

ON REGULARIZATION METHODS FOR INVERSE PROBLEMS OF DYNAMIC TYPE

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□ *In this paper, we consider new regularization methods for linear inverse problems of dynamic type. These methods are based on dynamic programming techniques for linear quadratic optimal control problems. Two different approaches are followed: a continuous and a discrete one. We prove regularization properties and also obtain rates of convergence for the methods derived from both approaches. A numerical example concerning the dynamic EIT problem is used to illustrate the theoretical results.*

Keywords Dynamic inverse problems; Dynamic programming; Regularization.

AMS Subject Classification 65J20; 47A52; 65J22.

1. INTRODUCTION

1.1. Inverse Problems of Dynamic Type

We begin by introducing the notion of dynamic inverse problems. Roughly speaking, these are inverse problems in which the measuring process—performed to obtain the data—is time dependent. As usual, the problem data corresponds to indirect information about an unknown parameter, which has to be reconstructed. The desired parameter is allowed to be itself time dependent.

Let X, Y be Hilbert spaces. We consider the inverse problem of finding $u : [0, T] \rightarrow X$ from the equation

$$F(t)u(t) = y(t), \quad t \in [0, T], \quad (1.1)$$

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where $y : [0, T] \rightarrow Y$ are the dynamic measured data and $F(t) : X \rightarrow Y$ are linear ill-posed operators indexed by the parameter $t \in [0, T]$. Notice that $t \in [0, T]$ corresponds to a (continuous) temporal index. The linear operators $F(t)$ map the unknown parameter $u(t)$ to the measurements $y(t)$ at the time point t during the finite time interval $[0, T]$. This is called a *dynamic inverse problem*.

If the properties of the parameter u do not change during the measuring process, the inverse problem in (1.1) reduces to the simpler case $F(t)u = y(t)$, $t \in [0, T]$, where $u(t) \equiv u \in X$. We shall refer to this as *static inverse problem*.

As one would probably expect at this point, a discrete version of (1.1) can also be formulated. The assumption that the measuring process is discrete in time leads to the *discrete dynamic inverse problems*, which are described by the model

$$F_k u_k = y_k, \quad k = 0, \dots, N \quad (1.2)$$

and correspond to phenomena in which only a finite number of measurements y_k are available. As in the (continuous) dynamic inverse problems, the unknown parameter can also be assumed to be constant during the measurement process. In this case, we shall refer to this problems as *discrete static inverse problems*.

Because the operators $F(t)$ are ill-posed, at each time point $t \in [0, T]$ the solution $u(t)$ does not depend on a stable way on the right-hand side $y(t)$. Therefore, regularization techniques have to be used in order to obtain a stable solution $u(t)$. It is convenient to consider *time-dependent regularization techniques*, which take into account the fact that the parameter $u(t)$ evolves continuously with the time.

In this paper, we shall concentrate our attention on the (continuous and discrete) dynamic inverse problems. The analysis of the static problems follows in a straightforward way, as it represents a particular subclass of the dynamic problems.

1.2. Some Relevant Applications

As a first example of dynamic inverse problem, we present the *dynamical source identification problem*: Let $u(x, t)$ be a solution to

$$\Delta_x u(x, t) = f(x, t) \quad \text{in } \Omega,$$

where $f(x, t)$ represents an unknown source that moves around and might change shape with time t . The inverse problem in this case is to reconstruct f from single or multiple measurements of Dirichlet and Neumann data ($u(x, t), \partial_n u(x, t)$), on the boundary $\partial\Omega$ over time $t \in [0, T]$.

Again, using the dynamic programming technique for this discrete linear quadratic control problem, we are able to derive an iterative regularization method for the inverse problem (1.2). In this discrete framework, the dynamic programming approach consists basically of the Bellman optimality principle and the dynamic programming equation.

1.4. Literature Overview and Outline of the Paper

Continuous and discrete regularization methods for inverse problems have been quite well studied in the past two decades and one can find relevant information, for example, in [9–12, 18, 24] and in the references therein.

So far, dynamic programming techniques have been mostly applied to solve particular inverse problems. In [15], the inverse problem of identifying the initial condition in a semilinear parabolic equation is considered. In [16], the same authors consider a problem of parameter identification for systems with distributed parameters. In [14], the dynamic programming methods are used in order to formulate an abstract functional analytical method to treat general inverse problems.

Concerning dynamic inverse problems, regularization methods were considered for the first time in [21, 22]. There, the authors analyze discrete dynamic inverse problems and propose a procedure called *spatio temporal regularizer* (STR), which is based on the minimization of the functional

$$\Phi(u) := \sum_{k=0}^N \|F_k u_k - y_k\|_{L^2}^2 + \lambda^2 \sum_{k=0}^N \|u_k\|_{L^2}^2 + \mu^2 \sum_{k=0}^{N-1} \frac{\|u_{k+1} - u_k\|_{L^2}^2}{(t_{k+1} - t_k)^2}. \quad (1.6)$$

Notice that the term with factor λ^2 corresponds to the classical (spatial) Tikhonov-Philips regularization, while the term with factor μ^2 enforces the temporal smoothness of u_k .

A characteristic of this approach is the fact that the whole solution vector $\{u_k\}_{k=0}^N$ has to be computed at a time. Therefore, the corresponding system of equations to evaluate $\{u_k\}$ has very large dimension. In the STR regularization, the associated system matrix is decomposed and rewritten into a Sylvester matrix form. The efficiency of this approach is based on fast solvers for the Sylvester equation.

This paper is organized as follows: In Section 2 we derive the solution methods discussed in this paper. In Section 3 we analyze some regularization properties of the proposed methods. In Section 4 we present numerical realizations of the discrete regularization method as well as a discretization of the continuous regularization method. For comparison purposes, we consider a dynamic EIT problem, similar to the one treated in [22].

2. DERIVATION OF THE REGULARIZATION METHODS

We begin this section considering a particular case, namely the dynamic inverse problems with constant operator. The analysis of this simpler problem allows us to illustrate the dynamic programming approach followed in this paper. In Subsections 2.2 and 2.3, we consider general dynamic inverse problems and derive a continuous and a discrete regularization method, respectively.

