

On the value function for nonautonomous optimal control problems with infinite horizon

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Abstract

In this paper we consider nonautonomous optimal control problems of infinite horizon type, whose control actions are given by L^1 -functions. We verify that the value function is locally Lipschitz. The equivalence between dynamic programming inequalities and Hamilton–Jacobi–Bellman (HJB) inequalities for proximal sub (super) gradients is proven. Using this result we show that the value function is a Dini solution of the HJB equation. We obtain a verification result for the class of Dini sub-solutions of the HJB equation and also prove a minimax property of the value function with respect to the sets of Dini semi-solutions of the HJB equation. We introduce the concept of viscosity solutions of the HJB equation in infinite horizon and prove the equivalence between this and the concept of Dini solutions. In the Appendix we provide an existence theorem. © 2006 Elsevier B.V. All rights reserved.

Keywords: Dynamic programming; Infinite horizon; Viscosity solutions; Dini solutions; Existence

1. Introduction

1.1. Formulation of the optimal control problem

Given the initial values $(\tau, z) \in [0, \infty) \times \mathbb{R}^n$, let us consider the following optimal control problem:

$$\left\{ \begin{array}{l} \text{Minimize} \quad J(u; \tau, z) := \int_{\tau}^{\infty} e^{-\delta t} l(t, x(t), u(t)) dt, \\ \text{subject to} \\ \quad u \in U_{\text{ad}}[\tau, \infty) := \{w \in L^1_{\text{loc}}([\tau, \infty); \mathbb{R}^m) \mid w(t) \\ \quad \in \Omega \text{ a.e. in } [\tau, \infty)\}, \\ \quad x(t) = z + \int_{\tau}^t f(s, x(s), u(s)) ds, \quad t \in [\tau, \infty). \end{array} \right.$$

The problem above is called $P_{\infty}(\tau, z)$. Here $l : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ are given functions; $\Omega \subset \mathbb{R}^m$ is the set of admissible control actions; $\delta > 0$

denotes the discount rate. An admissible control strategy u is a L^1_{loc} -function satisfying the control constraint $u(t) \in \Omega$ a.e. in $[\tau, \infty)$. By L^1_{loc} we mean the space of locally integrable functions $w : [0, \infty) \rightarrow \mathbb{R}^m$. We call state trajectory an absolutely continuous function $x : [0, \infty) \rightarrow \mathbb{R}^n$ that is a solution of the integral equation in $P_{\infty}(\tau, z)$.

A pair of functions $(x(\cdot), u(\cdot))$, consisting of an admissible control u and the corresponding state trajectory x , is called an admissible process. We are specially interested in the set of admissible processes with finite costs:

$$\mathcal{D}_{\tau, z; \infty} := \left\{ (x, u) \mid u \in U_{\text{ad}}[\tau, \infty); x(t) = z + \int_{\tau}^t f(r, x(r), u(r)) dr; J(u; \tau, z) < \infty \right\}.$$

We stress the fact that the cost functional in problem $P_{\infty}(\tau, z)$ is of infinite horizon type and that both the cost functional and the dynamic are allowed to be time dependent. This sort of control problem is particularly interesting for applications related to biological and economic sciences, since no reasonable bound can be placed on the time horizon. Several applications

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of this nature can be found in Carlson and Haurie [4], Clark [5], Seierstad and Sydsaetter [15], Sethi and Thompson [16], Flatten [11] and Baumeister and Leitão [3], among others.

1.2. Discussion of the main contributions

We are mainly concerned with the investigation of the value function for the family of control problems $P_\infty(\tau, z)$.

We managed to extend to nonautonomous optimal control problems of infinite horizon type some classical results of the dynamic programming theory, like dynamic programming inequalities, verification results, characterization of the value function as both a Dini and a viscosity solution of the HJB equation.³

The difficulty in the formulation and analysis of the HJB equation for problem $P_\infty(\tau, z)$ arises from the choice of the boundary condition at infinity for the partial differential equation. It turns out that this boundary condition can be formulated as uniform convergence to zero, as $t \rightarrow \infty$, in each compact set of \mathcal{R}^n . Under the assumption of existence of invariant sets for the trajectories, this decay condition is exactly what one needs to prove the verification result and the minimax property of the value function among Dini semi-solutions of the HJB equation.

Another feature of the present approach is the choice of controllers as L^1 -functions rather than L^∞ -functions, which are more common in the existing literature (see, e.g., [1,9]).

The main tool used in this text to analyze the HJB equation in infinite horizon is the concept of Dini solutions. The analysis of Dini solutions of the HJB equation for finite horizon is considered in Clarke et al. [6], Rapaport and Vinter [14] and Vinter and Wolenski [17], among others. In finite horizon, the analysis of the HJB equation for nonautonomous control problems is developed in Rapaport and Vinter [14] and Frankowska et al. [12].

Dynamic programming for autonomous infinite horizon control problems is considered by many authors [13,1,9]. Halkin [13] considers some economic applications of infinite horizon which are modeled by control problems with continuous controllers. Bardi and Capuzzo-Dolcetta [1] characterizes the value function as a viscosity solution of the HJB equation. They also provide verification results and minimax properties. Da Lio [9] obtains some regularity results for the value function and also characterizes it as a stable viscosity solution of the HJB equation.

However, dynamic programming for nonautonomous control problems of infinite horizon has not been exploited in the literature. This paper makes a first effort trying to close up this gap by setting up a framework in which the dynamic programming approach can still be carried out.

As part of our developments we address the question of existence of optimal processes for $P_\infty(\tau, z)$. An existence theorem for problems of infinite horizon type is given by Baum [2]. However, this result is not applicable in our situation.

1.3. Outline of the paper

The paper is organized as follows. In Section 2, we verify some properties of the value function, including (local) Lipschitz continuity. In Section 3, we introduce several concepts of generalized derivatives and gradients. The main purpose of this section is the definition of the Dini solutions of the HJB equation. In Section 4, we analyze some monotonicity properties of functions related to the HJB equation. In Section 5, we prove the equivalence between certain HJB inequalities for Dini-gradients and proximal-gradients. In the sequel we use this result to prove that the value function is a Dini solution of the HJB equation. In Section 6, we derive a verification result for the Dini sub-solutions of the HJB equation and prove a minimax property for the value function (the uniqueness of Dini solutions follows from this property). In Section 7, we verify the equivalence between the concepts of Dini and viscosity solutions of the HJB equation. This characterizes the value function as a viscosity solution of the HJB equation. The Appendix is devoted to discussing the issue of existence of optimal processes for $P_\infty(\tau, z)$.

