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On level set type methods for elliptic Cauchy problems

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Abstract

Two methods of level set type are proposed for solving the Cauchy problem for an elliptic equation. Convergence and stability results for both methods are proven, characterizing the iterative methods as regularization methods for this ill-posed problem. Some numerical experiments are presented, showing the efficiency of our approaches and verifying the convergence results.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

We start by introducing the inverse problem under consideration. Let $\Omega \subset \mathbb{R}^d, d = 2, 3$, be an open bounded set with piecewise Lipschitz boundary $\partial\Omega$. Further, we assume that $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, where Γ_i are two open connected disjoint parts of $\partial\Omega$. We denote by P the elliptic operator defined in Ω by

$$P(u) := -\sum_{i,j=1}^{d} D_i(a_{i,j} D_j u), \tag{1}$$

where the real functions $a_{i,j} \in L^\infty(\Omega)$ are such that the matrix $A(x) := (a_{i,j})_{i,j=1}^d$ satisfies $\xi^t A(x)\xi > \alpha||\xi||^2$, for all $\xi \in \mathbb{R}^d$ and for a.e. $x \in \Omega$, where $\alpha > 0$.

We denote by the *elliptic Cauchy problem* (CP) the following boundary value problem (BVP):

$$(CP) \qquad \begin{cases} Pu = f & \text{in } \Omega \\ u = g_1 & \text{at } \Gamma_1 \\ u_\nu = g_2 & \text{at } \Gamma_1. \end{cases}$$

The functions $g_1, g_2 \colon \Gamma_1 \to \mathbb{R}$ are given *Cauchy data*, and $f \colon \Omega \to \mathbb{R}$ is a known source term in the model.

As a solution of the elliptic Cauchy problem we consider every $H^m(\Omega)$-distribution, which solves the weak formulation of the elliptic equation in Ω and also satisfies the Cauchy data at Γ_1 in the sense of the trace operator ($m \in \mathbb{N}$ still has to be chosen). Note that if we know either the Neumann or the Dirichlet trace of u at Γ_2, then u can be computed as a solution of a mixed BVP in a stable way. Therefore, it is enough to consider the task of determining a trace (Dirichlet or Neumann) of u at Γ_2.

It is well known that elliptic Cauchy problems are not well posed in the sense of Hadamard. A famous example given by Hadamard in the early 1920s [16, 23] shows that one cannot expect the solution of the CP to depend continuously on the data. For Lipschitz bounded domains $\Omega \subset \mathbb{R}^2$, the severely ill-posedness of the CP was recently investigated in [3] using a Steklov–Poincaré approach. The existence of solutions for arbitrary Cauchy data (g_1, g_2) cannot be assured as a direct argumentation with the Schwartz reflection principle shows [15] (the Cauchy data (g_1, g_2) are called *consistent* if the corresponding problem (CP) has a solution). It has been recently shown [2] that in the case $m = 1$, there exists a dense subset M of $H^{1/2}(\Gamma_1) \times \left[H_{00}^{1/2}(\Gamma_1)\right]'$ such that the CP has a $H^1(\Omega)$ solution for Cauchy data $(g_1, g_2) \in M$. As concerns the uniqueness of the solution for the CP, it is possible to extend the Cauchy–Kowalewsky and Holmgren theorems to the H^m-context and prove the uniqueness of weak solutions (see, e.g., [12] for the case $m = 1$). For classical uniqueness results, we refer the reader to [8]. A weak uniqueness result for nonlinear Cauchy problems can be found in [21].

A variety of numerical methods for solving the CP can be found in the literature. An optimization approach based on least squares and Tikhonov regularization was used in [13]. In [20] Maz'ya *et al* proposed an iterative algorithm based on the successive solutions of well-posed mixed BVPs. A generalization of this method, based on the fixed point theory, was derived in [23]. A further generalization for nonlinear elliptic Cauchy problems can be found in [21]. In [18, 22] the Backus–Gilbert method was used to solve the CP. A Mann iterative regularization method was proposed for the CP in [12]. In [17] a method of conjugate gradient type was investigated. Finite element approximations, based on an optimal control formulation of the CP, were discussed in [9]. An application of the quasi-reversibility method for the CP was considered in [4, 5]. In [2] Ben Abda *et al* introduced an energy functional, which depends on both unknown traces of the solution of the CP at Γ_2.

