

LETTER TO THE EDITOR

On the relation between constraint regularization, level sets, and shape optimization

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Abstract

We consider regularization methods based on the coupling of Tikhonov regularization and projection strategies. From the resulting constraint regularization method we obtain level set methods in a straightforward way. Moreover, we show that this approach links the areas of asymptotic regularization to inverse problems theory, scale-space theory to computer vision, level set methods, and shape optimization.

1. Introduction

The major goal of this paper is to highlight the relation between the following areas:

- (i) regularization for inverse and ill-posed problems, in particular
 - (a) Tikhonov regularization for constraint operator equations;
 - (b) asymptotic regularization;
- (ii) scale-space theory in *computer vision*;
- (iii) shape optimization.

The general context is to solve the constraint ill-posed operator equation:

$$F(u) = y, \tag{1}$$

where u is in the admissible class

$$U := \{u : u = P(\phi) \text{ and } \phi \in \mathcal{D}(P)\}.$$

The constraint equation can be formulated as an unconstrained equation

$$F(P(\phi)) = y. \tag{2}$$

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Assuming that the operator equation is ill-posed it has to be regularized for a stable solution.

Classical results on convergence and stability of regularization (see e.g. [7, 17, 18]) such as

- (i) existence of a regularized solution
- (ii) stability of the regularized approximations
- (iii) approximation properties of the regularized solutions

are applicable if P is

- (i) bounded and linear or
- (ii) nonlinear, continuous, and weakly closed.

In order to link constraint regularization methods, shape optimization, level sets, and inverse scale-space, we require discontinuous operators P , and thus the classical framework of regularization theory is not applicable yet.

Tikhonov regularization for solving the unconstrained equation (1) consists in approximation of the solution of (1) by the minimizer u_α of the functional

$$\|F(u) - y\|^2 + \alpha \|u - u_*\|^2.$$

If F is differentiable, then

$$F'(u_\alpha)^*(F(u_\alpha) - y) + \alpha(u_\alpha - u_*) = 0, \quad (3)$$

where $F'(u_\alpha)^*$ denotes the adjoint \cdot^* of the derivative of F at u_α . Equation (3) is the optimality condition for a minimizer of the Tikhonov functional. Using the formal setting $\Delta t := 1/\alpha$, $u(\Delta t) := u_\alpha$, and $u(0) := u_*$ we find

$$F'(u(\Delta t))^*(F(u(\Delta t)) - y) + \frac{u(\Delta t) - u(0)}{\Delta t} = 0.$$

Thus $u_\alpha = u(\Delta t)$ can be considered as the solution of one implicit time step with step-length $\Delta t = \frac{1}{\alpha}$ for solving

$$\frac{\partial u}{\partial t} = -F'(u)^*(F(u) - y), \quad (4)$$

and we end up with the *inverse scale-space method* (see e.g. [10, 24]). We note that the inverse scale-space method corresponds to the *asymptotic regularization method* as introduced by Tautenhahn [28, 29].

The terminology ‘inverse scale-space’ is motivated from scale-space theory in *computer vision*: images contain structures at a variety of scales. Any feature can optimally be recognized at a particular scale. If the optimal scale is not available *a priori*, it is desirable to have an image representation at multiple scales.

A *scale-space* is an image representation at a continuum of scales, embedding the image u into a family

$$\{T_t(u) : t \geq 0\}$$

of gradually simplified versions satisfying:

- (i) *Recursivity*:

$$T_0(u) = u.$$

- (ii) *Causality*:

$$T_{t+s}(u) = T_t(T_s(u)) \quad \text{for all } s, t \geq 0.$$

(iii) *Regularity*:

$$\lim_{t \rightarrow 0^+} T_t(u) = u.$$

For more background on the topic of scale-space theory we refer to [13, 15, 19, 30].

The ill-posedness of inverse problems prohibits such a representation in scales of images and the concept has to be replaced by inverse scale-space theory, which includes approximate causality together with:

(i) *Inverse recursivity*:

$$T_\infty(y) = u^\dagger.$$

(ii) *Inverse regularity*:

$$\lim_{t \rightarrow \infty^-} T_t(y) = u^\dagger.$$

Here y is the input data and u^\dagger is a solution of (1). As shown in [24], (4) is an inverse scale-space method.

