le \varepsilon,$$

where $\varepsilon > 0$ is the noise level. This assumption is especially interesting in the case where $(f_\varepsilon, g_\varepsilon)$ correspond to measured data. In this case we consider the iteration residual and use the generalized *discrepancy principle* to provide a stopping rule for the algorithm, i.e the iteration is stopped at the step k_ε such that

$$k_\varepsilon := \min\{k \in \mathbb{N};\ \|z_\varepsilon - (I-\mathcal{T}_l)\varphi_k\| \le \mu\varepsilon\}$$

for some $\mu > 1$ fixed. If we do not make any further (regularity) assumption on the solution $\bar{\varphi}$, we cannot prove convergence rates for the iterates $\|\varphi_k - \bar{\varphi}\|$. However, it is possible to obtain rates of convergence for the residuals, as follows

Proposition 12 *If $\mu > 1$ is fixed, the stopping rule defined by the* discrepancy principle *satisfies $k_\varepsilon = O(\varepsilon^{-2})$.*

The proof of this result is similar to the one known for the *Landweber iteration* (see, e.g. [EHN, Section 6.1]).

If appropriate regularity assumptions are made on the fixed point $\bar{\varphi}$, it is possible to obtain convergence rates also for the approximate solutions. This additional assumptions are stated here in the form of *source conditions* (see, e.g., [EHN] or [EnSc]). Since linear Cauchy problems are exponentially ill-posed (see, e.g., [Le]), the source condition take the form

$$\bar{\varphi} - \varphi_1 = f(I - \mathcal{T}_l)\psi, \tag{20}$$

where ψ is some function in $H_{00}^{1/2}(\Gamma_2)'$ and f is defined by

$$f(\lambda) := \begin{cases} (\ln(e/\lambda))^{-p}, & \lambda > 0 \\ 0, & \lambda = 0 \end{cases}$$

with $p > 0$ fixed. This choice of f corresponds to the so-called source conditions of logarithmic-type. Under these assumptions we can prove the following rates:

Proposition 13 *Let k_ε be the stopping rule determined by the discrepancy principle with $\mu > 2$. If the fixed point $\bar{\varphi}$ satisfies the source condition* (20)*, then we have*

i) $k_\varepsilon = O(\varepsilon^{-1}(-\ln\sqrt{\varepsilon})^{-p})$;
ii) $\|\bar{\varphi} - \varphi_k\| = O((-\ln\sqrt{\varepsilon})^{-p})$.

This result is a consequence of [EnLe, Theorem 15] (see also [EnSc] for the non linear Landweber iteration). The interest in the source conditions of logarithmic-type for this fixed point equation is motivated by the fact that it can be interpreted in the sense of H^s regularity. Indeed, it is shown in [EnLe] that condition (20) is equivalent to assume, in a special case where the spectral decomposition of the operator $\mathcal{T}_l$ is known, that $\bar{\varphi} - \varphi_1$ is in the Sobolev space H^p.

6.2 Analysis of the second approach

Let T be the operator defined in (16). We assume the Cauchy data (f, g) to be consistent and denote by $\bar{\varphi}$ the fixed point of T. We start by defining the operator

$$\overline{T}\varphi := \begin{cases} T\varphi, & \|T\varphi\| \le \|\varphi\| \\ \frac{\|\varphi\|}{\|T\varphi\|}T\varphi, & \|T\varphi\| \ge \|\varphi\| \end{cases} \tag{21}$$

The operator $\overline{T}$ is continuous, nonlinear and further satisfies $\|\overline{T}\varphi\| \le \|\varphi\|$, for all $\varphi \in H^{1/2}_{00}(\Gamma_2)'$. As one can easily check, every fixed point of T is also a fixed point of $\overline{T}$. The reciprocal, however, is not true. What we can prove is the following:

Proposition 14 *An element $\varphi \in H^{1/2}_{00}(\Gamma_2)'$ is a fixed point of T iff it is a fixed point of $\overline{T}$ and $\|T\varphi\| \le \|\varphi\|$.*

