KACZMARZ METHODS FOR REGULARIZING NONLINEAR ILL-POSED EQUATIONS I: CONVERGENCE ANALYSIS

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Abstract. In this article we develop and analyze novel iterative regularization techniques for the solution of systems of nonlinear ill–posed operator equations. The basic idea consists in considering separately each equation of this system and incorporating a loping strategy. The first technique is a Kaczmarz–type method, equipped with a novel stopping criteria. The second method is obtained using an embedding strategy, and again a Kaczmarz–type approach. We prove well-posedness, stability and convergence of both methods.

1. Introduction

In this article we investigate regularization methods for solving *linear* and *nonlinear systems of ill–posed operator equations*. Many practical *inverse problems* are naturally formulated in such a way [2, 5, 7, 8, 14, 18, 17, 21, 20, 26].

We consider the problem of determining some physical quantity x from data $(y^i)_{i=0}^{N-1}$, which are functionally related by

$$(1) \qquad F_i(x) = y^i, \quad i = 0, \dots, N-1.$$

Here $F_i : D_i \subseteq X \to Y$ are operators between Hilbert spaces X and Y. We are specially interested in the situation where the data is not exactly known, i.e., we have only an approximation $y^{\delta,i}$ of the exact data, satisfying

$$(2) \qquad \|y^{\delta,i} - y^i\| < \delta^i.$$

Standard methods for the solution of such systems are based on rewriting (1) as a single equation

$$(3) \qquad F(x) = \mathbf{y}, \quad i = 0, \dots, N-1,$$

where $F := 1/\sqrt{N} \cdot (F_0, \dots, F_{N-1})$ and $\mathbf{y} = 1/\sqrt{N} \cdot (y^0, \dots, y^{N-1})$. There are at least two basic concepts for solving ill posed equations of the form (3): *Iterative* regularization methods (cf., e.g., [15, 10, 6, 1, 12]) and *Tikhonov type* regularization methods [25, 24, 23, 19, 6]. However these methods become inefficient if N is large or the evaluations of $F_i(x)$ and $F_i'(x)^*$ are expensive. In such a situation Kaczmarz–type methods [11, 20] which cyclically consider each equation in (1) separately, are much faster [21] and are often the method of choice in practice. On the other hand, only few theoretical results about regularizing properties of Kaczmarz methods are available, so far.

The *Landweber–Kaczmarz approach* for the solution of (1), (2) analyzed in this article consists in incorporating a bang-bang relaxation parameter in the classical Landweber–Kaczmarz method [13], combined with a new stopping rule. Namely,

$$(4) \qquad x_{n+1} = x_n - \omega_n F_{[n]}'(x_n)^* (F_{[n]}(x_n) - y^{\delta,[n]}),$$

with

$$(5) \qquad \omega_n := \omega_n(\delta, y^\delta) = \begin{cases} 1 & \|F_{[n]}(x_n) - y^{\delta,[n]}\| > \tau \delta^{[n]} \\ 0 & \text{otherwise} \end{cases},$$

where $\tau > 2$ is an appropriate chosen positive constant and $[n] := n \mod N \in \{0, \dots, N-1\}$. The iteration terminates if all ω_n become zero within a cycle, that is if $\|F_i(x_n) - y^{\delta,i}\| \le \tau \delta^i$ for all $i \in \{0, \dots, N-1\}$. We shall refer to this method as *loping Landweber–Kaczmarz method* (LLK). Its worth mentioning that, for noise free data, $\omega_n = 1$ for all n and therefore, in this special situation, our iteration is identical to the classical Landweber–Kaczmarz method

$$(6) \qquad x_{n+1} = x_n - F_{[n]}'(x_n)^* (F_{[n]}(x_n) - y^{\delta,[n]}),$$

which is a special case of [21, Eq. (5.1)].

However, for noisy data, the LLK method is fundamentally different to (6): The parameter ω_n effects that the iterates defined in (4) become stationary and *all components* of the residual vector $\|F_i(x_n) - y^{\delta,i}\|$ fall below some threshold, making (4) a convergent regularization method. The convergence of the residuals in the maximum norm better exploits the error estimates (2) than standard methods, where only *squared average* $1/N \cdot \sum_{i=0}^{N-1} \|F_i(x_n) - y^{\delta,i}\|^2$ of the residuals falls below a certain threshold. Moreover, especially after a large number of iterations, ω_n will vanish for some n. Therefore, the computational expensive evaluation of $F_{[n]}[x_n]^*$ might be loped, making the Landweber–Kaczmarz method in (4) a fast alternative to conventional regularization techniques for system of equations.