2.1. A Tutorial Approach: The Constant Operator Case

In this subsection, we derive a family of regularization operators for the dynamic inverse problem in (1.1), in the particular case where the operators $F(t)$ does not change during the measurement process, that is, $F(t) = F : X \rightarrow Y$, $t \in [0, T]$. The starting point of our approach is the constrained optimization problem in (1.4). We shall consider a very simple dynamic, which does not depend on the state u , but only on the control v , namely: $u' = v$, $t \geq 0$. In this case, the control v can be interpreted as a *velocity function*. The pairs (u, v) formed by a trajectory and the corresponding control function are called *admissible processes* for the control problem.

Next we define the residual function $\varepsilon(t) := Fu(t) - y(t)$ associated with a given trajectory u . Notice that this residual function evolves according to the dynamic

$$\varepsilon' = Fu(t) - y'(t) = Fv(t) - y'(t), \quad t \geq 0.$$

With this notation, problem (1.4) can be rewritten in the form

$$\begin{cases} \text{Minimize } J(\varepsilon, v) = \frac{1}{2} \int_0^T \langle \varepsilon(t), L(t)\varepsilon \rangle + \langle v(t), M(t)v(t) \rangle dt \\ \text{s.t. } \varepsilon' = Fv(t) - y'(t), \quad t \geq 0, \quad \varepsilon(0) = F(0)u_0 - y(0). \end{cases} \quad (2.1)$$

The next result states a parallel between solvability of the optimal control problem (1.4) and the auxiliary problem (2.1).

Proposition 2.1. *If (\bar{u}, \bar{v}) is an optimal process for problem (1.4), then the process $(\bar{\varepsilon}, \bar{v})$, with $\bar{\varepsilon} := F\bar{u}(t) - y(t)$, will be an optimal process for problem (2.1). Conversely, if $(\bar{\varepsilon}, \bar{v})$ is an optimal process for problem (2.1), with $\varepsilon(0) = Fu_0 - y(0)$, for some $u_0 \in X$, then the corresponding process (\bar{u}, \bar{v}) is an optimal process for problem (1.4).*

In the sequel, we derive the dynamic programming approach for the optimal control problem in (2.1). We start by introducing the first

Hamilton function $H : [0, T] \times X^3 \rightarrow \mathbb{R}$, defined by

$$H(t, \varepsilon, \lambda, v) := \langle \lambda, Fv \rangle - \langle \lambda, y'(t) \rangle + \frac{1}{2}[\langle \varepsilon, L(t)\varepsilon \rangle + \langle v, M(t)v \rangle].$$

Notice that the variable λ plays the role of a Lagrange multiplier in the above definition. According to the Pontryagin's maximum principle, the Hamilton function furnishes a necessary condition of optimality for problem (2.1). Furthermore, because (in this particular case) this function is convex in the control variable, this optimality condition also happens to be sufficient. From the maximum principle we know that, along an optimal trajectory, the equality

$$0 = \frac{\partial H}{\partial v}(t, \varepsilon(t), \lambda(t), v(t)) = F^* \lambda(t) + M(t)v(t) \quad (2.2)$$

holds. This means that the optimal control \bar{v} can be obtained directly from the Lagrange multiplier $\lambda : [0, T] \rightarrow X$, by solving the system

$$M(t)\bar{v}(t) = -F^* \lambda(t), \quad \forall t.$$

Therefore, the key task is actually the evaluation of the Lagrange multiplier. This leads us to the HJ equation. Substituting the above expression for \bar{v} in (2.2), we can define the second Hamilton function $\mathcal{H} : \mathbb{R} \times X^2 \rightarrow \mathbb{R}$

$$\mathcal{H}(t, \varepsilon, \lambda) := \min_{v \in X} \{H(t, \varepsilon, \lambda, v)\} = \frac{1}{2} \langle \varepsilon, L(t)\varepsilon \rangle - \langle \lambda, y'(t) \rangle - \frac{1}{2} \langle \lambda, FM(t)^{-1}F^* \lambda \rangle.$$

Now, let $V : [0, T] \times X \rightarrow \mathbb{R}$ be the value function for problem (2.1), that is,

$$V(t, \xi) := \min \left\{ \frac{1}{2} \int_t^T \langle \varepsilon(s), L(s)\varepsilon(s) \rangle + \langle v(s), M(s)v(s) \rangle ds \mid (\varepsilon, v) \text{ admissible process for (2.1) with initial condition } \varepsilon(t) = \xi \right\}. \quad (2.3)$$

Our interest in the value function comes from the fact that this function is related to the Lagrange multiplier λ by: $\lambda(t) = V_\varepsilon(t, \bar{\varepsilon})$, where $\bar{\varepsilon}$ is an optimal trajectory. From the control theory we know that the value function is a solution of the HJ equation

$$\begin{aligned} 0 &= V_t(t, \varepsilon) + \mathcal{H}(t, \varepsilon, V_\varepsilon(t, \varepsilon)) \\ &= V_t + \frac{1}{2} \langle \varepsilon, L(t)\varepsilon \rangle - \langle V_\varepsilon, y'(t) \rangle - \frac{1}{2} \langle V_\varepsilon, FM(t)^{-1}F^* V_\varepsilon \rangle. \end{aligned} \quad (2.4)$$

Now, making the ansatz: $V(t, \varepsilon) = \frac{1}{2}\langle \varepsilon, Q(t)\varepsilon \rangle + \langle b(t), \varepsilon \rangle + g(t)$, with $Q : [0, T] \rightarrow \mathbb{R}$, $b : [0, T] \rightarrow X$ and $g : \mathbb{R} \rightarrow \mathbb{R}$, we are able to rewrite (2.4) in the form

$$\begin{aligned} & \frac{1}{2}\langle \varepsilon, Q'(t)\varepsilon \rangle + \langle b'(t), \varepsilon \rangle + g'(t) + \frac{1}{2}\langle \varepsilon, L(t)\varepsilon \rangle - \langle Q(t)\varepsilon + b(t), y'(t) \rangle \\ & - \frac{1}{2}\langle Q(t)\varepsilon + b(t), FM(t)^{-1}F^*[Q(t)\varepsilon + b(t)] \rangle = 0. \end{aligned} \quad (2.5)$$

This is a polynomial equation in ε , therefore the quadratic, the linear, and the constant terms must vanish. The quadratic term yields for Q the Riccati equation:

$$Q'(t) = -L(t) + QFM(t)^{-1}F^*Q. \quad (2.6)$$

From the linear term in (2.5), we obtain an evolution equation for b

$$b' = Q(t)FM(t)^{-1}F^*b + Q(t)y'(t) \quad (2.7)$$

and from the constant term in (2.5), we derive an evolution equation for g

$$g' = \frac{1}{2}\langle b(t), FM(t)^{-1}F^*b(t) \rangle + \langle b(t), y'(t) \rangle. \quad (2.8)$$

Notice that the cost of all admissible processes for an initial condition of the type (T, ε) is zero. Therefore we have to consider the system equations (2.6), (2.7), (2.8) with the final conditions

$$Q(T) = 0, \quad b(T) = 0, \quad g(T) = 0. \quad (2.9)$$

Notice that this system can be solved separately, first for Q , then for b , and finally for g .