In this work we show that the inverse scale-space method for the constrained inverse problem (2) with appropriate P is a *level set method*. Level set methods have been developed by Osher and Sethian [20] (see also [27]). Recently, level set methods have been successfully applied for the solution of inverse problems (see e.g. [3, 5, 12, 16, 21–23]).

Moreover, we show that the shape derivative in form optimization and the level set derivative correspond. For simplicity of presentation we concentrate on highlighting this link by considering a particular example from [11].

2. Derivation of the level set method

In this section we consider the constraint optimization problem of solving (1) on the set of piecewise constant functions which attain two values, which we fix for the sake of simplicity of presentation to 0 and 1. Typical examples include parameter identification problems where the value 1 denotes an inclusion.

Let $\Omega \subseteq \mathbb{R}^n$ be bounded with boundary $\partial\Omega$ Lipschitz. Set

$$\mathcal{P} := \{u : u = \chi_{\tilde{\Omega}} : \tilde{\Omega} \subseteq \Omega\} \cap L^2(\Omega),$$

then the unconstrained inverse problem consists in solving (2) with

$$P : H^1(\Omega) \rightarrow \mathcal{P}$$

$$\phi \mapsto \frac{1}{2} + \frac{1}{2} \operatorname{sgn}(\phi) =: \frac{1}{2} + \frac{1}{2} \begin{cases} 1 & \text{for } \phi \geq 0 \\ -1 & \text{for } \phi < 0. \end{cases}$$

Moreover, let for the sake of simplicity of presentation,

$$F : L^2(\Omega) \rightarrow L^2(\Omega)$$

be Fréchet-differentiable. It is as well possible to consider the operator F in various Hilbert space settings such as, for instance, $F : H^1(\Omega) \rightarrow L^2(\partial\Omega)$. Since it does not make any methodological differences we concentrate on an operator on $L^2(\Omega)$. Also the space $H^1(\Omega)$ is chosen more or less arbitrarily; we have selected these spaces in such a way that the typical distance functions for smooth domains are contained in $H^1(\Omega)$.

Tikhonov regularization for this problem consists in minimizing the functional

$$\int_{\Omega} (F(P(\phi)) - y)^2 + \alpha \int_{\Omega} ((\phi - \phi_*)^2 + |\nabla(\phi - \phi_*)|^2). \quad (5)$$

Since the functional (5) may not attain a minimum, we consider the ‘minimizer’ in a generalized setting, as

$$\phi_\alpha = \lim_{\varepsilon \rightarrow 0^+} \phi_{\varepsilon, \alpha},$$

where the limit has to be understood appropriately, and $\phi_{\varepsilon, \alpha}$ minimizes the functional

$$\int_{\Omega} (F(P_\varepsilon(\phi)) - y)^2 + \alpha \int_{\Omega} ((\phi - \phi_*)^2 + |\nabla(\phi - \phi_*)|^2). \quad (6)$$

We use

$$P_\varepsilon(t) := \begin{cases} 0 & \text{for } t < -\varepsilon, \\ 1 + \frac{t}{\varepsilon} & \text{for } t \in [-\varepsilon, 0], \\ 1 & \text{for } t > 0, \end{cases}$$

for approximating P as $\varepsilon \rightarrow 0^+$. In this case we have

$$P'(t) = \lim_{\varepsilon \rightarrow 0^+} P'_\varepsilon(t) = \delta(t).$$

Here and in the following $\delta(t)$ denotes the one-dimensional δ -distribution. Moreover, we denote

$$u_\alpha := \lim_{\varepsilon \rightarrow 0^+} P_\varepsilon(\phi_{\varepsilon, \alpha}).$$

Note that we do *not* require that $u_\alpha = P(\phi_\alpha)$. The proposed methodology to define generalized solutions $u_\alpha = \lim_{\varepsilon \rightarrow 0^+} P(\phi_{\varepsilon, \alpha})$ is a standard way in *phase transition problems* (see e.g. [2]).

In the following we derive an optimality condition for a minimizer of (5), which is considered the limit $\varepsilon \rightarrow 0^+$ of the minimizers of the functionals (6). For this purpose it is convenient to recall some basic results from the *Morse theory* of surfaces. The particular results are collected from [9]. We emphasize that in this paper we only apply the Morse theory to the compact, smooth subset of \mathbb{R}^2 , which of course can be considered as surfaces.