The second regularization strategy considered in this article is an embedding approach, which consists in rewriting (1) into an system of equations on the space X^N

$$(7) \qquad F_i(x^i) = y^i, \quad i = 0, \dots, N-1,$$

with the additional constraint

$$(8) \qquad \sum_{i=0}^{N-1} \|x^{i+1} - x^i\|^2 = 0,$$

where we set $x^N := x^0$. Notice that if x is a solution of (1), then the *constant vector* $(x^i = x)_{i=0}^{N-1}$ is a solution of system (7), (8), and vice versa. This system of

equations is solved using a block Kaczmarz strategy of the form

$$(9) \qquad \mathbf{x}_{n+1/2} = \mathbf{x}_n - \omega_n \mathbf{F}'(\mathbf{x}_n)^* (\mathbf{F}(\mathbf{x}_n) - \mathbf{y}^\delta)$$

$$(10) \qquad \mathbf{x}_{n+1} = \mathbf{x}_{n+1/2} - \omega_{n+1/2} \mathbf{G}(\mathbf{x}_{n+1/2}),$$

where $\mathbf{x} := (x^i)_i \in X^N$, $\mathbf{y}^\delta := (y^{\delta,i})_i \in Y^N$, $\mathbf{F}(\mathbf{x}) := (\mathrm{F}_i(x^i))_i \in Y^N$,

$$(11) \qquad \begin{aligned} \omega_n &= \begin{cases} 1 & \|\mathbf{F}(\mathbf{x}_n) - \mathbf{y}^\delta\| > \tau\delta \\ 0 & \text{otherwise} \end{cases}, \\[2mm] \omega_{n+1/2} &= \begin{cases} 1 & \|\mathbf{G}(\mathbf{x}_{n+1/2})\| > \tau\epsilon(\delta) \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

with $\delta := \max\{\delta^i\}$. The strictly increasing function $\epsilon : [0, \infty) \to [0, \infty)$ satisfies $\epsilon(\delta) \to 0$, as $\delta \to 0$, and guaranties the existence of a finite stopping index. A natural choice is $\epsilon(\delta) = \delta$. Moreover, up to a positive multiplicative constant, $\mathbf{G}$ corresponds to the steepest descent direction of the functional

$$(12) \qquad \mathcal{G}(\mathbf{x}) := \sum_{i=0}^{N-1} \|x^{i+1} - x^i\|^2$$

on X^N. Notice that (10) can also be interpreted as a Landweber–Kaczmarz step with respect to the equation

$$(13) \qquad \lambda D(\mathbf{x}) = 0,$$

where $D(\mathbf{x}) = (x^{i+1} - x^i)_i \in X^N$ and λ is a small positive parameter such that $\|\lambda D\| \leq 1$. Since equation (1) is embedded into a system of equations on a higher dimensional function space we call the resulting regularization technique *embedded Landweber–Kaczmarz* (ELK) method. As shown in Section 3, (9), (10) generalizes the Landweber method for solving (3).

The article is outlined as follows. In Section 2 we investigate the LLK method with the novel parameter stopping rule. We prove well–posedness, stability and convergence, as the noise level tends to zero. Moreover, we show that all components of the residual vector fall below a certain threshold. In Section 3 we analyze the ELK method. In particular, we make use of the results in Section 2 to prove that the ELK method is well posed, convergent and stable.

2. ANALYSIS OF THE LOPING LANDWEBER–KACZMARZ METHOD

In this section we present the convergence analysis of the *loping Landweber–Kaczmarz* (LLK) method. The novelty of our approach consists in omitting an update in the Landweber Kaczmarz iteration, within one cycle, if the corresponding i–th residual is below some threshold, see (5). Consequently, the LLK method is not stopped until all residuals are below the specified threshold. Therefore, it is the natural counterpart of the Landweber–Kaczmarz iteration [11, 20] for ill–posed problems.