Once we have solved the initial value problem (2.6)–(2.9), the Lagrange multiplier is given by $\lambda(t) = Q(t)\bar{\varepsilon}(t) + b(t)$ and the optimal control is obtained in the form of the feedback control $\bar{v}(t) = -M^{-1}(t)F^*[Q(t)\bar{\varepsilon}(t) + b(t)]$. Therefore, the optimal trajectory of problem (1.4) is given by

$$\bar{u}' = -M^{-1}(t)F^*(Q(t)[F\bar{u}(t) - y(t)] + b(t)), \quad \bar{u}(0) = u_0. \quad (2.10)$$

By choosing appropriately a family of operators $\{M_\alpha, L_\alpha\}_{\alpha>0}$, it is possible to use the corresponding optimal trajectories \bar{u}_α , defined by the initial value problem (2.10) in order to define a family of reconstruction operators $R_\alpha : L^2((0, T); Y) \rightarrow H^1((0, T); X)$, by

$$R_\alpha(y) := u_0 - \int_0^t M_\alpha^{-1}(s)F^*(Q(s)[F\bar{u}_\alpha(s) - y(s)] + b(s))ds. \quad (2.11)$$

We shall return to the operators $\{R_\alpha\}$ in Section 3 and prove that the family of operators defined in (2.11) is a regularization method for (1.4) (see, e.g., [9]).

Remark 2.2. It is possible to simplify the above equations to compute the optimal trajectory \bar{u} . If we introduce the function $\eta(t) := F^*Q(t)y(t) - F^*b(t)$, then we can write $\bar{u}' = -M^{-1}(t)F^*Q(t)F\bar{u} + M^{-1}(t)\eta$. Furthermore, using the equations for Q' and b' , we have $\eta' = -F^*Ly(t) + F^*Q(t)FM^{-1}(t)\eta$. Thus, solving (2.10) is equivalent to solve the system

$$\bar{u}' = -M^{-1}(t)F^*Q(t)F\bar{u} + M^{-1}(t)\eta, \quad \eta' = -F^*Ly(t) + F^*Q(t)FM^{-1}(t)\eta.$$

This system can again be solved separately, first for η [backwards in time, with $\eta(T) = 0$] and then for \bar{u} (forward in time). Notice that the computation of both $b(t)$ and $g(t)$ is not required to build this system. Furthermore, we do not need the derivative of the data $y(t)$.

2.2. Dynamic Inverse Problems

In the sequel, we consider the dynamic inverse problem described in (1.1). As in the previous subsection, we shall look for a continuous regularization strategy.

We start by considering the constrained optimization problem (1.4), where $F(t)$, $u(t)$, and $y(t)$ are defined as in (1.1), $v(t) \in X$, $t \in [0, T]$, $L(t) : Y \rightarrow Y$, $M(t) : X \rightarrow X$, $A(t) \equiv I : X \rightarrow X$, $B(t) \equiv 0$, and $u_0 \in X$.

Following the footsteps of the previous subsection, we define the first Hamilton function $H : [0, T] \times X^3 \rightarrow \mathbb{R}$ by

$$H(t, u, \lambda, v) := \langle \lambda, v \rangle + \frac{1}{2}[\langle F(t)u - y(t), L(t)(F(t)u - y(t)) \rangle + \langle v, M(t)v \rangle].$$

Thus, it follows from the maximum principle $0 = \partial H / \partial v(t, u(t), \lambda(t))$, $v(t) = \lambda(t) + M(t)v(t)$, and we obtain a relation between the optimal control and the Lagrange parameter; namely, $\bar{v}(t) = -M^{-1}(t)\lambda(t)$.

As before, we define the second Hamilton function $\mathcal{H} : \mathbb{R} \times X^2 \rightarrow \mathbb{R}$

$$\mathcal{H}(t, u, \lambda) := \frac{1}{2}\langle F(t)u - y(t), L(t)(F(t)u - y(t)) \rangle - \frac{1}{2}\langle \lambda, M(t)^{-1}\lambda \rangle.$$

Because $\lambda(t) = \partial V / \partial u(t, u)$, where $V : [0, T] \times X \rightarrow \mathbb{R}$ is the value function of problem (1.4), it is enough to obtain V . This is done by solving the HJ equation [see (2.4)]

$$0 = V_t + \frac{1}{2}\langle F(t)u - y(t), L(t)(F(t)u - y(t)) \rangle - \frac{1}{2}\langle V_u, M(t)^{-1}V_u \rangle.$$

As in Subsection 2.1, we make the ansatz $V(t, u) = \frac{1}{2}\langle u, Q(t)u \rangle + \langle b(t), u \rangle + g(t)$, with $Q : [0, T] \rightarrow \mathbb{R}$, $b : [0, T] \rightarrow X$ and $g : \mathbb{R} \rightarrow \mathbb{R}$. Then, we are able to rewrite the HJ equation above in the form of a polynomial equation in u . Arguing as in (2.5), we conclude that the quadratic, the linear, and the constant terms of this polynomial equation must all vanish. Thus we obtain

$$Q' = Q^*M(t)^{-1}Q - F^*(t)L(t)F(t), \quad b' = Q(t)^*M(t)^{-1}b + F^*(t)L(t)y(t). \quad (2.12)$$

The final conditions $Q(T) = 0$, $b(T) = 0$ are derived just like in the previous subsection.¹ Once the above system is solved, the optimal control \bar{u} is obtained by solving

$$\bar{u}'(t) = -M^{-1}(t)V_u(t, u) = -M^{-1}(t)[Q(t)\bar{u}(t) + b(t)] \quad (2.13)$$

with initial condition $\bar{u}(0) = u_0$.