2. Lipschitz continuity of the value function

Let us consider again the family of optimal control problems $P_\infty(\tau, z)$, for initial conditions $(\tau, z) \in \mathcal{R} \times \mathcal{R}^n$. In the sequel we shall consider problem $P_\infty(\tau, z)$ under the following assumptions:

- (A1) There exists $K_1 > 0$, such that for every fixed $t \in [0, \infty)$, $u \in \Omega$

$$|l(t, x, u) - l(t, z, u)| + |f(t, x, u) - f(t, z, u)| \leq K_1|x - z|,$$

holds for all $x, z \in \mathcal{R}^n$.

- (A2) There exists a positive scalar K_2 such that f and l satisfy the linear growth condition

$$|l(t, x, u)| + |f(t, x, u)| \leq K_2(1 + |x|),$$

for all $t \in [0, \infty)$, $x \in \mathcal{R}^n$, $u \in \Omega$.

- (A3) The functions $l(\cdot, \cdot, \cdot)$, $f(\cdot, \cdot, \cdot)$ are continuous.
 (A4) The set of points $\bar{f}(t, x, \Omega) := \{(e^{-\delta t} l(t, x, v), f(t, x, v))^T \mid v \in \Omega\}$ is a convex set in \mathcal{R}^{n+1} for all t, x (here T means transposition).
 (A5) Ω is compact.
 (A6) Given $(\tau, z) \in [0, \infty) \times \mathcal{R}^n$, there exists a bounded (invariant) set $\mathcal{S}_{\tau, z} \subset \mathcal{R}^n$ such that every admissible process $(x(\cdot), u(\cdot)) \in \mathcal{D}_{\tau, z; \infty}$ satisfies $x(t) \in \mathcal{S}_{\tau, z}$, for $t \geq \tau$.

The function \bar{f} in (A4) is called the extended velocity vector related to the control problem $P_\infty(\tau, z)$.

We start by defining a key function in the theory of dynamic programming, namely, the value function.

³ For an account of different types of solutions of the HJB equation, see Bardi and Capuzzo-Dolcetta [1], Clarke et al. [6] and Crandall and Lions [8].

Definition 1. The mapping

$$V : [0, \infty) \times \mathcal{R}^n \longrightarrow \mathcal{R},$$

$$(\tau, z) \longmapsto \inf \{J(u; \tau, z) \mid (x, u) \in \mathcal{D}_{\tau, z; \infty}\} \quad (2.1)$$

is called the *value function* associated with the family of problems $P_\infty(\tau, z)$.

Next, we define the Hamilton-function as the application

$$H : [0, \infty) \times \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^m \longrightarrow \mathcal{R},$$

$$(t, x, \lambda, u) \longmapsto \langle \lambda, f(t, x, u) \rangle + e^{-\delta t} l(t, x, u). \quad (2.2)$$

We also define the (control independent) auxiliary function

$$\mathcal{H} : [0, \infty) \times \mathcal{R}^n \times \mathcal{R}^n \longrightarrow \mathcal{R},$$

$$(t, x, \lambda) \longmapsto \inf_{u \in \Omega} \{H(t, x, \lambda, u)\}. \quad (2.3)$$

Instead of (A6), we shall consider in this section the assumption:

($\tilde{A}6$) The Lipschitz constant K_1 in (A1) and the discount rate δ satisfy $K_1 - \delta < 0$. Furthermore, given $(\tau, z) \in [0, \infty) \times \mathcal{R}^n$, there exists a neighborhood $\mathcal{V}_{\tau, z}$ of (τ, z) and a bounded (invariant) set $\mathcal{S}_{\tau, z} \subset \mathcal{R}^n$ such that for every initial condition $(\tilde{\tau}, \tilde{z}) \in \mathcal{V}_{\tau, z}$, the admissible processes $(x(\cdot), u(\cdot)) \in \mathcal{D}_{\tilde{\tau}, \tilde{z}; \infty}$ satisfy $x(\cdot) \in \mathcal{S}_{\tau, z}$.

Remark 2. The assumption concerning the relationship between the Lipschitz constant K_1 and the discount rate δ looks somehow artificial. Nevertheless, it has been used by several authors, see, e.g., Bardi and Capuzzo-Dolcetta [1], Wirth [18] and Colonius [7].

Under the extra assumption above, it is possible to verify some important properties of the value function V , which we summarize in Lemma 4. First, we state an auxiliary lemma. The proof of this result does not contain new ideas and will be omitted.

Lemma 3. *Suppose the assumptions (A1), (A2) are satisfied. Let $(\tau, z), (\tau, z_1) \in [0, \infty) \times B_R(0)$ and $u \in U_{\text{ad}}[\tau, \infty)$. Let x, x_1 be the associated states. Then*

$$|x(t) - x_1(t)| \leq e^{K_1(t-\tau)} |z - z_1|, \quad t \geq \tau. \quad (2.4)$$

Furthermore, we have

$$|x(t) - z| \leq K_2(1 + R)e^{K_2(t-\tau)} |t - \tau|, \quad t \geq \tau. \quad (2.5)$$

In the next lemma we need to combine (admissible) controls defined in different time intervals. Let $u_1 \in U_{\text{ad}}[s_1, \infty)$, $u_2 \in U_{\text{ad}}[s_2, \infty)$, where $0 \leq s_1 \leq s_2 < \infty$. We define the *concatenation*

$$u_1 \sqcap^{s_2} u_2 := \begin{cases} u_1(t), & t \in [s_1, s_2), \\ u_2(t), & t \in [s_2, \infty). \end{cases}$$

Lemma 4. *Let assumptions (A1)–(A5) and ($\tilde{A}6$) hold. Then we have:*

- (a) *The value function is well defined on $[0, \infty) \times \mathcal{R}^n$.*
- (b) *The value function is locally Lipschitz continuous.*