The following assumptions are standard in the convergence analysis of iterative regularization methods [6, 10, 12]. We assume that F_i is *Fréchet differentiable* and that there exists $\rho > 0$ with

$$(14) \qquad \|\mathrm{F}'_i(x)\|_Y \leq 1, \qquad x \in B_\rho(x_0) \subset \bigcap_{i=0}^{N-1} D_i.$$

Here $B_\rho(x_0)$ denotes the closed ball of radius ρ around the starting value x_0, D_i is the domain of F_i, and $F_i'(x)$ is the Fréchet derivative of F_i at x.

Moreover, we assume that the *local tangential cone condition*

$$
(15) \quad \|F_i(x) - F_i(\bar{x}) - F_i'(x)(x - \bar{x})\|_Y \le \eta\|F_i(x) - F_i(\bar{x})\|_Y ,
$$

$$
x, \bar{x} \in B_\rho(x_0) \subset D_i
$$

holds for some $\eta < 1/2$. This is a central assumption in the analysis of iterative methods for the solution of nonlinear ill–posed problems [6, 12].

In the analysis of the LLK method we assume that τ (used in the definition (5) of ω_n) satisfies

$$
(16) \quad \tau > 2\frac{1 + \eta}{1 - 2\eta} > 2 .
$$

Note that, for noise free data, the LLK method is equivalent to the classical Landweber–Kaczmarz method, since $\omega_n = 1$ for all $n \in \mathbb{N}$.

In the case of noisy data, iterative regularization methods require early termination, which is enforced by an appropriate stopping criteria. In order to motivate the stopping criteria, we derive in the following lemma an estimate related to the monotonicity of the sequence x_n defined in (4).

Lemma 2.1. *Let x be a solution of (1) where F_i are Fréchet differentiable in $B_\rho(x_0)$, satisfying (14), (15). Moreover, let x_n be the sequence defined in (4), (5). Then*

$$
(17) \quad \begin{aligned} &\|x_{n+1} - x\|^2 - \|x_n - x\|^2 \\ &\le \omega_n\|F_{[n]}(x_n) - y^{\delta,[n]}\|\left(2(1 + \eta)\delta^{[n]} - (1 - 2\eta)\|F_{[n]}(x_n) - y^{\delta,[n]}\|\right), \end{aligned}
$$

where $[n] := n \mod N$.

Proof. The proof follows the lines of [10, Proposition 2.2]. Notice that if ω_n is different from zero, inequality (17) follows analogously as in [10]. In the case $\omega_n = 0$, (17) follows from $x_n = x_{n+1}$. □

Motivated, by Lemma 2.1 we define the termination index $n_*^\delta = n_*^\delta(y^\delta)$ as the smallest integer multiple of N such that

$$
(18) \quad x_{n_*^\delta} = x_{n_*^\delta+1} = \cdots = x_{n_*^\delta+N} .
$$

Now we have the following monotonicity result:

Lemma 2.2. *Let x, F_i and x_n be defined as in Lemma 2.1 and n_*^δ be defined by (18). Then we have*

$$
(19) \quad \|x_{n+1} - x\| \le \|x_n - x\| , \quad n = 0, \ldots, n_*^\delta .
$$

Moreover, the stoping rule (18) implies $\omega_{n_*^\delta+i} = 0$ for all $i \in \{0, \ldots, N - 1\}$, i.e.,

$$
(20) \quad \|F_i(x_{n_*^\delta}) - y^{\delta,i}\| \le \tau\delta^i , \quad i = 0, \ldots, N - 1 .
$$

Proof. If $\omega_n = 0$, then (19) holds since the iteration stagnates. Otherwise, from the definitions of ω_n in (5) and τ in (16), it follows that

$$
(21) \quad 2(1 + \eta)\delta^i - (1 - 2\eta)\|F_{[n]}(x_n) - y^{\delta,[n]}\| < 0 ,
$$

and the right hand side in (17) becomes non–positive.