Proposition 2.1. *Let ϕ be a smooth function on a compact smooth surface M , and $\phi^{-1}[a, b] \subseteq M$ contain no critical point of ϕ . Then,*

- (i) *the level sets $\phi^{-1}(b)$ and $\phi^{-1}(a)$ are diffeomorphic (in particular they consist of the same number of smooth circles diffeomorphic to a standard circle) [9, proposition 6.2.1.]. In particular, the Hausdorff measure of $\phi^{-1}(t)$, $t \in [a, b]$ changes continuously.*
- (ii) *Moreover, for any $\rho \in [a, b]$, $\phi^{-1}(\rho)$ is a smooth compact 1-manifold [9, p 107]. In particular, $\phi^{-1}(\rho)$ can be parametrized by finitely many disjoint curves.*

Lemma 2.2. *Let ϕ be a smooth function, having no critical point in a compact neighbourhood M of the level set $\phi^{-1}(0)$. Then,*

$$\lim_{\varepsilon \rightarrow 0^+} P'_\varepsilon(\phi) = \frac{1}{|\nabla\phi|} \delta(\phi).$$

We recall that $\delta(\phi)$ is the one-dimensional δ -distribution centred at the level line in the normal direction.

Proof. In dimension one this is a well-known result, especially in physics (see [25, 26]). We sketch the proof adopted to level set functions in dimension two; for higher dimension the generalization is obvious.

From proposition 2.1 we know that the level set $\phi^{-1}(0)$ is a smooth compact 1-manifold, which can be parametrized by a curve⁴ $s(\tau)$, $\tau \in [0, 2\pi)$, i.e.

$$\phi^{-1}(0) := \{s(\tau) = (s_1(\tau), s_2(\tau)) : \tau \in [0, 2\pi)\}.$$

Here n is the normal vector to the level set, which can be characterized as

$$n(\tau) = -\frac{\nabla\phi}{|\nabla\phi|}(s(\tau)).$$

We choose the negative sign in the definition of the normal vector based on the following considerations: if ϕ is a monotonically increasing function in the normal direction to the level set pointing into the domain bounded by the level set, then $n(\tau)$, as defined above, points outside this domain.

The basic idea of the proof is to find a relation between a parameter ε and a parametric function $\psi : [0, 2\pi) \rightarrow \mathbb{R}$ such that the sets

$$\Omega_\psi := \{s(\tau) + \rho n(\tau) : \tau \in [0, 2\pi), \rho \in [0, \psi(\tau))\}$$

and $\phi^{-1}(-\varepsilon, 0]$ ‘asymptotically’ correspond.

By making a Taylor series expansion we find

$$\begin{aligned} \phi(\Omega_\psi) &= \phi\left(\left\{s(\tau) - \rho \frac{\nabla\phi}{|\nabla\phi|}(s(\tau)) : \tau \in [0, 2\pi), \rho \in [0, \psi(\tau))\right\}\right) \\ &= \left\{\phi\left(s(\tau) - \rho \frac{\nabla\phi}{|\nabla\phi|}(s(\tau))\right) : \tau \in [0, 2\pi), \rho \in [0, \psi(\tau))\right\} \\ &= \left\{\phi(s(\tau)) - \rho \frac{\nabla\phi}{|\nabla\phi|}(s(\tau)) \nabla\phi(s(\tau)) + O(\rho^2) : \tau \in [0, 2\pi), \rho \in [0, \psi(\tau))\right\} \\ &= \{-\rho |\nabla\phi|(s(\tau)) + O(\rho^2) : \tau \in [0, 2\pi), \rho \in [0, \psi(\tau))\}. \end{aligned}$$

If we choose

$$\psi(\tau) := \psi_\varepsilon(\tau) = \frac{\varepsilon}{|\nabla\phi(s(\tau))|},$$

and set

$$C_{\min} := \inf\{|\nabla\phi|(s(\tau)) : \tau \in [0, 2\pi)\},$$

then there exists a constant C such that

$$\Omega_- := \left[-\varepsilon + \varepsilon^2 \frac{C}{C_{\min}^2}, -\varepsilon^2 \frac{C}{C_{\min}^2}\right] \subseteq \phi(\Omega_\psi) \subseteq \left[-\varepsilon - \varepsilon^2 \frac{C}{C_{\min}^2}, \varepsilon^2 \frac{C}{C_{\min}^2}\right] =: \Omega_+.$$