Next we introduce an iterative method (based on the Mann iteration) to approximate the fixed points of $\overline{T}$. Given a segmenting matrix A, consider the algorithm

1. Choose $\varphi_1 \in H^{1/2}_{00}(\Gamma_2)'$;
2. For $k = 1, 2, \ldots$ do

   $\phi_k := \sum_{j=1}^{k} a_{kj}\,\varphi_j$;

   $\varphi_{k+1} := \overline{T}(\phi_k)$;

Notice that $\varphi_k, \phi_k \in H^{1/2}_{00}(\Gamma_2)'$, $k = 1, 2, \ldots$. We represent this iterative process by $(\varphi_1, A, \overline{T})$. Obviously this iteration coincides with the Picard iteration if one chooses $A = I$.[6]

Now we discuss an auxiliary lemma concerning uniformly convex spaces.[7]

Lemma 15 *Let X be an uniformly convex linear space with modulus of convexity δ. Further let $\varepsilon > 0$, $d > 0$ and $\lambda \in [0, 1]$ be given. If $\phi, \psi \in X$ are such that*

$$\|\psi\| \;\le\; \|\phi\| \;\le\; d \quad \textit{and} \quad \|\phi - \psi\| \;\ge\; \varepsilon,$$

then

$$\|(1-\lambda)\phi + \lambda\psi\| \;\le\; \|\phi\| \left(1 - 2\delta(d^{-1}\varepsilon)\, \min\{\lambda, 1-\lambda\}\right).$$

Proof. See [Gr2]. □

Next we prove for our iterative method, a result analogous to the one stated Lemma 10. Notice that, different to that lemma, we do not assume nonexpansivity. Instead we use the weaker property $\|\overline{T}\varphi\| \le \|\varphi\|$ of $\overline{T}$.

Proposition 16 *Let $\overline{T}$ be the operator defined in* (21) *and A a segmenting matrix such that $\sum_{k=1}^{\infty} d_k(1-d_k)$ diverges. Further let $\varphi_1 \in H^{1/2}_{00}(\Gamma_2)'$ and $\{\phi_k\}$ be the sequence generated by the iteration $(\varphi_1, A, \overline{T})$. Then, there exists a subsequence $\{\phi_{k_j}\}$ such that $\{(I - \overline{T})\phi_{k_j}\}$ converges (strongly) to zero.*

---

[6] In the linear case the choice $A = I$ corresponds to the Maz'ya iteration; see [EnLe].

[7] See [Ad] for the corresponding definitions.

Proof. Given $\varphi_1 \in H_{00}^{1/2}(\Gamma_2)'$, let $\{\varphi_k\}$ and $\{\phi_k\}$ be the sequences generated by the iteration $(\varphi_1, A, \overline{T})$. Setting $\psi_k := \varphi_{k+1}$, for $k \geq 0$, it follows from the definition of $\overline{T}$ and from the segmenting property (19) that

$$\|\psi_k\| \;\leq\; \|\phi_k\| \;\leq\; \|\varphi_1\|\,,\; k = 1, 2, \ldots \tag{22}$$

and

$$\phi_{k+1} \;=\; (1 - d_k)\,\phi_k + d_k\,\psi_k\,,\; k = 1, 2, \ldots\,. \tag{23}$$

Since $(I - \overline{T})\phi_k = \phi_k - \psi_k, k \geq 1$, it is enough to prove that $0 \in \mathrm{cl}\{\phi_k - \psi_k\}$, where $\mathrm{cl}\,\{M\}$ denotes the (strong) closure of the set M in $H_{00}^{1/2}(\Gamma_2)'$.