Proof. Notice that the existence of an optimal process is guaranteed by Theorem 19. Then, (a) follows. The assertion (b) is the most interesting one. To prove it, it is enough to verify that for every $T > 0$, $R > 0$, there exists a constant C such that for all $(s_1, z_1), (s_2, z_2) \in [0, T] \times \bar{B}_R(0)$ the following inequality holds:

$$|V(s_1, z_1) - V(s_2, z_2)| \leq C(|s_1 - s_2| + |z_1 - z_2|). \quad (2.6)$$

Since $[0, T] \times \bar{B}_R(0)$ is compact, it follows from (A6) that there exists a bounded invariant set $\mathcal{S}_{T, R} \subset \mathcal{R}^n$ such that for every initial condition $(\tau, z) \in [0, T] \times \bar{B}_R(0)$, the admissible processes $(x(\cdot), u(\cdot)) \in \mathcal{D}_{\tau, z; \infty}$ satisfy $x(\cdot) \in \mathcal{S}_{T, R}$, i.e.,

$$|x(t)| \leq M_{T, R}, \quad t \in [\tau, \infty). \quad (2.7)$$

Let $(s_1, z_1), (s_2, z_2) \in [0, \infty) \times B_R(0)$ be given, with $s_1 \leq s_2$. Given $\varepsilon > 0$, choose $u_1 \in U_{\text{ad}}[s_1, \infty)$, $u_2 \in U_{\text{ad}}[s_2, \infty)$ such that

$$V(s_i, z_i) \geq J(u_i; s_i, z_i) - \varepsilon, \quad i = 1, 2.$$

Define $\tilde{u}_1 := u_1 \sqcap^{s_2} u_2$, $\tilde{u}_2 := u_1|_{[s_2, \infty)}$. Let $x_1, x_2, \tilde{x}_1, \tilde{x}_2$ be the corresponding states with $x_1(s_1) = \tilde{x}_1(s_1) = z_1$, $x_2(s_2) = \tilde{x}_2(s_2) = z_2$.⁴ Then, it follows that

$$\begin{aligned} V(s_1, z_1) - V(s_2, z_2) &\leq V(s_1, z_1) - J(u_2; s_2, z_2) + \varepsilon \\ &\leq J(\tilde{u}_1; s_1, z_1) - J(u_2; s_2, z_2) + \varepsilon \end{aligned}$$

and by symmetry

$$\begin{aligned} |V(s_1, z_1) - V(s_2, z_2)| &\leq \varepsilon + |J(\tilde{u}_1; s_1, z_1) - J(u_2; s_2, z_2)| \\ &\quad + |J(u_1; s_1, z_1) - J(\tilde{u}_2; s_2, z_2)|. \end{aligned} \quad (2.8)$$

To obtain (2.6) we have to estimate the terms

$$|J(\tilde{u}_1; s_1, z_1) - J(u_2; s_2, z_2)|, \quad |J(u_1; s_1, z_1) - J(\tilde{u}_2; s_2, z_2)|.$$

From the definition of J and assumptions (A2), (A3) it follows that

$$\begin{aligned} |J(\tilde{u}_1; s_1, z_1) - J(u_2; s_2, z_2)| &\leq \left| \int_{s_1}^{s_2} e^{-\delta r} l(r, \tilde{x}_1(r), \tilde{u}_1(r)) dr \right| \\ &\quad + \left| \int_{s_2}^{\infty} e^{-\delta r} l(r, \tilde{x}_1(r), \tilde{u}_1(r)) - e^{-\delta r} l(r, x_2(r), u_2(r)) dr \right| \\ &\leq K_2 \int_{s_1}^{s_2} e^{-\delta r} (1 + |x_1(r)|) dr \\ &\quad + K_1 \int_{s_2}^{\infty} e^{-\delta r} |\tilde{x}_1(r) - x_2(r)| dr. \end{aligned} \quad (2.9)$$

⁴ Notice that $J(\tilde{u}_1; s_1, z_1)$ and/or $J(\tilde{u}_2; s_2, z_2)$ may be infinite.

From the Gronwall lemma, the first term on the right-hand side of (2.9) can be estimated by $\tilde{C}|s_1 - s_2|$, with some constant \tilde{C} depending on K_2 , T and R . We now estimate the last term. It follows from (2.4) that

$$|\tilde{x}_1(r) - x_2(r)| \leq e^{K_1(r-s_2)} |\tilde{x}_1(s_2) - z_2| \leq e^{K_1(r-s_2)} (|\tilde{x}_1(s_2) - z_1| + |z_1 - z_2|),$$

for $r \geq s_2$. Furthermore, it follows from (2.5) that $|\tilde{x}_1(s_2) - z_1| \leq C|s_2 - s_1|$, with $C = K_2(1 + R)e^{K_2T}$. Hence,

$$|\tilde{x}_1(r) - x_2(r)| \leq Ce^{K_1(r-s_2)} (|s_1 - s_2| + |z_1 - z_2|), \quad r > s_2.$$

Substituting in (2.9) and using the assumption $K_1 < \delta$, we obtain

$$|J(\tilde{u}_1; s_1, z_1) - J(u_2; s_2, z_2)| \leq C(|s_1 - s_2| + |z_1 - z_2|).$$

The term $|J(u_1; s_1, z_1) - J(\tilde{u}_2; s_2, y_2)|$ can be estimated analogously and it follows from (2.8) that

$$|V(s_1, z_1) - V(s_2, z_2)| \leq \varepsilon + C(|s_1 - s_2| + |z_1 - z_2|),$$

where the constant C is independent of $z_1, z_2, s_1, s_2, \varepsilon$ and grows exponentially with T . By taking the limit $\varepsilon \downarrow 0$, the lemma follows. \square

3. Dini solutions of the HJB equation

In this section we discuss a first concept of solution for the HJB equation

$$\partial_t v(t, x) + \mathcal{H}(t, x, \partial_x v(t, x)) = 0, \quad (t, x) \in (0, \infty) \times \mathcal{R}^n, \quad (3.1)$$

where \mathcal{H} is the function defined in (2.3).

As it is well known, one should not consider C^1 -solutions of (3.1) since such solutions do not exist in general: no matter how regular the control problem is, the value function might not be differentiable everywhere. Thus, a concept of nonsmooth solution for the HJB equation is necessary in this context and will be introduced in the sequel.