Set $\tau = \frac{C}{C_{\min}^2}$. Then, for $v \in C(\overline{\Omega})$, it follows from the *co-area formula* [8] that

$$\begin{aligned} \left| \int_{\phi^{-1}(-\varepsilon, 0)} v - \int_{\Omega_\psi} v \right| &\leq \frac{\max |v|}{C_{\min}} \left\{ \int_{\phi^{-1}(-\varepsilon - \tau\varepsilon^2, -\varepsilon + \tau\varepsilon^2)} |\nabla\phi| + \int_{\phi^{-1}(-\tau\varepsilon^2, \tau\varepsilon^2)} |\nabla\phi| \right\} \\ &\leq \frac{\max |v|}{C_{\min}} \left\{ \int_{-\varepsilon - \tau\varepsilon^2}^{-\varepsilon + \tau\varepsilon^2} \mathcal{H}^1(\phi^{-1}(\rho)) d\rho + \int_{-\tau\varepsilon^2}^{\tau\varepsilon^2} \mathcal{H}^1(\phi^{-1}(\rho)) d\rho \right\} \end{aligned}$$

where $\mathcal{H}^1(\phi^{-1}(\rho))$ is the one-dimensional Hausdorff measure of the set $\phi^{-1}(\rho)$. According to proposition 2.1, $\mathcal{H}^1(\phi^{-1}(\rho))$ is uniformly bounded. This implies that

$$\left| \int_{\phi^{-1}(-\varepsilon, 0)} v - \int_{\Omega_\psi} v \right| = O(\varepsilon^2),$$

⁴ For the sake of simplicity of presentation we assume that the level set is parametrized by just one curve. The general case of finitely many disjoint curves is analogous.

and consequently

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} P'_\varepsilon(\phi)v = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\phi^{-1}(-\varepsilon, 0)} v = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\Omega_{\psi_\varepsilon}} v = \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega_{\psi_\varepsilon}} \frac{1}{\psi_\varepsilon} \frac{1}{|\nabla\phi|} v.$$

This shows that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} P'_\varepsilon(\phi)v &= \lim_{\varepsilon \rightarrow 0^+} \int_0^{2\pi} \frac{1}{|\nabla\phi|(s(\tau))} \frac{1}{\psi_\varepsilon(\tau)} \int_0^{\psi_\varepsilon} v \left| \det \begin{bmatrix} s'_1(\tau) + \rho n'_1(\tau) & n_1(\tau) \\ s'_2(\tau) + \rho n'_2(\tau) & n_2(\tau) \end{bmatrix} \right| d\rho d\tau \\ &= \int_{\Omega} \delta(\phi) \frac{v}{|\nabla\phi|}. \end{aligned}$$

□

Lemma 2.2 is central to derive the optimality condition for a minimizer of (5). From the definition of a minimizer of (6) it follows that for all $h \in H^1(\Omega)$

$$\int_{\Omega} (F(u_{\varepsilon, \alpha}) - y) F'(u_{\varepsilon, \alpha}) P'_\varepsilon(\phi_{\varepsilon, \alpha}) h + \alpha \int_{\Omega} ((\phi_{\varepsilon, \alpha} - \phi_*) h + \nabla(\phi_{\varepsilon, \alpha} - \phi_*) \nabla h) = 0. \quad (7)$$

We denote by $F'(u)^*$, $P'_\varepsilon(\phi)^*$ the L^2 -adjoints of $F'(u)$, $P'_\varepsilon(\phi)$, respectively, i.e. for all $v, w \in L^2(\Omega)$

$$\int_{\Omega} w(F'(u)v) = \int_{\Omega} (F'(u)^*w)v \quad \text{and} \quad \int_{\Omega} w(P'_\varepsilon(\phi)v) = \int_{\Omega} (P'_\varepsilon(\phi)^*w)v.$$