Now, suppose there exists $\varepsilon > 0$ such that

$$\|\phi_k - \psi_k\| \geq \varepsilon\,,\; k = 1, 2, \ldots\,. \tag{24}$$

Thus, it follows from Lemma 15, together with (22) and (23)

$$\begin{aligned}\|\phi_{k+1}\| &= \|(1 - d_k)\,\phi_k + d_k\,\psi_k\|\\ &\leq \|\phi_k\|\,\big(1 - 2\delta(\|\varphi_1\|^{-1}\varepsilon)\,\min\{d_k, 1 - d_k\}\big).\end{aligned}$$

Repeating inductively the argumentation we obtain

$$\|\phi_{k+1}\| \;\leq\; \|\phi_1\| \prod_{j=1}^{k} \big(1 - 2\delta(\|\varphi_1\|^{-1}\varepsilon)\,\min\{d_j, 1 - d_j\}\big). \tag{25}$$

However, since $d_j(1 - d_j) \leq \min\{d_j, 1 - d_j\}$, for $j \geq 1$, it follows from the assumption on the series $\sum d_j(1 - d_j)$ that

$$\sum_{j=1}^{\infty} \min\{d_j, 1 - d_j\} \;=\; \infty.$$

Consequently, the product on the right hand side of (25) converges to zero and hence $\lim \|\phi_k\| = \lim \|\psi_k\| = 0$. However, this contradicts (24), completing the proof. □

Notice that the assumption $\sum_{k=1}^{\infty} d_k(1 - d_k) = \infty$ in Theorem 16 can be replaced by the requirement $\sum_{j=1}^{\infty} \min\{d_j, 1 - d_j\} = \infty$. The proof remains the same.

Remark 17 In [Gr2, Lemma 2], C. Groetsch obtains an analogous result for an iterative process introduced by W. Kirk (see [Ki]). In that paper, fixed points of a nonexpansive operator T are approximated by a sequence of the type $x_{n+1} = S_n x_n$, where the operators S_n are defined by

$$S_n \;:=\; \alpha_{n0} I + \alpha_{n1} T + \cdots + \alpha_{nk} T^k\,,\; n = 1, 2, \ldots\,,$$

with $\alpha_{ij} \geq 0$, $\alpha_{n1} \geq \alpha > 0$, $\sum_{j=1}^{k} \alpha_{nj} = 1$, $j = 1, \ldots, k$ and $\sum_{n=1}^{\infty} \min\{\alpha_{n0}, 1 - \alpha_{n0}\} = \infty$.

One should notice that in [Gr2, Lemma 2] as well as in Lemma 10, one needs for the proof the fact that the operator T is nonexpansive. However, we have only used the property $\|\overline{T}\varphi\| \leq \|\varphi\|$, for all φ.

Remark 18 If for a particular coefficient function q the operator $\overline{T}$ happens to be nonexpansive, then one can argue as in [EnLe, Corollary 7] to prove that, under the same assumptions of Proposition 16, the sequence $\{\phi_k\}$ generated by the iteration $(\varphi_1, A, \overline{T})$ converges (strongly) to a fixed point of $\overline{T}$ for every $\varphi_1 \in H_{00}^{1/2}(\Gamma_2)'$.

7 Numerical experiments

In the next paragraphs we present some numerical results, which correspont to the implementation of the iterative method proposed in Section 6.2. The first two examples concern consistent Cauchy problems (exact data) in a square domain. In the third example we consider a problem with noisy data.

The computation was performed on the Silicon Graphics SGI-machines (based on R12000 processors; 32-bit code) at the Spezialforschungsbereich F013. The elliptic mixed boundary value problems were solved using the NETLIB software package PLTMG (see [Ba]).

7.1 A problem with harmonic solution

Let $\Omega \subset \mathbb{R}^2$ be the open rectangle $(0, 1) \times (0, 1/2)$ and define the following subsets of $\partial\Omega$:

$$\Gamma_1 := \{(x, 0); x \in (0, 1)\}, \quad \Gamma_2 := \{(x, 1/2); x \in (0, 1)\},$$

$$\Gamma_3 := \{(0, y); y \in (0, 1/2)\}, \quad \Gamma_4 := \{(1, y); y \in (0, 1/2)\}.$$