Remark 5. The HJB equation (3.1) is of nonstationary type since the control problem is nonautonomous. If the functions f, l do not depend explicitly on t , then, by a simple time transformation, (3.1) can be transformed into a stationary equation.

For the rest of this section let Φ be a mapping from $[0, \infty) \times \mathcal{R}^n$ to \mathcal{R} . This mapping should be considered as candidate for the value function.

Definition 6. The lower Dini derivative of a function Φ at (t, x) in the direction $(\alpha, \xi) \in \mathcal{R}^{1+n}$, is defined by

$$D^- \Phi(t, x; \alpha, \xi) := \liminf_{r \downarrow 0; w \rightarrow \xi} \frac{\Phi(t + r\alpha, x + rw) - \Phi(t, x)}{r}.$$

Before proceeding we make a remark on the definition of lower Dini directional derivatives. Recently, it has been widely

accepted that a lower Dini directional derivative must be defined through the lower limit with changes in both variables (t, x) , as it is defined above. However, since we are dealing with Lipschitz functions, the more general definition reduces to a limit on $r \downarrow 0$ alone, as according to Vinter and Wolenski [17, Lemma 5.1].

One should notice that the change of the $\lim \inf$ in Definition 6 by a $\lim \sup$ allows the definition of the upper Dini derivative of Φ (denoted by $D^+ \Phi$).

Next, we define the Dini generalized gradient of a function.

Definition 7. The Dini sub-gradient of a function Φ at $(t, x) \in [0, \infty) \times \mathcal{R}^n$ is defined by

$$\partial_D \Phi(t, x) := \{(\alpha, \xi) \in \mathcal{R}^{1+n}; D^- \Phi(t, x; u, v) \geq u\alpha + \langle v, \xi \rangle, \forall (u, v) \in \mathcal{R}^{1+n}\}.$$

Analogously, the Dini super-gradient of Φ at $(t, x) \in [0, \infty) \times \mathcal{R}^n$ is defined by

$$\partial^D \Phi(t, x) := \{(\alpha, \xi) \in \mathcal{R}^{1+n}; D^+ \Phi(t, x; u, v) \leq u\alpha + \langle v, \xi \rangle, \forall (u, v) \in \mathcal{R}^{1+n}\}.$$

It is easy to see that $\partial^D \Phi(t, x) = -\partial_D(-\Phi)(t, x)$. Now we are ready to introduce the concept of Dini semi-solutions of the HJB equation, which is our main interest in this paper.

Definition 8. We say that a function Φ is a Dini sub-solution of the HJB equation if it is continuous and satisfies

$$(a) \text{ for all } (\tau, z) \in [0, \infty) \times \mathcal{R}^n, \quad \alpha + \mathcal{H}(\tau, z, \xi) \geq 0 \quad \forall (\alpha, \xi) \in \partial^D \Phi(\tau, z), \quad (3.2)$$

(b) $\Phi(\tau, z) \rightarrow 0$, as $\tau \rightarrow \infty$, uniformly on each compact subset $K \subset \mathcal{R}^n$.

If Φ is continuous, satisfies (b), and further

$$(c) \text{ for all } (\tau, z) \in [0, \infty) \times \mathcal{R}^n, \quad \alpha + \mathcal{H}(\tau, z, \xi) \leq 0 \quad \forall (\alpha, \xi) \in \partial_D \Phi(\tau, z), \quad (3.3)$$

then it is called Dini super-solution of the HJB equation. If Φ satisfies (a)–(c), is called Dini solution of the HJB equation.

We close this section introducing another concept of generalized gradients, namely the proximal gradients. This will allow us to establish a relationship between the concepts of Dini and viscosity solutions of the HJB equation.

Definition 9. The proximal sub-gradient of a function Φ at (t, x) , denoted by $\partial_P \Phi(t, x)$, is the set of all vectors $(\alpha, \xi) \in \mathcal{R}^{1+n}$, such that there exists $\sigma > 0$ and a neighborhood U of (t, x) with

$$\Phi(\tau, y) \geq \Phi(t, x) + \alpha(\tau - t) + \langle \xi, y - x \rangle - \sigma(\|\tau - t\|^2 + \|y - x\|^2), \quad (3.4)$$

for all $(\tau, y) \in U$. Analogously, the *proximal super-gradient* of a function Φ at (t, x) , denoted by $\partial^P \Phi(t, x)$, is the set of all vectors $(\alpha, \zeta) \in \mathbb{R}^{1+n}$, such that there exists $\sigma > 0$ and a neighborhood U of (t, x) with

$$\Phi(\tau, y) \leq \Phi(t, x) + \alpha(\tau - t) + \langle \zeta, y - x \rangle + \sigma(\|\tau - t\|^2 + \|y - x\|^2), \quad (3.5)$$

for all $(\tau, y) \in U$.

It is worth noticing that the proximal super-gradient can be alternatively defined by $\partial^P \Phi(t, x) = -\partial_P(-\Phi)(t, x)$. Notice also that $\partial_P \Phi(t, x) \subset \partial_D \Phi(t, x)$.

4. Monotonicity properties

In this section we verify some monotonicity properties related to the solutions of HJB inequalities. We close the section with a monotonicity property involving the value function.

Let f, l be as defined in Section 1. For this section we suppose that assumptions (A1)–(A5) hold. Further, let $\Phi : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function. The pairs (Φ, f) are called systems. A system (Φ, f) is said to be *weakly decreasing* when, for every $z \in \mathbb{R}^n$, there exists an admissible process $(x, u) \in \mathcal{D}_{0,z;\infty}$ satisfying $\Phi(t, x(t)) \leq \Phi(0, x(0))$, $t \geq 0$. We say that a system (Φ, f) is *strongly increasing* in $[0, \infty)$ when for every interval $[a, b] \subset [0, \infty)$, each admissible process $(x, u) \in \mathcal{D}_{a,x(a),\infty}$ satisfies $\Phi(t, x(t)) \leq \Phi(b, x(b))$, $t \in [a, b]$. Weakly increasing and strongly decreasing systems are defined in an analogous way.