Since $P'_\varepsilon(\phi)$ is self-adjoint, i.e. $P'_\varepsilon(\phi)^* = P'_\varepsilon(\phi)$, it follows that

$$\begin{aligned} P'_\varepsilon(\phi_{\varepsilon, \alpha}) F'(u_{\varepsilon, \alpha})^* (F(u_{\varepsilon, \alpha}) - y) + \alpha(I - \Delta)(\phi_{\varepsilon, \alpha} - \phi_*) &= 0 \quad \text{on } \Omega, \\ \frac{\partial(\phi_{\varepsilon, \alpha} - \phi_*)}{\partial n} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (8)$$

Thus $u_\alpha = \lim_{\varepsilon \rightarrow 0^+} u_{\varepsilon, \alpha}$ and $\phi_\alpha = \lim_{\varepsilon \rightarrow 0^+} \phi_{\varepsilon, \alpha}$ satisfies

$$\delta(\phi_\alpha) \frac{F'(u_\alpha)^*(F(u_\alpha) - y)}{|\nabla\phi_\alpha|} + \alpha(I - \Delta)(\phi_\alpha - \phi_*) = 0. \quad (9)$$

For the sake of simplicity of presentation we assume that the operator F is of such quality that $F'(u)^*(F(u) - y)$ is continuous on Ω . Note that in general this may not be the case since $F'(u)^*(F(u) - y) \in H^1(\Omega)$.

Therefore, it follows from (9) that

$$(I - \Delta)^{-1} \left(\delta(\phi_\alpha) \frac{F'(u_\alpha)^*(F(u_\alpha) - y)}{|\nabla\phi_\alpha|} \right) + \alpha(\phi_\alpha - \phi_*) = 0.$$

Set $\alpha = \frac{1}{\Delta t}$ and set $\phi_\alpha = \phi(t)$, $\phi_* = \phi(0)$ and accordingly $u(t) = P(\phi(t))$. Then, by taking the formal limit $\Delta t \rightarrow 0^+$ we get the asymptotic regularization method

$$\frac{\partial\phi}{\partial t} = -(I - \Delta)^{-1} \left(\delta(\phi(t)) \frac{F'(u(t))^*(F(u(t)) - y)}{|\nabla\phi(t)|} \right). \quad (10)$$

The right-hand side v of (10) solves the equation

$$\begin{aligned} (I - \Delta)v &= -\delta(\phi(t)) \frac{F'(u(t))^*(F(u(t)) - y)}{|\nabla\phi(t)|}, \\ \frac{\partial v}{\partial n} &= 0. \end{aligned} \quad (11)$$

Using potential theory (see e.g. [6, 14]), a solution v_1 of the homogeneous problem

$$\Delta v_1(t) = \delta(\phi(t)) \frac{F'(u(t))^*(F(u(t)) - y)}{|\nabla\phi(t)|},$$

is given by the *single-layer potential*

$$v_1(x) = - \int_{\phi(t)^{-1}(0)} \frac{F'(u(t))^*(F(u(t)) - y)(z)\gamma(x, z)}{|\nabla\phi(t)(z)|} dz,$$

where

$$\gamma(x, y) = \begin{cases} \frac{1}{2\pi} \ln\left(\frac{1}{|x - y|}\right) & \text{in } \mathbb{R}^2, \\ \frac{1}{4\pi} \frac{1}{|x - y|} & \text{in } \mathbb{R}^3 \end{cases} \quad (12)$$

is the *single-layer potential*.

Then, $v = v_1 + v_2$ solves (11) where v_2 solves

$$v_2 - \Delta v_2 = -v_1 \quad \text{on } \Omega \quad \frac{\partial v_2}{\partial n} = -\frac{\partial v_1}{\partial n} \quad \text{on } \partial\Omega.$$

Equation (10) is a *level set method* describing the evolution of the level set function ϕ . The zero-level set of ϕ , i.e. the set $\{\phi = 0\}$, describes the boundary of the inclusions to be recovered.

Remark 2.3. An adequate approximation of P is central in our considerations. The family of functions

$$Q_\varepsilon(t) := \begin{cases} 0 & \text{for } t < -\varepsilon, \\ \frac{t + \varepsilon}{2\varepsilon} & \text{for } t \in [-\varepsilon, \varepsilon], \\ 1 & \text{for } t > \varepsilon, \end{cases}$$

approximates the δ -distribution too. The point-wise limit of Q_ε is

$$P(t) := \begin{cases} 0 & \text{for } t < 0, \\ \frac{1}{2} & \text{for } t = 0, \\ 1 & \text{for } t > 0, \end{cases}$$

which is not in \mathcal{P} if the n -dimensional Lebesgue measure of $\phi^{-1}(0)$ is greater than zero. This would not be appropriate for our problem setting.