To prove the second assertion we use (17) for $n = n_*^\delta + i$, for $i \in \{0, \ldots, N-1\}$. By noting that $x_{n_*^\delta + i} = x_{n_*^\delta}$ and $[n_*^\delta + i] = i$, we obtain

$$0 \leq \omega_{n_*^\delta + i} \cdot \|F_i(x_{n_*^\delta}) - y^{\delta,i}\| \left(2(1+\eta)\delta^i - (1-2\eta)\|y^{\delta,i} - F_i(x_{n_*^\delta})\| \right),$$

for $i \in \{0, \ldots, N-1\}$. Suppose $\omega_{n_*^\delta + i} \neq 0$, then $2(1+\eta)\delta^i - (1-2\eta)\|y^{\delta,i} - F_i(x_{n_*^\delta})\| \geq 0$, which contradicts the definition of $\omega_{n_*^\delta + i}$. $\qquad\square$

Note that for $n > n_*^\delta$, $\omega_n \equiv 0$ and therefore $x_n = x_{n_*^\delta}$. This shows that the Landweber–Kaczmarz method becomes stationary after n_*^δ.

Remark 1. Similar to the nonlinear Landweber iteration one obtains the estimate

$$(22) \qquad \frac{n_*^\delta \cdot \left(\tau \min_i(\delta^i) \right)^2}{N} \leq \sum_{n=0}^{n_*^\delta - 1} \omega_n \|y^{\delta,[n]} - F_{[n]}(x_n)\|^2 \leq \frac{\tau \|x - x_{n_*^\delta}\|^2}{(1-2\eta)\tau - 2(1+\eta)}.$$

Here we use the notation of Lemma 2.1.

From Remark 1 it follows that, in the case of noisy data, $n_*^\delta < \infty$ and the iteration terminates after a finite number of steps. Next, we state the main result of this section, namely that the Landweber–Kaczmarz method is a convergent regularization method.

Theorem 2.3. *Assume that* F_i *are Fréchet-differentiable in* $B_\rho(x_0)$, *satisfy (14), (15) and the system (1) has a solution in* $B_{\rho/2}(x_0)$. *Then*

1. *For exact data* $y^{\delta,i} = y^i$, *the sequence* x_n *in (4) converges to a solution of (1). Moreover, if* $x^\dagger$ *denotes the unique solution of (1) with minimal distance to* x_0 *and*

$$(23) \qquad \mathcal{N}(F_i'(x^\dagger)) \subseteq \mathcal{N}(F_i'(x)), \quad x \in B_\rho(x_0), \ i \in \{0, \ldots, N-1\},$$

   then $x_n \to x^\dagger$.
2. *For noisy data the loping Landweber–Kaczmarz iterates* $x_{n_*^\delta}$ *converge to a solution of (1) as* $\delta \to 0$. *If in addition (23) holds, then* $x_{n_*^\delta}$ *converges to* $x^\dagger$ *as* $\delta \to 0$.

Proof. The proof of the first item is analogous to the proof in [13, Proposition 4.3] (see also [12]). We emphasize that, for exact data, the iteration (4) reduces to the classical Landweber–Kaczmarz method, which allows to apply the corresponding result of [13].

The proof of the second item is analogous to the proof of the corresponding result for the Landweber iteration as in [10, Theorem 2.9]. For the first case within this proof, (20) is required. For the second case we need the monotony result from Lemma 2.2. $\qquad\square$

In the case of noisy data (i.e. the second item of Theorem 2.3), it has been shown in [13] that the Landweber–Kaczmarz iteration

$$(24) \qquad x_{n+1} = x_n - F_{[n]}'(x_n)^*(F_{[n]}(x_n) - y^{\delta,[n]}),$$

is convergent if it is terminated after the $\tilde{n}^\delta$–th step, where $\tilde{n}^\delta$ is the smallest iteration index that satisfies

$$(25) \qquad \|F_{[\tilde{n}^\delta]}(x_{\tilde{n}^\delta}) - y^{\delta,[\tilde{n}^\delta]}\| \leq \tau \delta^{[\tilde{n}^\delta]}.$$

Therefore, in general, only one of the components of the residual vector $\left(\| F_i(x_{\tilde{n}^\delta}) - y^{\delta,i} \| \right)_i$ is smaller than $\tau \delta^i$, namely the *active component* $\| F_{[\tilde{n}^\delta]}(x_{\tilde{n}^\delta}) - y^{\delta,[\tilde{n}^\delta]} \|$. However, the argumentation in [13] is incomplete, in the sense that the case when $\tilde{n}^\delta$ stagnates as $\delta \to 0$, has not been considered. Hence, [13, Theorem 4.4] requires the additional assumption that $\tilde{n}^\delta \to \infty$, as $\delta \to 0$, which is usually the case in practice.