It is easy to check that (Φ, f) is strongly increasing in $[0, \infty)$ iff the function $t \mapsto \Phi(t, x(t))$ is nondecreasing for every admissible process (x, u) of $P_\infty(0, x(0))$. In the sequel we analyze a relation between the monotonicity of systems and HJB inequalities.

Proposition 10. *Let $\varphi : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function and \mathcal{H} be the Hamiltonian function in (2.3). The following assertions hold true:*

(a) *A function $[s, \infty) \ni t \mapsto \varphi(t, x(t)) - \int_t^\infty e^{-\delta\sigma} l(\sigma, x(\sigma), u(\sigma)) d\sigma \in \mathbb{R}$ is nondecreasing along every process $(x, u) \in \mathcal{D}_{s,z;\infty}$ iff*

$$\inf_{(\alpha, \zeta) \in \partial_P \varphi(s, z)} \{\alpha + \mathcal{H}(s, z, \zeta)\} \geq 0, \quad (4.1)$$

for all $(s, z) \in [0, \infty) \times \mathbb{R}^n$.

(b) *A function $[s, \infty) \ni t \mapsto \varphi(t, x(t)) - \int_t^\infty e^{-\delta\sigma} l(\sigma, x(\sigma), u(\sigma)) d\sigma \in \mathbb{R}$ is nonincreasing along every process $(x, u) \in \mathcal{D}_{s,z;\infty}$ iff*

$$\sup_{(\alpha, \zeta) \in \partial_P \varphi(s, z)} \{\alpha + \hat{\mathcal{H}}(s, z, \zeta)\} \leq 0, \quad (4.2)$$

for all $(s, z) \in [0, \infty) \times \mathbb{R}^n$.

(c) *A function $[s, \infty) \ni t \mapsto \varphi(t, x(t)) - \int_t^\infty e^{-\delta\sigma} l(\sigma, x(\sigma), u(\sigma)) d\sigma \in \mathbb{R}$ is nonincreasing along some process*

$(x, u) \in \mathcal{D}_{s,z;\infty}$ iff

$$\sup_{(\alpha, \zeta) \in \partial_P \varphi(s, z)} \{\alpha + \mathcal{H}(s, z, \zeta)\} \leq 0, \quad (4.3)$$

for all $(s, z) \in [0, \infty) \times \mathbb{R}^n$.

The notation $\hat{\mathcal{H}}$ means the substitution of the minimization by a maximization in the definition of \mathcal{H} , i.e.,

$$\hat{\mathcal{H}}(s, z, \zeta) := \sup_{u \in \Omega} \{\langle \zeta, f(s, z, u) \rangle + e^{-\delta s} l(s, z, u)\}.$$

Proof. We prove only assertion (a) of the proposition, the others being analogous. Let $\phi : [0, \infty) \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by

$$\phi(t, x, y) := \varphi(t, x) - y.$$

As a consequence of the proximal sub-gradient inequality, we have

$$\begin{aligned} (\alpha, \zeta, \zeta_0) \in \partial_P \phi(t, x, y) \\ \iff (\alpha, \zeta) \in \partial_P \varphi(t, x) \quad \text{and} \quad \zeta_0 = -1. \end{aligned} \quad (4.4)$$

Define the extended vector field $\hat{f} := (-e^{-\delta \cdot} l, f)^T$. Note that, because of (A1)–(A5), the function \hat{f} is continuous; $\hat{f}(t, x, \Omega)$ is a convex compact set for $t, x \in [0, \infty) \times \mathbb{R}^n$; \hat{f} satisfies a linear growth condition with respect to x .

First, we prove the necessity of condition (4.1). From the hypothesis, it follows that ϕ is monotone increasing along every admissible process for \hat{f} . This is equivalent to the system (ϕ, \hat{f}) being strongly increasing. Then, it follows from Clarke et al. [6, Exercise 4.6.4] that

$$\alpha + \inf_{u \in \Omega} \{\langle \zeta, f(t, x, u) \rangle - \langle \zeta_0, e^{-\delta t} l(t, x, u) \rangle\} \geq 0, \quad (4.5)$$

for all $(t, x, y) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}$, and all $(\alpha, \zeta, \zeta_0) \in \partial_P \phi(t, x, y)$. This last inequality, together with (4.4), guarantee that (4.1) is satisfied for all $(t, x) \in [0, \infty) \times \mathbb{R}^n$ and all $(\alpha, \zeta) \in \partial_P \varphi(t, x)$.

Next, we prove the sufficiency of condition (4.1). Note that this condition, together with (4.4), guarantees that (4.5) is satisfied for all $(t, x, y) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}$ and all $(\alpha, \zeta, \zeta_0) \in \partial_P \phi(t, x, y)$. Now, it follows from Clarke et al. [6, Exercise 4.6.4] that the system (ϕ, \hat{f}) is strongly increasing. This is equivalent to the desired monotonicity of the function $[s, \infty) \ni t \mapsto \varphi(t, x(t)) - \int_t^\infty e^{-\delta\sigma} l(\sigma, x(\sigma), u(\sigma)) d\sigma \in \mathbb{R}$. \square

In the sequel we verify that the real function defined by

$$[s, \infty) \ni t \mapsto V(t, x(t)) - \int_t^\infty e^{-\delta\sigma} l(\sigma, x(\sigma), u(\sigma)) d\sigma \in \mathbb{R}$$

is nondecreasing along every admissible process $(x, u) \in \mathcal{D}_{s,z;\infty}$. Choose a fixed $t \in [s, \infty)$ and $r > 0$. Thus, for every

process $(x, u) \in \mathcal{D}_{s,z;\infty}$, we have

$$\begin{aligned} & V(t+r, x(t+r)) - \int_{t+r}^{\infty} e^{-\delta\sigma} l(\sigma, x(\sigma), u(\sigma)) d\sigma \\ & - \left[V(t, x(t)) - \int_t^{\infty} e^{-\delta\sigma} l(\sigma, x(\sigma), u(\sigma)) d\sigma \right] \\ & = V(t+r, x(t+r)) - V(t, x(t)) \\ & \quad + \int_t^{t+r} e^{-\delta\sigma} l(\sigma, x(\sigma), u(\sigma)) d\sigma \geq 0 \end{aligned}$$

(the inequality follows from Bellman’s optimality principle). The desired monotony follows now from this inequality. An immediate consequence of this monotonicity (cf. Proposition 10) is the fact that the value function satisfies (4.1).