Following the ideas of the previous tutorial subsection, we shall choose a family of operators $\{M_\alpha, L_\alpha\}_{\alpha>0}$ and use the corresponding optimal trajectories \bar{u}_α in order to define a family of reconstruction operators $R_\alpha : L^2((0, T); Y) \rightarrow H^1((0, T); X)$,

$$R_\alpha(y) := u_0 - \int_0^t M_\alpha^{-1}(s)[Q(s)\bar{u}(s) + b(s)]ds.$$

The regularization properties of the operators $\{R_\alpha\}$ will be analyzed in Section 3.

2.3. Discrete Dynamic Inverse Problems

In this subsection, we use the optimal control problem (1.5) as starting point to derive a discrete regularization method for the inverse problem in (1.2).

In the framework of discrete dynamic inverse problems, we have a trajectory, represented by the sequence u_k , which evolves according to the dynamic

$$u_{k+1} = A_k u_k + B_k v_k, \quad k = 0, 1, \dots, N$$

where the operators A_k and B_k still have to be chosen and $\{v_k\}_{k=0}^{N-1}$, is the control of the system. As in the continuous case, we shall consider a simpler

¹Because function g is not needed for the computation of the optimal trajectory, we omit the expression of the corresponding dynamic.

dynamic: $u_{k+1} = u_k + v_k$, $k = 0, 1, \dots$ (i.e., $A_k = B_k = I$). In the objective function J of (1.5) we choose $M_k = \alpha I$, $\alpha \in \mathbb{R}^+$, for all k .

In the sequel, we derive the dynamic programming approach for the optimal control problem in (1.5). We start by introducing the value function (or Lyapunov function) $V : \mathbb{R} \times X \rightarrow \mathbb{R}$

$$V(k, \xi) := \min\{J_k(u, v) \mid (u, v) \in Z_k(\xi) \times X^{N-k}\},$$

where

$$J_k(\varepsilon, v) := \frac{1}{2} \left[\langle F_N u_N - y_N, L_N(F_N u_N - y_N) \rangle + \sum_{j=k}^{N-1} \langle F_j u_j - y_j, L_j(F_j u_j - y_j) \rangle + \alpha \langle v_j, v_j \rangle \right] \quad (2.14)$$

and $Z_k(\xi) := \{u \in X^{N-k+1} \mid u_k = \xi, u_{j+1} = u_j + v_j, j = k, \dots, N-1\}$. [Compare with the definition in (2.3)]. The Bellman principle for this discrete problem reads

$$V(k, \xi) = \min_{v \in X} \left\{ V(k+1, \xi + v) + \frac{1}{2} \langle F_k \xi - y_k, L_k(F_k \xi - y_k) \rangle + \frac{\alpha}{2} \langle v, v \rangle \right\}. \quad (2.15)$$

The optimality equation (2.15) is the discrete counterpart of the HJ equation (2.4). Now we make the ansatz for the value function: $V(k, \xi) = \frac{1}{2} \langle \xi, Q_k \xi \rangle + \langle b_k, \xi \rangle + g_k$. Notice that the value function satisfies the boundary condition: $V(N, \xi) = \frac{1}{2} \langle F_N \xi - y_N, L_N(F_N \xi - y_N) \rangle$. Therefore,

$$Q_N = F_N^* L_N F_N \quad b_N = -F_N^* L_N y_N. \quad (2.16)$$

As in the continuous case, the optimality equation has to be solved backwards in time ($k = N-1, \dots, 0$) recursively. A straightforward calculation shows that the minimizer of (2.15) is given by $\bar{v} = -(Q_{k+1} + \alpha I)^{-1} (Q_{k+1} \xi + b_{k+1})$. Substituting in (2.15), we obtain a recursive formula to compute Q_k , b_k , and u_k :

$$Q_{k-1} = \alpha(Q_k + \alpha I)^{-1} Q_k + F_{k-1}^* L_{k-1} F_{k-1} \quad k = N \dots 2 \quad (2.17)$$

$$b_{k-1} = \alpha(Q_k + \alpha I)^{-1} b_k - F_{k-1}^* L_{k-1} y_{k-1} \quad k = N \dots 2 \quad (2.18)$$

$$u_{k+1} = (Q_{k+1} + \alpha I)^{-1} (\alpha u_k - b_{k+1}) \quad k = 0 \dots N-1 \quad (2.19)$$

Together with the end conditions (2.16) and an arbitrary initial condition u_0 , these recursions can be solved backwards for Q_k , b_k and forwards for u_k .

In the sequel, we verify that the iteration in (2.17), (2.18), (2.19) is well defined.

Lemma 2.3. *The recursion (2.17) with the condition (2.16), defines a sequence of self-adjoint positive semi-definite operators Q_k . In particular, $(Q_k + \alpha I)^{-1}$ exists and is bounded for all k . Moreover,*

$$\|Q_k\| \leq \alpha + \max_k \|F_k\|^2.$$

Proof. Because a sum of two bounded self-adjoint operators is symmetric, it follows by induction that Q_k are self-adjoint for all k . Denote by $\sigma(Q_k)$ its spectrum, we can prove by induction that

$$\sigma(Q_k) \subset [0, \alpha + \max_k \|F_k\|^2].$$

Indeed, if Q_{k+1} has this property, then $(Q_{k+1} + \alpha I)^{-1}$ exists, and

$$B_{k+1} := (\alpha(Q_{k+1} + \alpha I)^{-1} Q_{k+1})$$

is positive semidefinite and bounded by $\|B_{k+1}\| \leq \alpha$. Hence, by the minimax characterization of the spectrum we obtain

$$\begin{aligned} \sigma(Q_k) &\geq \inf_{\|x\| \leq 1} (x, Q_k x) \geq \inf_{\|x\| \leq 1} (x, B_k x) + \inf_{\|x\| \leq 1} (x, F_k^* F_k x) \geq 0 \\ \sigma(Q_k) &\leq \sup_{\|x\| \leq 1} (x, Q_k x) \leq \sup_{\|x\| \leq 1} (x, B_k x) + \sup_{\|x\| \leq 1} (x, F_k^* F_k x) \leq \alpha + \|F_k\|^2, \end{aligned}$$

concluding the proof. □

3. REGULARIZATION PROPERTIES

Before we examine the regularization properties of the methods derived in Section 2, let us state a result about existence and uniqueness of the Riccati equations (2.12).