3. Analysis of the embedded Landweber–Kaczmarz method

In the *embedded Landweber–Kaczmarz* (ELK) method for the solution of (1), $x \in X$ is substituted by a vector $\mathbf{x} = (x^i)_{i=0}^{N-1}$. In (9) each component of $\mathbf{x}$ is updated independently according to one of the system equations. In the balancing step (10), the difference between the components of $\mathbf{x}$ is minimized.

In order to determine $\mathbf{x}_{n+1/2}$, each of its components $x_{n+1/2}^i$ can be evaluated independently:

$$x_{n+1/2}^i = x_n^i - \omega_n F_i'(x_n^i)^* \left(F_i(x_n^i) - y^{\delta,i} \right), \ i = 0, \ldots, N-1.$$

In the balancing step (10), $\mathbf{x}_{n+1}$ is determined from $\mathbf{x}_{n+1/2}$ by a matrix multiplication with the sparse matrix $I_{X^N} - \omega_{n+1/2} \mathbf{G}$, where

$$\mathbf{G} = \lambda^2 \begin{pmatrix} 2I & -I & 0 & & -I \\ -I & 2I & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & \ddots & \ddots & 2I & -I \\ -I & & 0 & -I & 2I \end{pmatrix} \in \mathcal{L}(X^N, X^N).$$

Here λ is a small positive parameter such that $\| \lambda D \| \leq 1$, and the operator $\mathbf{G}$ is a discrete variant of $-\lambda^2$ times the second derivative operator and therefore penalizes for varying components. As already mentioned in the introduction, the the balancing step (10) is a Landweber–Kaczmarz step with respect to the equation (13). The operator D is linear and bounded, which guaranties the existence of a positive constant λ such that λD satisfies (14), which will be needed in the analysis of the embedded Landweber method. The iteration defined in (9), (10) is terminated when for the first time

(26) $$\mathbf{x}_{n_\star^\delta+1} = \mathbf{x}_{n_\star^\delta+1/2} = \mathbf{x}_{n_\star^\delta}.$$

The artificial noise level $\epsilon : [0, \infty) \to [0, \infty)$ satisfies $\epsilon(\delta) \to 0$, as $\delta \to 0$ and guaranties the existence of a finite stopping index in the ELK method.

In the sequel we shall apply the results of the Section 2 to prove convergence of the ELK method. As initial guess we use a constant vector $\mathbf{x}_0 := (x_0)_i$ whose components are identical to x_0. Moreover, our convergence analysis will again require the scaling assumption (14) and the tangential cone condition (15) to be satisfied near x_0.

Remark 2 (Comparison with the classical Landweber iteration)**.** Let $F := 1/\sqrt{N} \cdot (F_0, \ldots, F_{N-1})$ and $\mathbf{y}^\delta := 1/\sqrt{N} \cdot (y^{\delta,0}, \ldots, y^{\delta,N-1})$. The Landweber iteration for
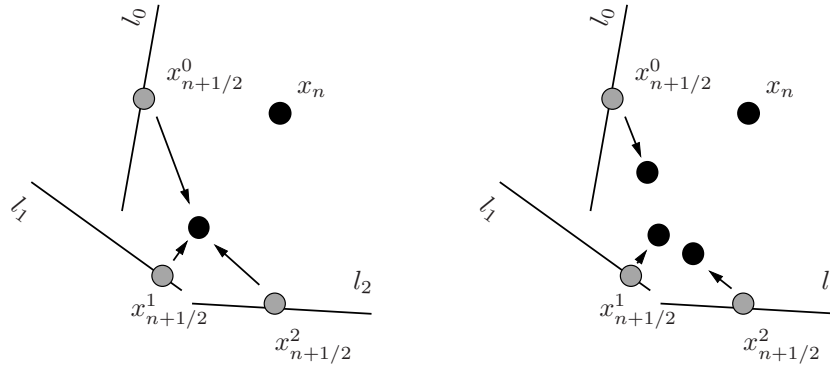
FIGURE 1. Landweber versus ELK method for solving $a_i \cdot x = y^i$, $i \in \{0, 1, 2\}$ with $\|a_1\| = 1$. In this case $x^i_{n+1/2} := x_n - a^*_i(a_i \cdot x_n - y^i)$ is the orthogonal projection of x_n on $l_i := \{x \in \mathbb{R}^2 : a_i \cdot x = y^i\}$. Each step of the Landweber iteration (left) generates a single element in X, namely the average $1/3 \sum_{i=0}^{2} x^i_{n+1/2}$. In contrast, a cycle in the embedded Landweber–Kaczmarz method (right) generates a vector in X^N, where each component of $\mathbf{x}_{n+1}$ is a linear combination of $x^i_{n+1/2}$.