Let $q(t) = 1 + t^2$ and $\bar{u} : \bar{\Omega} \to \mathbb{R}$ be the harmonic function

$$\bar{u}(x, y) \ := \ x^2 - y^2 + 5x + 2y - 3xy.$$

We consider the Cauchy problem

$$\begin{cases} -\nabla \cdot (q(u)\nabla u) = h\,, \text{ in } \Omega \\ u = f\,, \text{ at } \Gamma_1 \\ q(u)u_\nu = g\,, \text{ at } \Gamma_1 \\ u = \bar{u}\,, \text{ at } \Gamma_3 \cup \Gamma_4 \end{cases}$$

where the Cauchy data

$$f(x) = x^2 + 5x, \quad g(x) = (1 + (x^2 + 5x)^2)\,(3x - 2)$$

is given at Γ_1 and the right hand side $h : \Omega \to \mathbb{R}$ is given by

$$h(x, y) = -2\bar{u}(x, y)\,|\nabla \bar{u}(x, y)|^2.$$

We aim to reconstruct the (Dirichlet) trace of u at Γ_2. The problem was artificially constructed and one can easily check that the desired trace is given by

$$\bar{\varphi} = x^2 + \tfrac{7}{2}x + \tfrac{3}{4} \; (= \bar{u}_{|\Gamma_2}).$$

We used in the iteration the Cesàro matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \cdots \\ 1/2 & 1/2 & 0 & \cdots & 0 \cdots \\ 1/3 & 1/3 & 1/3 & \cdots & 0 \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \;\; \vdots \end{pmatrix}.$$

As initial guess, $\varphi_1 \equiv 0$ was chosen (at the end points $x = 0$ and $x = 1$, φ_1 must take the values 3/4 and 21/4 respectively, in order to be compatible with the other boundary conditions). Each mixed boundary value problem was solved using a (multi-grid) finite element method, with linear elements and a uniform mesh with 1 073 nodes (33 nodes on Γ_2). We used the stopping rule $\|\varphi_k - \varphi_{k-1}\|_{L^2} \le 10^{-2}$ (the same used in [EnLe] for the linear case).

In Figure 1 we present the results corresponding to the Mann iteration for the operator $\mathcal{T}$ (first approach). In this case the transformation Q (see Section 4) is given by $Q(t) = t + t^3/3$ and it's inverse can be analytically calculated. The dotted (blue) line represents the exact solution, the dashed (black) line represents the sequence φ_k and the solid (red) line represents the sequence ϕ_k, both generated by $(0, A, \mathcal{T})$.

In Figure 2 we present the results corresponding to the Mann iteration for the operator $\bar{T}$ (second approach). The dotted (blue) line represents the

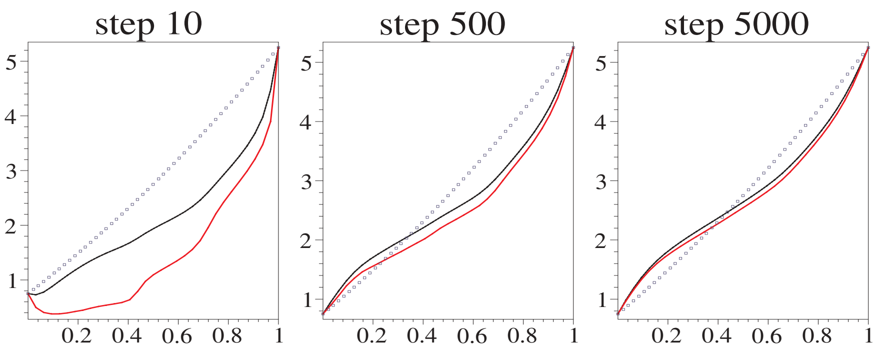

Fig. 1. Iteration for a Cauchy problem with harmonic solution: First approach

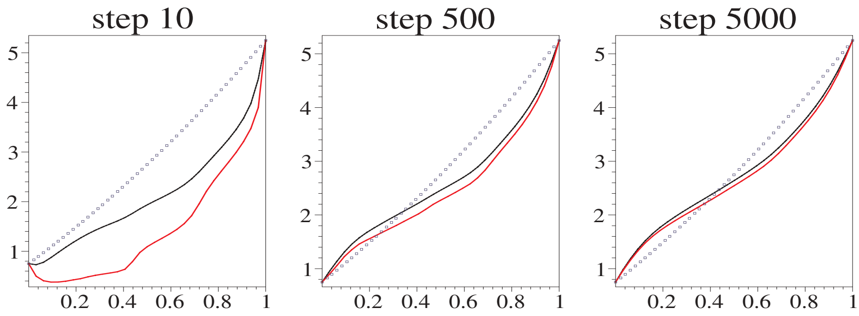

Fig. 2. Iteration for a Cauchy problem with harmonic solution: Second approach

exact solution, the dashed (black) line represents the sequence φ_k and the solid (red) line represents the sequence ϕ_k, both generated by $(0, A, \bar{T})$.