Theorem 3.1. *If $F, L, M \in C([0, T], \mathcal{L}(X, Y))$, then the Riccati equation (2.12) has a unique symmetric positive semidefinite solution in $C^1([0, T], \mathcal{L}(X))$.*

Proof. In [2], the uniqueness and positivity of a weak solution to (2.12) in the form

$$\tilde{Q}(t) = \int_t^T \tilde{Q}(s)^* M^{-1}(s) \tilde{Q}(s) - F(s)^* L(s) F(s) ds, \quad (3.1)$$

is proved. If F, L, M are continuous then, by Lebesgues theorem, \tilde{Q} is continuously differentiable, and hence a strong solution. The symmetry of \tilde{Q} follows from the uniqueness, because \tilde{Q}^* satisfies the same equation as \tilde{Q} . Existence of a solution to (2.12), (2.13) is standard, as these are linear equations (cf. [23]). \square

Remark 3.2. It is well-known in control theory that the existence of a solution to (2.12) can be constructed from the functional

$$V(t, \xi) := \min_{\substack{u(t)=\xi \\ u \in H^1([t, T], X)}} \frac{1}{2} \int_t^T \langle F(s)u(s) - y(s), L(s)[F(s)u(s) - y(s)] \\ + \langle u'(s), M(s)u'(s) \rangle ds. \tag{3.2}$$

This functional is quadratic in u and, from the Tikhonov regularization theory (see, e.g., [9]), it admits a unique solution u , and is quadratic in ξ . Furthermore, the leading quadratic part $(\xi, Q(t)\xi)$ is a solution to the Riccati equation.

Next we consider regularization properties of the method derived in Subsection 2.2. The following lemma shows that the solution u of (2.13) satisfies the necessary optimality condition for the functional

$$J(u) = \frac{1}{2} \int_0^T \langle F(s)u(s) - y(s), L(s)[F(s)u(s) - y(s)] \rangle + \langle u'(s), M(s)u'(s) \rangle ds \tag{3.3}$$

(notice that this is the cost functional $J(u, v)$ in (1.4) with $v = u'$).

Lemma 3.3. *Let $Q(t), b(t), u(t)$ be defined by (2.12), (2.13), together with the boundary conditions $Q(T) = 0, b(T) = 0$, and $u(0) = u_0$. Then, $u(t)$ solves*

$$F^*(t)L(t)F(t)u(t) - M(t)u(t)'' = F^*(t)L(t)y(t), \tag{3.4}$$

together with the boundary conditions $u(0) = u_0, u'(T) = 0$.

Proof. Equation (3.4) follows from Equations (2.12), (2.13) by differentiation:

$$-M(t)u''(t) = \frac{d}{dt}(Qu(t) + b(t)) = Q(t)'u(t) + b'(t) + Q(t)u'(t) \\ = -F(t)^*L(t)F(t)u(t) + F(t)^*L(t)y(t)$$

The boundary condition $u(0) = u_0$ holds by definition and the identity $u'(T) = 0$ follows from (2.13) and the boundary conditions for Q and b . \square

Because the cost functional in (3.3) is quadratic, the necessary first order conditions are also sufficient. Thus, the solution $u(t)$ of (3.4) is actually a minimizer of this functional. Including the boundary conditions we obtain the following corollary:

Corollary 3.4. *The solution $u(t)$ of (3.4) is a minimizer of the Tikhonov functional in (3.3) over the linear manifold*

$$\mathcal{H} := \{u \in H^1([0, T], X) \mid u(0) = u_0\}.$$

In particular, this means that the above procedure is a regularization method for the inverse problem (1.1). Bellow we summarize a stability and convergence result. The proof uses classical techniques from the analysis of Tikhonov type regularization methods (cf. [9], [10]) and thus is omitted.

Theorem 3.5. *Let $M(t) \equiv \alpha I$, $\alpha > 0$, $L(t) > 0$, $t \in [0, T]$ and J_α be the corresponding Tikhonov functional given by (3.3).*

Stability: Let the data $y(t)$ be noise free and denote by $u_\alpha(t)$ the minimizer of J_α . Then, for every sequence $\{\alpha_k\}_{k \in \mathbb{N}}$ converging to zero, there exists a subsequence $\{\alpha_{k_j}\}_{j \in \mathbb{N}}$, such that $\{u_{\alpha_{k_j}}\}_{j \in \mathbb{N}}$ is strongly convergent. Moreover, the limit is a minimal norm solution.

Convergence: Let $\|y^\delta(t) - y(t)\| \leq \delta$. If $\alpha = \alpha(\delta)$ satisfies

$$\lim_{\delta \rightarrow 0} \alpha(\delta) = 0 \quad \text{and} \quad \lim_{\delta \rightarrow 0} \delta^2 / \alpha(\delta) = 0.$$

Then, for a sequence $\{\delta_k\}_{k \in \mathbb{N}}$ converging to zero, there exists a sequence $\{\alpha_k := \alpha(\delta_k)\}_{k \in \mathbb{N}}$ such that u_{α_k} converges to a minimal norm solution.

A result similar to the one stated in Corollary 3.4 holds for the discrete case:

Lemma 3.6. *Let Q_k, b_k, u_k be defined by (2.17), (2.18), and (2.19), together with the boundary conditions (2.16). Then u_k satisfies*

$$F_k^* L_k F_k u_k - \alpha(u_{k-1} - 2u_k + u_{k+1}) = F_k^* L_k y_k, \quad k = 1 \dots n \quad (3.5)$$

together with the boundary condition $u(0) = u_0, u_{n+1} = u_n$.