the solution of $\mathrm{F}(x) = \mathbf{y}^\delta$, see (3), is [10, 6]

$$x_{n+1} = x_n - \mathrm{F}'[x_n]^*(\mathrm{F}(x_n) - \mathbf{y}^\delta)$$

$$= x_n - \frac{1}{N} \cdot \sum_{i=0}^{N-1} \mathrm{F}'_i(x_n)^*(\mathrm{F}_i(x_n) - y^{\delta,i})$$

$$= \frac{1}{N} \cdot \sum_{i=0}^{N-1} \left(x_n - \mathrm{F}'_i(x_n)^*(\mathrm{F}_i(x_n) - y^{\delta,i}) \right).$$

If we set $x^i_{n+1/2} := x_n - \mathrm{F}'_i(x_n)^*(\mathrm{F}_i(x_n) - y^{\delta,i})$ then the Landweber method can be rewritten in form similar to (9), (10), namely

$$\begin{aligned} x^i_{n+1/2} &= x_n - F'_i(x_n)^*(F_i(x_n) - y^{\delta,i}), \\ (27) \qquad x_{n+1} &= \frac{1}{N} \cdot \sum_{i=0}^{N-1} x^i_{n+1/2}. \end{aligned}$$

Hence, the distinction between the Landweber and the ELK method is that (27) in the Landweber method makes all components equal, whereas the balancing step (10) in the embedded Landweber–Kaczmarz method leaves them distinct.

In order to illustrate the idea behind the ELK method, we exemplarily consider the case $N = 3$. In this case the mechanism of the embedded iteration is explained in Figure 1 in contrast to the Landweber method.

In the next theorem we prove that the termination index is well defined, as well as convergence and stability of the ELK method.

Theorem 3.1. *Assume that the operators* F_i *are Fréchet-differentiable in* $B_\rho(x_0)$ *and satisfy (14), (15). Moreover, we assume that (1) has a solution in* $B_{\rho/2}(x_0)$. *Then we have:*

1. *For exact data* $y^{\delta,i} = y^i$, *the sequence* $\mathbf{x}^n$ *in (9), (10) converges to a constant vector* $(x)_i$, *where* x *is a solution (1) in* $B_{\rho/2}(x_0)$. *Additionally, if the operators* F_i *satisfy (23), then the sequence* $\mathbf{x}^n$ *converges to the constant vector* $\mathbf{x}^\dagger = (x^\dagger)_i$, *where* $x^\dagger$ *is the unique solution of minimal distance to* x_0.
2. *For noisy data* $\delta > 0$, *(26) defines a finite termination index* $n_\star^\delta$. *Moreover, the embedded Landweber–Kaczmarz iteration* $\mathbf{x}^{n_\star^\delta}$ *converges to a constant vector* $\mathbf{x} = (x)_i$, *where* x *is a solution (1) in* $B_{\rho/2}(x_0)$, *as* $\delta \to 0$. *If in addition (23) holds, then each component of* $\mathbf{x}^{n_\star^\delta}$ *converges to* $x^\dagger$, *as* $\delta \to 0$.

Proof. In order to prove the first item we apply Theorem 2.3, item 1 to the system (7), (13). From (14) it follows that $\|\mathbf{F}[\mathbf{x}]\| \le 1$ for $\mathbf{x} \in B_\rho(x_0)^N$. Moreover, since D is bounded linear, $\|\lambda D\| \le 1$ for sufficiently small λ. The tangential cone condition (15) for F_i implies

$$\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\bar{\mathbf{x}}) - \mathbf{F}'(\mathbf{x})(\mathbf{x} - \bar{\mathbf{x}})\| \le \eta \|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\bar{\mathbf{x}})\|,$$
$$\mathbf{x}, \bar{\mathbf{x}} \in B_\rho(x_0)^N.$$

Moreover, since λD is a linear operator, the tangential cone condition is obviously satisfied for λD with the same η. Therefore, by applying Theorem 2.3, item 1 we conclude that $\mathbf{x}_n$ converges to a solution $\tilde{\mathbf{x}}$ of (1), (13). From (13) it follows that $\tilde{\mathbf{x}} = (\tilde{x})_i$ is a constant vector. Therefore, $F_i(\tilde{x}) = y^i$, proving the assertion.