In this section we have elaborated on the interaction between constraint regularization methods and level set methods. We have shown that our level set method can be considered as an inverse scale-space method, respectively asymptotic regularization method. In contrast to standard results on asymptotic regularization methods and inverse scale-space methods (see [10, 28, 29]), here the situation is more involved, since the regularizer of the underlying regularization functional (5) is considered as an approximation of the minimizers of the functional (6), i.e. it is a Γ -limit (see e.g. [1]).

One of the most significant advantages of level set methods is that the topology of the zero-level set may change over time. So far, this situation has not been covered by our derivation of level set methods, where we essentially relied on proposition 2.1 and lemma 2.2. In the case that a topology change occurs, the Morse index of the level set function ϕ changes and proposition 2.1, and consequently lemma 2.2, are not applicable. Moreover, in this case the single-layer potential representations (12) are no longer valid (see e.g. [4, 14]), since the topology changes result in a domain with cusps. The effect of topology changes on the level set methods are the subject of ongoing research. In this letter we are interested in revealing interactions between constraint regularization techniques, level set methods, and shape optimization. To show the interaction part we rely on some explicit calculations of the shape derivative in [11], where inclusions are considered smooth without cusps. Thus in order to compare level set evolution and shape derivative, we find it desirable to limit our considerations and neglect topology changes.

2.1. Relation to other level set methods

Equation (10) is a Hamilton–Jacobi type equation of the form

$$\frac{\partial \phi}{\partial t} + V \nabla \phi = 0 \quad (13)$$

with velocity

$$V = \frac{(I - \Delta)^{-1} \left(\delta(\phi) \frac{F'(u(t))^*(F(u(t)) - y)}{|\nabla \phi(t)|} \right)}{|\nabla \phi(t)|} \frac{\nabla \phi}{|\nabla \phi(t)|}.$$

The numerical solution of (10) is similar to the implementation of well-established level set methods, e.g., as considered by Santosa [23], who suggested a velocity

$$V = -F'(u(t))^*(F(u(t)) - y) \frac{\nabla \phi(t)}{|\nabla \phi(t)|}.$$

The differential equation

$$\frac{\partial \phi}{\partial t} = F'(u(t))^*(F(u(t)) - y) |\nabla \phi(t)| \quad (14)$$

is solved explicitly in time, which results in

$$\frac{\phi(t + \Delta t) - \phi(t)}{\Delta t} = F'(u(t))^*(F(u(t)) - y) |\nabla \phi(t)|.$$

After several numerical time-steps the iterates are *updated*. In our level-set approach such an update is inherent, since in each step the data are normalized by the operator $(I - \Delta)^{-1}$.

2.2. Relation to shape optimization

In this section we show that the term

$$\delta(\phi) \frac{F'(u)^*(F(u) - y)}{|\nabla \phi|}$$

is the steepest descent direction of the functional $\|F(u) - y\|^2$ with respect to the *shape* of the level set $\phi^{-1}(0)$.

It is illustrative to show this relation by example. To this end we consider the *inverse potential problem* of recovery of a object $D \subseteq \mathbb{R}^2$ in

$$\Delta v = \chi(D) \quad \text{in } \Omega \text{ with } v = 0 \text{ on } \partial\Omega.$$

In this context

$$\begin{aligned} F : L^2(\Omega) &\rightarrow L^2(\Omega) \\ f &\mapsto \Delta^{-1} f \text{ with homogeneous Dirichlet data.} \end{aligned}$$

The numerical recovery of shape of the inclusion D from Neumann boundary measurements was considered in [11]. For the sake of simplicity of presentation, here we are interested in the shape derivative of F , while Hettlich and Rundell considered the operator $T \circ F$, where T is the Neumann trace operator. Since T is linear, the shape derivative of $T \circ F$ is completely determined by the shape derivative of F , and thus we do not impose any restriction on the consideration by considering the simpler problem.