Equation (3.5) is the necessary (and by convexity also sufficient) condition for a minimizer of J_0 in (2.14). This proves the following corollary:

Corollary 3.7. *The sequence u_k is a minimizer of the Tikhonov functional (2.14) over all (w_k) with $w_0 = u_0$.*

4. APPLICATION TO DYNAMIC EIT PROBLEM

After a spacial discretization of the operator equation (1.1), the differential equations (2.12), (2.13) can be solved by standard methods for ordinary differential equations, such as the Euler-Method or Runge-Kutta-Methods. Choosing $M(t) \equiv Id : X \rightarrow X$ and $L(t) \equiv \alpha^{-1}Id : Y \rightarrow Y$ in (2.12), (2.13) we obtain

$$\begin{aligned} Q'(t) &= -\alpha^{-1}F(t)^*F(t) + Q(t)^*Q(t) \\ b'(t) &= Q(t)^*b(t) + \alpha^{-1}F(t)^*y(t) \\ u'(t) &= -Qu(t) - b(t) \end{aligned}$$

From a computational point of view, the first of these is the most expensive one, as it is nonlinear and involves matrix products. Once $Q(t)$ is known, the equations for b, u are linear and only involve matrix-vector multiplications.

The simplest approach is to use an explicit Euler method for solving the equation for Q backwards in time ($t_k = \frac{k}{n_T}T, \Delta t = \frac{1}{n_T}T$).

$$Q_{k-1} = Q_k - \Delta t(-\alpha^{-1}F(t_k)^*F(t_k) + Q_k(t)^*Q_k(t)) \quad k = n - 1, \dots, 0 \tag{4.1}$$

$$b_{k-1} = b_k - \Delta t(Q_k b_k + \alpha^{-1}F(t_k)^*y(t_k)) \tag{4.2}$$

$$u_{k+1} = u_k + \Delta t(-Q_k u_k - b_k), \tag{4.3}$$

with $Q_n = 0, b_n = 0$. It is well-known that an explicit method is conditionally stable. The iteration matrix for (4.1) is $(I - \Delta t Q_k)$. An analogy to Landweber iteration [9] a stability criterion is that

$$\Delta t \leq \|Q_k\|^{-1}. \tag{4.4}$$

This condition is satisfied if Δt is small enough, as the following theorem states:

Theorem 4.1. *Let the following CFL-condition be satisfied*

$$\alpha^{-1}(\Delta t)^2 \max_{t \in [0, T]} \|F(t)\|^2 \leq \frac{1}{2}. \tag{4.5}$$

Then (4.1) defines a sequence of positive definite selfadjoint operators Q_k such that (4.4) hold.

Proof. It is trivial that Q_{k-1} is self-adjoint if Q_k is. The iteration can be written as

$$\Delta t Q_{k-1} = (I - \Delta t Q_k) \Delta t Q_k + \alpha^{-1}(\Delta t)^2 F_k^* F_k.$$

If the spectrum σ of Q_k satisfies $\sigma(\Delta t Q_k) \subset [0, 1]$, then the right-hand side of the iteration is a sum of two positive definite operators and hence the left-hand side is also positive definite. Moreover,

$$\|\Delta t Q_{k-1}\| \leq \frac{1}{2} + \alpha^{-1}(\Delta t)^2 \|F_k\|^2.$$

If $\alpha^{-1}(\Delta t)^2 \|F_k\|^2 \leq \frac{1}{2}$ holds, then we obtain by induction that $\sigma(\Delta t Q_{k-1}) \subset [0, 1]$ for all k , which implies (4.4). \square

It follows from the last theorem that Δt has to be chosen proportional to $\sqrt{\alpha}$. If the regularization parameter is small, this requires very small time-steps. In this case, an alternative is to use the discrete versions (2.17), (2.18), (2.19), which are quite similar to an implicit Euler schema. Contrary to the explicit Euler steps, it does not require any restriction on Δt .

In this section, we apply our regularization method to a dynamic inverse problem, namely the linearized impedance tomography problem, that is, one is faced with the problem of determining a time-dependent diffusion coefficient $\tilde{\gamma}(x, t)$ in the equation

$$\nabla \cdot (\tilde{\gamma}(\cdot, t) \nabla u) = 0 \quad \text{in } \Omega \tag{4.6}$$

from the Neumann-to-Dirichlet operator:

$$\Lambda_{\tilde{\gamma}} : \frac{\partial}{\partial n} u|_{\partial\Omega} \rightarrow u|_{\partial\Omega} \quad u \text{ solution to the Neumann problem (4.6).}$$

We consider $\Lambda_{\tilde{\gamma}}$ an operator mapping a subspace $L^2(\partial\Omega)$ into itself. Because the Neumann data have to satisfy the compatibility condition $\int_{\partial\Omega} \frac{\partial}{\partial n} u = 0$, the domain of definition of $\Lambda_{\tilde{\gamma}}$ has to incorporate this condition. It is well-known (see, e.g., [13]) that $\Lambda_{\tilde{\gamma}}$ is a compact operator between Hilbert-spaces, hence we can consider it an element of the space of Hilbert-Schmidt operators H and use the Hilbert-Schmidt norm on this space. The parameter-to-data operator can be written as $F : X \subset L^2([0, T] \times \Omega) \rightarrow H$, $F(\tilde{\gamma}) := \Lambda_{\tilde{\gamma}}$.

The subset X is the set of $\tilde{\gamma}$ such that $\tilde{\gamma}$ is bounded from below and above by positive constants, which is necessary to ensure ellipticity of (4.6). Because the operator F is nonlinear, for a successful application of the dynamic algorithm we will consider a linearization around 1, using $F(\tilde{\gamma}) - F(1) \sim F'(1)(\tilde{\gamma} - 1)$. Notice that $F(1)$ can be computed *a priori*, therefore we consider the data to our problem to be $y = F(\tilde{\gamma}) - F(1)$ and the corresponding unknown $\gamma(x, t) = \tilde{\gamma}(x, t) - 1$. This gives the linearized problem

$$F'(1)\gamma = y,$$

where γ, y both depend on time. Hence, we can solve this problem within the framework developed in Subsection 2.1.