From the results in Figure 2, one can conclude that the convergence rate decays very fast with the iteration. This can be in part explained by the linear convex combination used to compute ϕ_k in the Mann iteration (note that $\phi_{k+1} - \phi_k = \frac{1}{k+1}\varphi_{k+1} - \frac{1}{k(k+1)}\sum_{j=1}^{k} \varphi_k$). We consider the following alternative to improve the convergence rate:

- Choose $\varepsilon' > 0$ larger then the precision ($\varepsilon > 0$) to be achieved;
- Use the stopping rule $\|\phi_k - \phi_{k-1}\|_{L^2} \leq \varepsilon'$;
- Restart the iteration with $\varphi_1 = \phi_k$;
- The iteration should be renewed restarted until $\|T\varphi_k - \varphi_k\|_{L^2} \leq \varepsilon$.

Notice that we restart the iteration every time the mean value ϕ_k stops changing significantly. The restart procedure should be repeated until φ_k (or alternatively ϕ_k) approximates the fixed point of $\overline{T}$ with the desired precision. In Figure 3 we present the results corresponding to this restart strategy (the meaning of the curves is the same as in the previous Figures). For comparison purposes, we restarted the iteration after every 50 steps. Thus, in order to compute the results in Figure 3, we had to evaluate 100, 200 and 300 iteration steps respectively.

One should notice that the result obtained after the third restart (middle picture in Figure 3) required 200 iteration steps to be computed and already gives us a much better approximation to the actual solution than the one obtained after 5000 steps with the Mann method.

7.2 A problem with non harmonic solution

Let $\Omega \subset \mathbb{R}^2$ and $\Gamma_i \subset \partial\Omega$, $i = 1, \ldots, 4$ be defined as in the previous section. Let $q(t) = 2 + \sin t$. We consider the Cauchy problem

$$\begin{cases} -\nabla \cdot (q(u)\nabla u) = h, & \text{in } \Omega \\ u = f, & \text{at } \Gamma_1 \\ q(u)u_\nu = g, & \text{at } \Gamma_1 \\ u = \bar{u}, & \text{at } \Gamma_3 \cup \Gamma_4 \end{cases}$$

with Cauchy data

$$f(x) = \cos \pi x, \quad g(x) = -\big(2 + \sin(\cos \pi x)\big) \cos(\pi x)$$

given at Γ_1 and right hand side $h : \Omega \to \mathbb{R}$ given by

$$h(x, y) = (2 + \sin \bar{u})(\pi^2 - 1)\bar{u} - \cos \bar{u}\, |\nabla \bar{u}|^2,$$

where $\bar{u} : \bar{\Omega} \to \mathbb{R}$ is defined by

$$\bar{u}(x, y) := \cos(\pi x) \exp(y).$$

As in the previous example, the Cauchy problem was so constructed, such that $\bar{\varphi}$ is known. Indeed we have $\bar{\varphi} = \bar{u}_{|\Gamma_2}$.

For the numerical computations we used the same segmenting matrix A and the same stopping rule as in the previous example. As initial guess, $\varphi_1 \equiv 4$ was chosen (at the end points $x = 0$ and $x = 1$ we must choose, for compatibility reasons, $\varphi_1(0) = \exp(0.5)$, $\varphi_1(1) = -\exp(0.5)$). Each mixed boundary value problem was solved using a (multi-grid) finite element method, with linear elements and a uniform mesh with 16 577 nodes (129 nodes on Γ_2).