Additionally, let $\mathbf{x}^\dagger$ denote the solution of (7), (13) with minimal distance to $(x_0)_i$. As an auxiliary result we show that $\mathbf{x}^\dagger = (x^\dagger)_i$, where $x^\dagger$ is the unique solution of (1) with minimal distance to x_0. Due to (13) we have $\mathbf{x}^\dagger = (\tilde{x})_i$, for some $\tilde{x} \in X$. Moreover, the vector $(x^\dagger)_i$ is a solution of (1), (13) and

$$\|\tilde{x} - x_0\|^2 = \frac{1}{N} \sum_{i=0}^{N-1} \|\tilde{x} - x_0\|^2 \le \frac{1}{N} \sum_{i=0}^{N-1} \|x^\dagger - x_0\|^2 = \|x^\dagger - x_0\|^2.$$

Therefore $\mathbf{x}^\dagger = (x^\dagger)_i$. Now, if (23) is satisfied, then

$$\mathcal{N}(\mathbf{F}'(\mathbf{x}^\dagger)) \subseteq \mathcal{N}(\mathbf{F}'(\mathbf{x})), \quad \mathbf{x} \in B_\rho(x_0)^N$$

and by applying Theorem 2.3 we conclude that $\mathbf{x}^n \to \mathbf{x}^\dagger$.

The proof of the second item follows from Theorem 2.3, item 2 in an analogous way as above. $\square$

As consequence of Theorem 3.1, if $n_\star^\delta$ is defined by (26), $\mathbf{x}^{n_\star^\delta} = (x_i^{n_\star^\delta})_i$, then

$$(28) \qquad x^{n_\star^\delta} := \sum_{i=0}^{N-1} x_i^{n_\star^\delta} \longrightarrow x^\dagger,$$

as $\delta \to 0$. However, Theorem 3.1 guaranties even more: All components $x_i^{n_\star^\delta}$ converge to $x^\dagger$ as the noise level tend to zero. Moreover, due to the averaging process in (28) the noise level in the actual regularized solution $x^{n_\star^\delta}$ becomes noticeable reduced.

4. Conclusion

We have suggested two novel Kaczmarz type regularization techniques for solving systems of ill-posed operator equations. For each one we proved convergence and stability results. The first technique is a variation of the Landweber–Kaczmarz method with a new stopping rule and a loping parameter that allows to skip some of the inner cycle iterations, if the corresponding residuals are sufficiently small. The second method derives from an embedding strategy, where the original system is rewritten in a larger space.

One advantage of Kaczmarz type methods is the fact that the resulting regularizing methods better explore the special structure of the model and the pointwise noise-estimate $\|y^i - y^{\delta,i}\| < \delta^i$. Moreover, for noisy data, it is often much faster [21, p.19] in practise than Newton–type methods. The key in this article to prove convergence, as the noise level tends to zero, was to introduce a bang–bang relaxation parameter ω_n in the iteration (4). Recently regularizing *Newton–Kaczmarz* methods, similar to (6), have been analyzed [4]. Their convergence analysis was based on the assumption [4, Eq. (3.14)] which in the linear case implies that, for exact data, a single equation in (1) would already be sufficient to find the solution of (1). Our strategy of incorporating a bang-bang relaxation parameter in (4), which can also be combined with Newton type iterations [9], overcomes this severe restriction. An analysis of loping Kaczmarz–type Levenberg–Marquard [9] and steepest–descent [22] regularization methods will be presented in a forthcoming publication.

Our methods allow fast implementation. The effectiveness is presented in a subsequent article: There we shall consider the Landweber–Kaczmarz methods of Sections 2, 3 applied to thermoacoustic tomography [5, 8, 26], semiconductor equations [18, 17] and Schlieren imaging [3, 16].

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