4.1. Discretization

We briefly comment about the discretization of the Neumann-to-Dirichlet operator. We use piecewise linear finite element functions on the boundary: $X_b := \{\sum_i g_i \tilde{\phi}_i(x) \mid x \in \partial\Omega\}$. The functions ϕ_i are the boundary-trace of the well-known Courant-element functions. Equation (4.6) is also solved by finite elements. Let ϕ_i be the piecewise linear and continuous ansatz functions on a triangular mesh. These ansatz functions form the basis for the finite-element space to solve (4.6) and also for the discretization of the space X , that is, γ is represented in the discrete setting by a sum of ϕ_i . If the Neumann data are in X_b , that is, $\frac{\partial}{\partial n} u = \sum_i g_i \tilde{\phi}_i$, then equation (4.6) corresponds to a discrete linear equation of the form

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} u_i \\ u_b \end{pmatrix} = \begin{pmatrix} 0 \\ Mg \end{pmatrix},$$

where the matrices $A_{11}, A_{12}, A_{21}, A_{22}$ are submatrices of the stiffness matrix $A_{i,j} = \int_{\Omega} \gamma \nabla \phi_i \nabla \phi_j$ with respect of a splitting of the indices into the interior and boundary components. The matrix M is coming from the contribution of the Neumann-data in the discretized equations:

$$M_{i,l} = \int_{\partial\Omega} \tilde{\phi}_l \tilde{\phi}_i \, d\sigma \tag{4.7}$$

In order to deal with the compatibility condition, we specify a reference boundary index i_* and set $g_{i_*} = 0$. The corresponding rows and columns in the matrices are canceled out. The variables connected with interior points can be eliminated from the discrete equation by taking the Schur-Complement; this gives the matrix

$$G := (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}M. \tag{4.8}$$

This matrix corresponds to a mapping $\tilde{\Lambda} : X_b^* \rightarrow X_b^*$, with

$$X_b^* := \left\{ \sum_i g_i \tilde{\phi}_i(x) \mid g_{i_*} = 0, x \in \partial\Omega \right\}.$$

Identifying the space X_b with the \mathbb{R}^n via $\sum_i g_i \tilde{\phi}_i(x) \Leftrightarrow (g_i)$, the discrete Neumann-to-Dirichlet operator is represented on \mathbb{R}^n by multiplication of the matrix G .

We calculate the Hilbert-Schmidt inner product for discrete Neumann-to-Dirichlet operators Λ_1, Λ_2 coming from the above discretizations.

These operators have the form $\Lambda_k \tilde{\phi}_i \rightarrow \sum_l (G_k)_{l,i} \tilde{\phi}_l$, $k \in \{1, 2\}$, where G_k is as in (4.8), corresponding to different coefficients γ . Note that G_k can be written as $G_k = S_k M$, where S_k is a symmetric matrix and M the boundary mass matrix (4.7).

The Hilbert-Schmidt inner product is defined as $(\Lambda_1, \Lambda_2) = \sum_i (\Lambda_1 e_i, \Lambda_2 e_i)$, where e_i is a orthonormal basis and (\cdot, \cdot) is the usual L^2 inner product. In our case we chose (e_i) orthonormal such that $\text{span}(e_i) = \text{span}(\tilde{\phi}_i)$. Each basis can be transformed into each other: $\tilde{\phi}_i = \sum_k \beta_{i,k} e_k$, $e_k = \sum_l \gamma_{k,l} \tilde{\phi}_l$.

Denote by B, Γ the matrices: $B = (\beta_{i,k}), \Gamma := (\gamma_{k,l})$. From the orthogonality of (e_i) , the following identities can be derived: $M = BB^T$, $\Gamma = B^{-1}$. Now Λe_k is given by $\Lambda e_k = \sum_l A_{k,l} \tilde{\phi}_l$ and further $A = \Gamma G^T$.

Finally, the Hilbert-Schmidt inner product can be calculated to (tr denotes the trace of a matrix):

$$\begin{aligned} (\Lambda_1, \Lambda_2) &= \text{tr}(\Gamma G_1^T M (\Gamma G_2^T)^T) = \text{tr}(G_1^T M G_2 \Gamma^T \Gamma) = \text{tr}(M S_1^T M S_2 M M^{-1}) \\ &= \text{tr}(M S_1^T M S_2) = \text{tr}(S_1^T M S_2 M) = \text{tr}(G_1 G_2), \end{aligned}$$

where we used $\Gamma^T \Gamma = M^{-1}$, and the symmetry of S, M , and the identity $\text{tr}(AB) = \text{tr}(BA)$.

4.2. Numerical Results

As test example for the linearized impedance tomography problem, we considered equation (4.6) on a unit square: $\Omega = [0, 1]^2$. As conductivity $\gamma(x, t)$ we used a piecewise constant function, with support on a moving circle:

$$\gamma(x, t) := 1 + 2\chi_{B_{x_t, 0.08}},$$

here χ denotes the characteristic function, $B_{x,r}$ denotes a circle with center at x and radius r . The time-varying center is chosen as

$$x_t := \begin{pmatrix} 0.4 - 0.2 \cos(2\pi t) \\ 0.5 - 0.2 \sin(2\pi t) \end{pmatrix}. \quad t \in [0, 1]$$

and is shown in Figure 1.

For the computation we used a uniform discretization, with 25 subdivisions of the interval $[0, 1]$ in each coordinate direction. The data are sampled at $t_i = \frac{i}{50}$ using 51 uniform distributed sample points of the interval $[0, 1]$.

We experimented both with the explicit Euler algorithm and the discrete version. However, the first one has the drawback of needing a

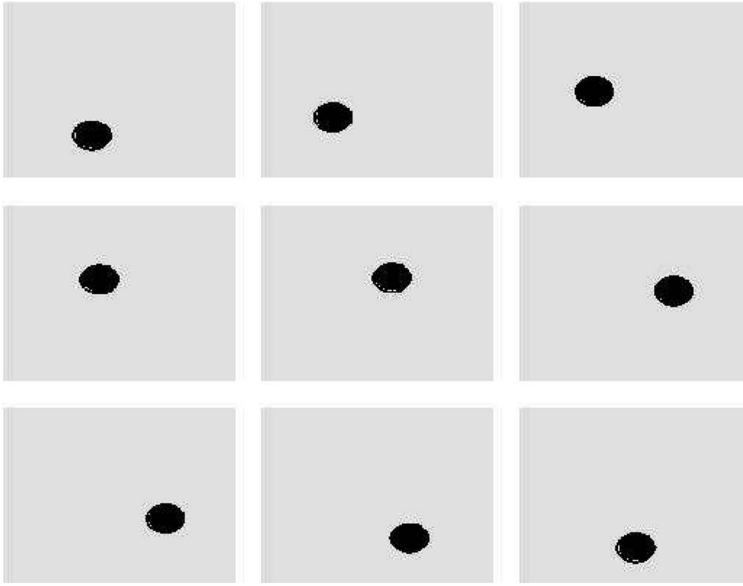


FIGURE 1 Exact solution of the dynamic EIT problem.