In Figure 4 we present the results corresponding to the Mann iteration for the operator $\bar{T}$ (the meaning of the curves is the same as in the Figures of Subsection 7.1).

Analogously as in the previous example, it is possible to accelerate the convergence of the iterative method by using a restart strategy. In Figure 5 we present the results obtained by restarting the iteration after every 50 steps

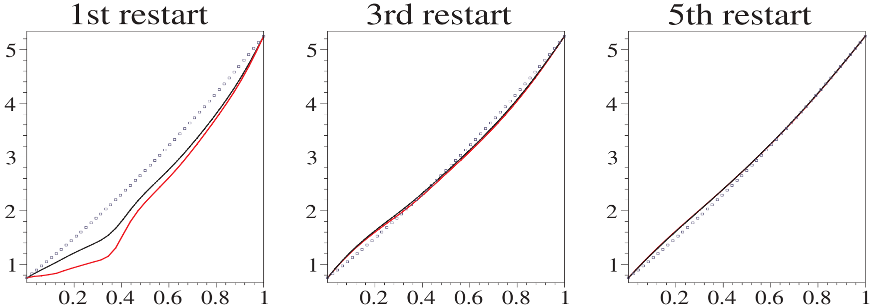

Fig. 3. Iteration with restart strategy for a Cauchy problem with harmonic solution: Second approach

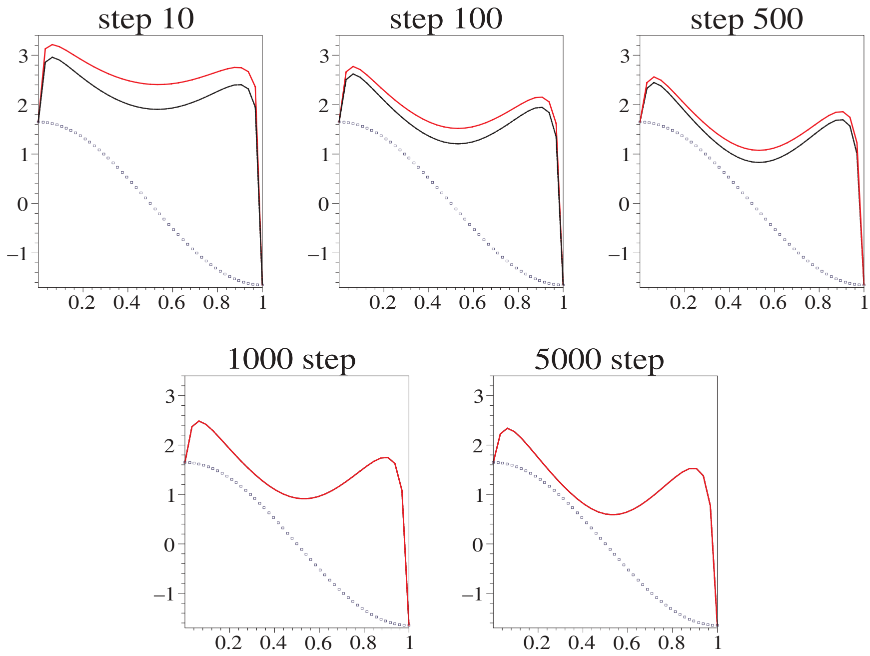

Fig. 4. Iteration for a Cauchy problem with non harmonic solution

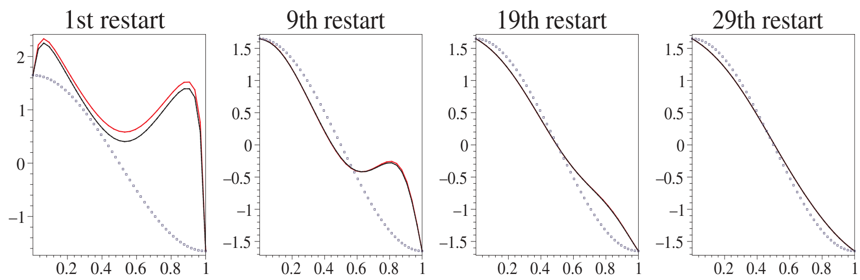

Fig. 5. Iteration with restart strategy (after every 50 steps) for a Cauchy problem with non harmonic solution

(in order to obtain the results in Figure 5 we had to evaluate 100, 500, 1000 and 1500 iteration steps respectively).