5. HJB inequalities

In this section we discuss some equivalences between HJB inequalities for proximal sub-gradients and Dini sub-gradients. We conclude the section by characterizing the value function as a Dini solution of the HJB equation.

Proposition 11. *Let assumptions (A1)–(A5) hold. If $\Phi : [0, \infty) \times \mathcal{R}^n \rightarrow \mathcal{R}$ is a continuous function, then*

- (a) $\sup_{(\alpha, \xi) \in \partial_P \Phi(s, z)} \{\alpha + \mathcal{H}(s, z, \xi)\} \leq 0, \forall (s, z)$
 $\iff \sup_{(\alpha, \xi) \in \partial_D \Phi(s, z)} \{\alpha + \mathcal{H}(s, z, \xi)\} \leq 0, \forall (s, z);$
- (b) $\inf_{(\alpha, \xi) \in \partial_P \Phi(s, z)} \{\alpha + \mathcal{H}(s, z, \xi)\} \geq 0, \forall (s, z)$
 $\iff \inf_{(\alpha, \xi) \in \partial^D \Phi(s, z)} \{\alpha + \mathcal{H}(s, z, \xi)\} \geq 0, \forall (s, z).$

Proof. We start by proving (a). The sufficiency follows from the inclusion $\partial_P \phi(t, x) \subset \partial_D \phi(t, x)$ for all (t, x) in the domain of ϕ . In order to prove the necessity, take an arbitrary $(\alpha, \beta) \in \partial_D \phi(s, z)$. It follows from Clarke et al. [6, Proposition 3.4.5] the existence of $(\alpha_n, \beta_n) \in (\alpha, \beta) + B_{1/n}(0)$ and $(s_n, z_n) \in [0, \infty) \times \mathcal{R}^n$, for $n \in \mathcal{N}$, such that $|s_n - s| + |z_n - z| < 1/n$, $(\alpha_n, \beta_n) \in \partial_P \phi(s_n, z_n)$ and $|\phi(s_n, z_n) - \phi(s, z)| < 1/n$. Therefore,

$$\alpha_n + \mathcal{H}(s_n, z_n, \beta_n) \leq 0 \quad \forall n.$$

Since \mathcal{H} is a continuous function, the desired result follows by taking the limit $n \rightarrow \infty$.

Next, we prove (b). The first inequality in (b) is satisfied for all $(\alpha, \xi) \in \partial_P \phi(s, z)$ if and only if the function

$$[s, \infty) \ni t \mapsto \varphi(t, x(t)) - \int_t^{\infty} e^{-\delta\sigma} f_0(\sigma, x(\sigma), u(\sigma)) d\sigma \in \mathcal{R}$$

is nondecreasing along every process $(x, u) \in \mathcal{D}_{s,z;\infty}$ (cf. Proposition 10). This is equivalent to the fact that

$$[s, \infty) \ni t \mapsto -\varphi(t, x(t)) + \int_t^{\infty} e^{-\delta\sigma} f_0(\sigma, x(\sigma), u(\sigma)) d\sigma \in \mathcal{R}$$

is nonincreasing along every process $(x, u) \in \mathcal{D}_{s,z;\infty}$. This, however, is equivalent to

$$\max_{(\alpha, \xi) \in \partial_P(-\varphi)(s, z)} \{\alpha + \tilde{\mathcal{H}}(s, z, \xi)\} \leq 0, \quad (5.1)$$

for all $(s, z) \in [0, \infty) \times \mathcal{R}^n$, where

$$\tilde{\mathcal{H}}(s, z, \xi) := \max_{u \in \Omega} \{\langle \xi, f(s, z, u) \rangle - e^{-\delta s} f_0(s, z, u)\}.$$

Analogously as in the proof of item (a), we use Clarke et al. [6, Proposition 3.4.5] to conclude that (5.1) is equivalent to

$$\max_{(\alpha, \xi) \in \partial_D(-\varphi)(s, z)} \{\alpha + \tilde{\mathcal{H}}(s, z, \xi)\} \leq 0, \quad (5.2)$$

for $(s, z) \in [0, \infty) \times \mathcal{R}^n$. Since $\partial_D(-\varphi)(s, z) = -\partial^D \varphi(s, z)$, it is easy to see that (5.2) is equivalent to

$$\min_{(\alpha, \xi) \in \partial^D \varphi(s, z)} \{\alpha + \mathcal{H}(s, z, \xi)\} \geq 0,$$

proving (b). \square

We are now ready to state and prove the main result of this section.

Theorem 12. *Suppose the assumptions (A1)–(A5) and (A6) are satisfied. Then the value function is a Dini solution of the HJB equation.*

Proof. First, we prove that the value function satisfies the decay condition in Definition 8. Since V is locally Lipschitz (see Lemma 4), it is enough to prove that $\lim_{\tau \rightarrow \infty} V(\tau, z) = 0$, for each $z \in \mathcal{R}^n$. Let $z \in \mathcal{R}^n$ and $\varepsilon > 0$ be given. For each $\tau \geq 0$, choose a process $(x_{\varepsilon, \tau}(\cdot), u_{\varepsilon, \tau}(\cdot)) \in \mathcal{D}_{\tau, z; \infty}$ satisfying

$$J(u_{\varepsilon, \tau}; \tau, z) - \varepsilon \leq V(\tau, z) \leq J(u_{\varepsilon, \tau}; \tau, z).$$

We now prove that

$$\lim_{\tau \rightarrow \infty} J(u_{\varepsilon, \tau}; \tau, z) = 0. \quad (5.3)$$

Indeed, from (A2) and the Gronwall lemma we obtain $|x_{\varepsilon, \tau}(t)| \leq M$ for $t \geq \tau$. Hence,

$$J(u_{\varepsilon, \tau}; \tau, z) \leq \int_{\tau}^{\infty} e^{-\delta t} K_2(1 + M) dt < \infty$$

and (5.3) follows.