CFL condition (4.5). For small α this requires a very fine discretization of the time-interval, which makes the method not very feasible. Hence for the numerical results, we used the discrete version, which is free of a CFL condition.

For the first example, we simulated data for the linearized problem, that is,

$$y = F'(1)(\gamma - 1).$$

The data were computed on a finer unstructured grid, in order to avoid inverse crimes. In Figure 2, we show a density plot of the results for different time-points.

For the second example, we used nonlinear data

$$y = F(\gamma) - F(1).$$

Again we computed this on a finer grid. Additionally, we added 5% random noise. Thus, we have in this case both an error due to noise and a systematic error coming from the fact that we used a linearized model for data corresponding to a nonlinear problem. Figure 3 shows the result for this case.

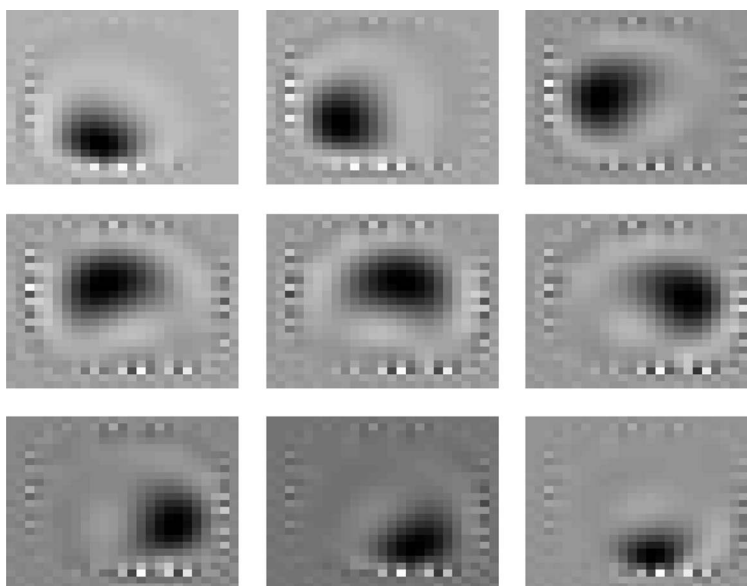


FIGURE 2 Reconstruction result for linearized data without noise.

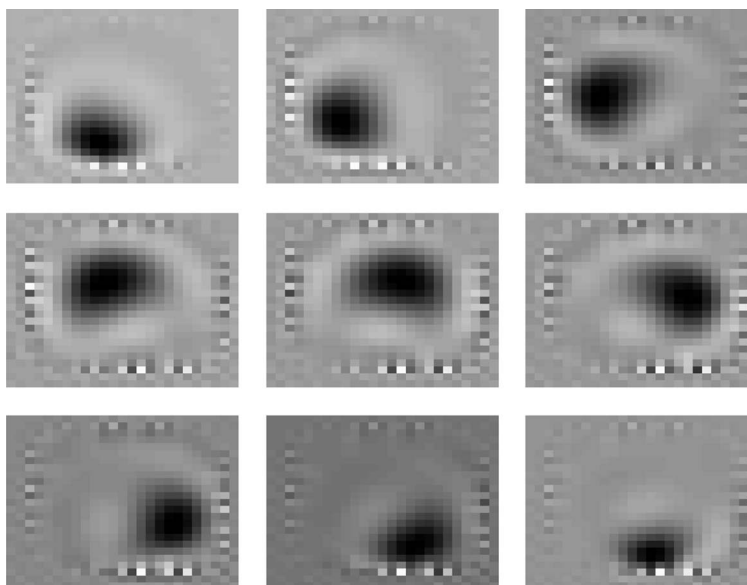


FIGURE 3 Reconstruction result for nonlinear data with 5% random noise.

5. CONCLUSIONS

Each method derived in this paper requires, in a first step, the solution of an evolutionary equation (of Hamilton-Jacobi type). In a second step, the components of the solution vector $\{u_k\}$ are computed one at a time. This strategy reduces significantly both the size of the systems involved in the solution method, as well as storage requirements needed for the numerical implementation. These points turn out to become critical for long time measurement processes.

Some detailed considerations about complexity: Assume that all $F(t_k)$ are discretized as $(n \times m)$ matrices. The main effort is the matrix multiplication for the update step for Q_k : In each step this requires $\mathcal{O}(n^3 + n^2m)$ calculations. Hence the overall complexity is of the order $\mathcal{O}(n_T(n^3 + n^2m))$ operations. If the discrete version is used, then in each step a matrix-inversion has to be performed, which is also of the order $\mathcal{O}(n^3)$, which leads to the same complexity as above. In contrast, the method in [21] requires $\mathcal{O}((n + n_T)^3 + (n_T + m)nn_T)$. Although this is only of cubic order in comparison with a quartic order complexity for the dynamic programming approach, it is cubic in n_T . Hence if n_T is large, the method proposed in this paper (which is linear in n_T) will be more effective than the method in [21].

The numerical results show the feasibility and the stability of our method. Note that the results are more smeared out at the center of the square, which is clear because the identification problem is less stable if the boundary is further away.

ACKNOWLEDGMENTS

The work of S.K. is supported by Austrian Science Foundation under grant SFB F013/F1317 and by NSF grant Nr. DMI-0327077. S.K. is on leave from the Industrial Mathematics Institute, Johannes Kepler University Linz, Austria.

Part of this paper was written during a sabbatical stay of A.L. at RICAM Institute (Linz). A.L. acknowledges support of the Austrian Academy of Sciences and of CNPq under grants 305823/2003-5 and 478099/2004-5.

The authors would like to thank Prof K. Kunisch (Graz) for the fruitful discussions about optimal control and optimization theory.

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