A similar restart strategy was suggested in [EnLe] (for the linear case) and produced nice results. It is worth mentioning that we have no analytical justification neither for the choice of the restart criterion nor for the improvement in the convergence rate.

7.3 A problem with noisy data

For this example we consider once more the Cauchy problem described in Section 7.1. To obtain the noisy Cauchy data we simply perturbed the

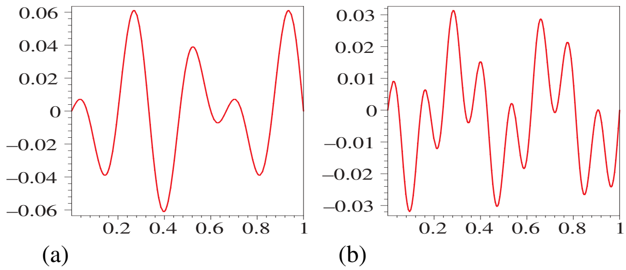

Fig. 6. Generation of noisy data; (a) Perturbation added to the Dirichlet data; (b) Perturbation added to the Neumann data

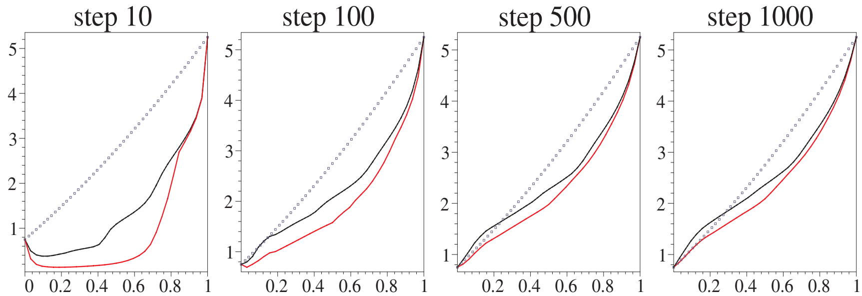

Fig. 7. Iteration for a Cauchy problem with noisy data

exact Cauchy data

$$(f, g) = \big(x^2 + 5x\,,\ (1 + (x^2 + 5x)^2)\,(3x - 2)\big)$$

using an error of level 1%.

In Figure 6 we present the perturbations added to the exact Dirichlet and Neumann data.

For the iteration, we used again the Cesàro matrix A in Section 7.1. The initial guess, stopping rule and mesh refinement used for the computation are the same as those used in that section.

The numerical results corresponding to the Mann iteration are presented in Figure 7: the dotted (blue) line corresponds to the exact solution; the dashed (black) line corresponds the iteration for exact Cauchy data (see Figure 2); the solid (red) line corresponds to the iteration for the noisy data.

It is worth mentioning that our numerical solver was not able to handle with the non linear mixed boundary value problems when we tried to use a larger error level. To contour this problem we could alternatively refine our mesh or increase the maximum number of Newton iterations in the solver. However this would interfere with the comparison of results presented in Figure 7.

8 Final remarks

Let us assume $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Now, if we want to solve a Cauchy problem with data given at Γ_1 plus some further boundary condition (Neumann, Dirichlet, ...) at Γ_3 (see the problems in Section 7). It is possible to adapt our iterative method for this type of problems. We just have to add the extra boundary condition at Γ_3 to both mixed boundary value problems at each iteration step. The over-determination of boundary data does not affect the analysis presented in this paper (see [Le] for the linear case).

The iterative methods proposed in Section 6 generate sequences of Neumann traces, which approximate the unknown Neumann boundary condition $q(u)u_{\nu|\Gamma_2}$. Alternatively, we could define an iterative method, which produces a sequence of Dirichlet traces (see the problems in Section 7). This was already suggested in the linear case in [Le]). The convergence proof for this iterative method is quite similar to the one discussed in this paper.

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