The operator F is linear and thus the Gateaux derivative of F at u in direction h satisfies $F'(u)h = F(h)$. Thus, the *level set derivative* is given by

$$v := F'(u)P'(\phi)h = F(P'(\phi)h) = \Delta^{-1} \left(\delta(\phi) \frac{h}{|\nabla \phi|} \right). \quad (15)$$

Let v_1 be the single-layer potential according to h on $\phi^{-1}(0)$, i.e.

$$v_1(x) = - \int_{\phi^{-1}(0)} \frac{1}{2\pi} \ln \frac{1}{|x-y|} \frac{h}{|\nabla\phi|}(y) dy.$$

This function satisfies

$$\Delta v_1 = \delta(\phi) \frac{h}{|\nabla\phi|} \quad \text{on } \Omega.$$

Let v_2 be the solution of

$$\Delta v_2 = 0 \quad \text{on } \Omega \quad \text{and} \quad v_1 = -v_2 \quad \text{on } \partial\Omega.$$

Then $v = v_1 + v_2$ solves

$$\Delta v = \delta(\phi) \frac{h}{|\nabla\phi|} \quad \text{on } \Omega \quad \text{and} \quad v = 0 \quad \text{on } \partial\Omega.$$

Moreover, the single-layer potential satisfies on the zero-level set

$$\begin{aligned} \left(\frac{\partial v_1}{\partial n} \right)_+ - \left(\frac{\partial v_1}{\partial n} \right)_- &= \frac{h}{|\nabla\phi|}, \\ (v_1)_+ &= (v_1)_-. \end{aligned}$$

Here $(\cdot)_+$, $(\cdot)_-$ denote the limits from outside, inside the domain bounded by the zero-level curves, respectively.

We recall that h is considered a perturbation of the level set *function*. A change in the level set function implies a change in the zero-level set, which eventually turns out to be the shape derivative.

To make this concrete, let s_{th} be the parametrizations of $(\phi+th)^{-1}(0)$, i.e. $(\phi+th)(s_{th}) = 0$. We make a Taylor ansatz with respect to the parametrization

$$s_{th} = s + t\tilde{h} + \mathcal{O}(t^2), \quad (16)$$

and a series expansion for ϕ and h , which gives

$$0 = (\phi + th)(s_{th}) = t\nabla\phi\tilde{h} + th(s) + \mathcal{O}(t^2).$$

This shows that on the zero-level set we have

$$\frac{h}{|\nabla\phi|} = - \frac{\nabla\phi}{|\nabla\phi|} \cdot \tilde{h} = n \cdot \tilde{h}.$$

Thus v satisfies the differential equation

$$\begin{aligned} \Delta v &= 0 \quad \text{on } \Omega \setminus \phi^{-1}(0), \\ v &= 0 \quad \text{on } \partial\Omega; \\ \left(\frac{\partial v}{\partial n} \right)_+ - \left(\frac{\partial v}{\partial n} \right)_- &= \tilde{h} \cdot n \quad \text{on } \phi^{-1}(0), \\ (v)_+ &= (v)_- \quad \text{on } \phi^{-1}(0). \end{aligned} \quad (17)$$

This is the shape derivative $F'(D)(\tilde{h})$ of F at $D = \{x : P(\phi) > 0\}$ in direction \tilde{h} as calculated by Hettlich and Rundell [11].

Our calculations show that the level set derivative $v := F'(u)P'(\phi)h$ can be computed from the shape derivative. Now, we point out that the converse is equally true. This is nontrivial since the arguments \tilde{h} appearing in the shape derivative are multidimensional functions, while the argument h in the level set derivative is one-dimensional.

Let \tilde{h} be expressed in terms of the local coordinate system n and τ , where n , τ are the normal, respectively tangential vectors on the zero-level set, i.e.

$$\tilde{h} = hn + h_\tau \tau.$$

The shape derivative is independent of the tangential component, which in particular implies that the shape derivative gradient descent deforms the shapes in the normal direction to the level curve. Thus, from (15) we find that

$$F'(D)(\tilde{h}) = F'(D)(hn) = F'(P\phi)h. \quad (18)$$

Thus, we have shown the following theorem.

Theorem 2.4. *By (18) the level set derivative $F'(u)P'(\phi)h = F(P'(\phi))h$ is uniquely determined from the shape derivative and vice versa.*

From theorem 2.4 we see that the level set derivative moves the zero-level set in the direction of the shape derivative.

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