Next, we prove that the value function satisfies (3.2) and (3.3). As already observed at the final part of Section 4, the value function V satisfies (4.1), which corresponds to the first inequality of Proposition 11(b). Thus, V also satisfies the second inequality, which corresponds to (3.2) (i.e., V is a Dini sub-solution).

Next, we prove that V is a Dini super-solution. It is enough to prove that V satisfies the first inequality of Proposition 11(a). Let $(s, z) \in [0, \infty) \times \mathcal{R}^n$ be given. Under assumptions (A1)–(A6), problem $P_{\infty}(s, z)$ has an optimal solution (see Appendix). Take (x, u) an optimal process for this problem. Thus, the function $[s, \infty) \ni t \mapsto V(t, x(t)) - \int_t^{\infty} e^{-\delta\sigma} l(\sigma, x(\sigma), u(\sigma)) d\sigma \in \mathcal{R}$ is constant. The desired inequality follows now from Proposition 10(c). \square

6. Verification functions and minimax results

In the first part of this section we use the Dini sub-solutions of the HJB equation in order to verify optimality of admissible processes. In the second part we concentrate our attention on some extremal properties of the value function, which involve the classes of Dini sub-solutions (respectively, super-solutions) of the HJB equation.

Proposition 13. *Let $(\bar{x}, \bar{u}) \in \mathcal{D}_{s,z;\infty}$ be given. If assumptions (A1)–(A5) hold and there exists a Dini sub-solution Φ of the HJB equation such that*

$$\Phi(s, z) = \int_s^\infty e^{-\delta t} l(t, \bar{x}(t), \bar{u}(t)) dt, \quad (6.1)$$

then (\bar{x}, \bar{u}) is an optimal process for problem $P_\infty(s, z)$.

Proof. Let Φ be a function satisfying the assumptions. Given an arbitrary admissible process $(x, u) \in \mathcal{D}_{s,z;\infty}$, it follows from Propositions 11(b) and 10(a) that

$$t \mapsto \Phi(t, x(t)) - \int_t^\infty e^{-\delta r} l(r, x(r), u(r)) dr$$

is a monotone nondecreasing function. Thus, for all $t > s$, we have

$$\begin{aligned} \Phi(t, x(t)) - \int_t^\infty e^{-\delta r} l(r, x(r), u(r)) dr \\ \geq \Phi(s, x(s)) - \int_s^\infty e^{-\delta r} l(r, x(r), u(r)) dr. \end{aligned}$$

From a simple algebraic calculation we deduce

$$\Phi(t, x(t)) + \int_s^t e^{-\delta r} l(r, x(r), u(r)) dr \geq \Phi(s, x(s)).$$

Taking the limit $t \rightarrow \infty$ and using the decay property of Φ , we conclude that

$$\begin{aligned} \int_s^\infty e^{-\delta r} l(r, x(r), u(r)) dr &\geq \Phi(s, x(s)) \\ &= \int_s^\infty e^{-\delta r} l(r, \bar{x}(r), \bar{u}(r)) ds, \end{aligned}$$

proving the optimality of (\bar{x}, \bar{u}) . \square

A function Φ satisfying the assumptions of the above proposition is called a *verification function* for process (\bar{x}, \bar{u}) . One should note that the value function V (which is a Dini sub-solution of the HJB equation from Theorem 12) is a verification function for every optimal process. Summarizing all these facts we have

Proposition 14. *Under assumptions (A1)–(A5) and $(\tilde{A}6)$, an admissible process $(\bar{x}, \bar{u}) \in \mathcal{D}_{s,z;\infty}$ is optimal iff there exists a corresponding verification function.*

In the sequel we investigate some minimax properties of the value function with respect to the sets of Dini semi-solutions of the HJB equation.

Proposition 15. *Under assumptions (A1)–(A6), the value function possesses the following maximal property:*

$$V(s, z) \geq \max\{\Phi(s, z); \Phi \text{ is a Dini sub-solution}\}.$$

Furthermore, the value function possesses the minimal property

$$V(s, z) \leq \min\{\Phi(s, z); \Phi \text{ is a Dini super-solution}\}.$$

Proof. It follows from Theorem 19 that the value function is well defined. Let Φ be a Dini sub-solution of the HJB equation. From item (b) of Proposition 11, we conclude that Φ satisfies (4.1). Thus, from item (a) of Proposition 10, it follows that the function

$$[s, \infty) \ni t \mapsto \Phi(t, x(t)) - \int_t^\infty e^{-\delta \sigma} l(\sigma, x(\sigma), u(\sigma)) d\sigma \in \mathcal{R}$$

is nondecreasing along any process $(x, u) \in \mathcal{D}_{s,z;\infty}$, for all $(s, z) \in [0, \infty) \times \mathcal{R}^n$. Then, for fixed (s, z) and a sub-optimal process $(x_\varepsilon, u_\varepsilon) \in \mathcal{D}_{s,z;\infty}$ we have

$$\begin{aligned} \Phi(t, x_\varepsilon(t)) - \int_t^\infty e^{-\delta \sigma} l(\sigma, x_\varepsilon(\sigma), u_\varepsilon(\sigma)) d\sigma \\ \geq \Phi(s, z) - J(u_\varepsilon; s, z) \\ \geq \Phi(s, z) - V(s, z) - \varepsilon. \end{aligned}$$

If we take the limit $t \rightarrow \infty$ and use the decay property of Φ , we obtain $\Phi(s, z) \leq V(s, z) + \varepsilon$, proving the desired inequality.

Now, let Φ be a Dini super-solution of the HJB equation. From item (a) of Proposition 11, we conclude that Φ satisfies (4.3). Thus, from item (c) of Proposition 10, follows the existence of a process $(x, u) \in \mathcal{D}_{s,z;\infty}$ (for each (s, z)), such that the function

$$[s, \infty) \ni t \mapsto \Phi(t, x(t)) - \int_t^\infty e^{-\delta \sigma} l(\sigma, x(\sigma), u(\sigma)) d\sigma \in \mathcal{R}$$

is nonincreasing. Then, for fixed (s, z) , we have

$$\begin{aligned} \Phi(t, x(t)) - \int_t^\infty e^{-\delta \sigma} l(\sigma, x(\sigma), u(\sigma)) d\sigma \\ \leq \Phi(s, z) - J(u_\varepsilon; s, z) \\ \leq \Phi(s, z) - V(s, z). \end{aligned}$$

Next, we take the limit $t \rightarrow \infty$ in the above inequality and obtain $\Phi(s, z) \geq V(s, z)$, concluding the proof. \square

Since the value function is a Dini solution of the HJB equation, Proposition 15 can be restated in the following form:

Corollary 16. *Under assumptions (A1)–(A6), the value function is the maximal Dini sub-solution (respectively, minimal Dini super-solution) of the HJB equation.*

An immediate consequence of Corollary 16 is the uniqueness of Dini solutions for the HJB equation.

7. Viscosity solutions

In this section we introduce the concept of viscosity solutions of the HJB equation. The main result of this section is a proof of the equivalence of the concepts of Dini and viscosity solutions of the HJB equation.

Definition 17. A function $v : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ is called *viscosity solution* of the partial differential equation

$$\begin{aligned} \partial_t v(t, x) + \mathcal{H}(t, x, \partial_x v(t, x)) \\ = 0 \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \end{aligned} \quad (7.1)$$

if it satisfies the following conditions:

- (a) v is continuous and $v(\tau, z) \rightarrow 0$, as $\tau \rightarrow \infty$, uniformly in each compact $K \subset \mathbb{R}^n$.
- (b) For every test function $\phi \in C^1((0, \infty) \times \mathbb{R}^n; \mathbb{R})$, such that $v - \phi$ has a local maximum at $(t, x) \in (0, \infty) \times \mathbb{R}^n$, we have

$$\partial_t \phi(t, x) + \mathcal{H}(t, x, \partial_x \phi(t, x)) \geq 0. \quad (7.2)$$

- (c) For every test function $\phi \in C^1((0, \infty) \times \mathbb{R}^n; \mathbb{R})$, such that $v - \phi$ has a local minimum at $(t, x) \in (0, \infty) \times \mathbb{R}^n$, we have

$$\partial_t \phi(t, x) + \mathcal{H}(t, x, \partial_x \phi(t, x)) \leq 0. \quad (7.3)$$

If the function v satisfies only (a) and (b), it is called *viscosity sub-solution*. If V satisfies only (a) and (c), it is called *viscosity super-solution*.

Next, we verify that a function is a viscosity solution iff it is a Dini solution of the HJB equation.

Theorem 18. *The following assertions hold true:*

- (i) *A function v is a viscosity sub-solution of the HJB equation iff*

$$\begin{aligned} \text{(a) for all } (s, z) \in [0, \infty) \times \mathbb{R}^n, \\ \{\alpha + \mathcal{H}(s, z, \xi)\} \geq 0 \quad \forall (\alpha, \xi) \in \partial^D v(s, z); \end{aligned} \quad (7.4)$$

- (b) $\lim_{s \uparrow \infty} v(s, z) = 0$ uniformly for z in compact subsets of \mathbb{R}^n ;

- (ii) *A function v is a viscosity super-solution of the HJB equation iff it satisfies assertion (b) of item (i) and, for all $(s, z) \in [0, \infty) \times \mathbb{R}^n$, we have*

$$\{\alpha + \mathcal{H}(s, z, \xi)\} \leq 0 \quad \forall (\alpha, \xi) \in \partial_D v(s, z). \quad (7.5)$$

Proof. We begin by proving (i). In the proof of the necessity, as well as in the proof of sufficiency, the decay condition is fulfilled by hypothesis. Therefore, it remains only to prove the equivalence between inequalities (7.4) and (7.2).

Let v be a continuous function satisfying (7.4) and ϕ be an arbitrary C^1 test function such that $v - \phi$ has a local maximum at $(s, z) \in [0, \infty) \times \mathbb{R}^n$. Consequently, $-(v - \phi)$ has a

local minimum at (s, z) , and we have $0 \in \partial_D(-v + \phi)(s, z)$. Thus, $-\nabla \phi(s, z) \in \partial_D(-v)(s, z)$. However, this is equivalent to $\nabla \phi(s, z) \in \partial^D v(s, z)$. Now, taking $(\alpha, \xi) = \nabla \phi(s, z)$ in (7.4), we conclude that ϕ satisfies (7.2).

Next, we prove the converse. Let v be a viscosity sub-solution of the HJB equation. Given $(\alpha, \xi) \in \partial^D v$, we have $(-\alpha, -\xi) \in \partial_D(-v)$. From Clarke et al. [6, Proposition 3.4.12] follows the existence of a C^1 function $-\phi$ such that $\nabla(-\phi)(s, z) = (-\alpha, -\xi)$ and $-v + \phi$ has a local minimum at (s, z) (i.e., $v - \phi$ has a local maximum at (s, z)). Now, since v is a viscosity sub-solution, (7.4) follows from (7.2) for this particular ϕ .

The proof of assertion (ii) is analogous and will be omitted. \square

As an immediate consequence of Theorem 18, we conclude that the value function is the unique viscosity solution of the HJB equation with a particular decay property. This theorem also characterizes Dini (semi) solutions, which were in focus of our attention in this paper, as an important tool to investigate nonautonomous optimal control problems of infinite horizon type. This last assertion is based on the knowledge that both concepts of solutions, Dini and viscosity, coincide, agreeing with the theory in finite or (autonomous) infinite horizon problems.

Appendix

In this appendix we discuss the issue of existence of an optimal control process for $P_\infty(\tau, z)$. The method of proof is standard, therefore we omit the proof and we make some comments instead.

Theorem 19. *Suppose (A1)–(A6) are satisfied. Then there exists an optimal process.*

The existence result in Theorem 19 is under compactness of the set of admissible control actions and convexity of the so-called extended velocity vector. We make the observation that the arguments of the proof are the usual in which we start with a minimizing sequence and under “compactness” of the trajectories on compact sub-intervals of $[\tau, \infty)$ we extract a sub-sequence, which converges to an absolutely continuous function uniformly on compact sub-intervals. This function, with the help of Filippov’s lemma [10], is proved to be a state trajectory corresponding to some admissible control. Together, they form a process that is optimal for $P_\infty(\tau, z)$. These are standard arguments in finite time interval that can be modified accordingly to be reproduced for infinite time horizon.

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