Our main goal in this paper is to apply level set methods for obtaining stable approximations of the solution of the CP. The first application of level set methods to inverse problems was proposed by Santosa [26]. These methods can be used in identification problems where the unknown parameter is a piecewise constant function, assuming one of only two possible values. The methods considered here are adequate to solve the CP in the special case where the Neumann trace of u at Γ_2 is known *a priori* to satisfy $u_\nu|_{\Gamma_2} = \chi_D$, for some $D \subset \Gamma_2$.

A related application corresponds to the inverse problem in corrosion detection [7, 19]. This problem consists of determining information about corrosion occurring on the inaccessible boundary part (Γ_2) of a specimen. The data for this inverse problem correspond to prescribed current flux (g_2) and voltage measurements (g_1) on the accessible boundary part (Γ_1), and the model is the Laplace equation with no source term ($P = \Delta$, $f = 0$). For simplicity, one assumes the specimen to be a thin plate ($\Omega \subset \mathbb{R}^2$) and $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Moreover, the unknown corrosion damage γ is assumed to be the characteristic function of some $D \subset \Gamma_2$, corresponding to the boundary condition $u_\nu + \gamma u = 0$ at Γ_2.

The numerical methods analyzed in this paper can be extended in a straightforward way to arbitrary elliptic Cauchy problems possessing a solution with a similar structure, i.e. whenever the assumption that $u_\nu|_{\Gamma_2}$ is a piecewise constant function assuming one of only two possible values (not necessarily zero and one) is valid. The proposed methods are inspired by the approaches followed in [6, 14] and relate to evolution flows of Hamilton–Jacobi type.

The paper is outlined as follows. In section 2, we write the elliptic Cauchy problem in the functional analytical framework of an (ill-posed) operator equation. This is the starting point for the level set approaches derived in the two subsequent sections. In section 3, we investigate a level set method for the CP based on the approach proposed in [6]. We prove convergence and stability of the proposed method, as well as a monotonicity result analogous to that known for the asymptotic regularization method [27]. In section 4, we derive a second level set method for the CP based on the ideas presented in [14]. First, we prove the existence of minima for a least-squares functional related to the CP. In the following, we prove convergence and stability results for our regularization strategy. The corresponding level set method is derived from an explicit Euler method for solving the evolution equation related to the first-order optimality condition of the least-squares functional. Section 5 is devoted to numerics. Three different experiments are provided, in order to illustrate the effectiveness of the level set method considered in section 4.

2. Formulation of the inverse problem

In this section, we rewrite the elliptic CP in the functional analytical framework of an operator equation. This is the starting point for the level set approaches derived in sections 3 and 4.

The functional analytical framework established in this section is similar to that derived in [23]. The difference is that, instead of looking for a fixed point operator, we follow an optimal control approach proposed by Lions [25]. We begin by defining the auxiliary problem:

$$\begin{cases} Pv = f & \text{in } \Omega \\ v = g_1 & \text{at } \Gamma_1 \\ v_\nu = \varphi & \text{at } \Gamma_2. \end{cases} \tag{2}$$

This mixed BVP defines the operator $T\colon \varphi \mapsto v_\nu|_{\Gamma_1}$. Note that if $\varphi = u_\nu|_{\Gamma_2}$, where u is the solution of the CP, then it would follow $T(\varphi) = g_2$. Thus, a simple least-squares approach [9, 19] for the CP consists of solving the optimization problem

$$\|T(\varphi) - g_2\|^2 \to \min.$$

Due to the superposition principle for linear elliptic BVPs [15], one can split the solution of (2) in $v = v_a + v_b$, where

$$Pv_a = 0 \quad \text{in} \quad \Omega, \qquad v_a = 0 \quad \text{at} \quad \Gamma_1, \qquad (v_a)_\nu = \varphi \quad \text{at} \quad \Gamma_2; \tag{3}$$

$$Pv_b = f \quad \text{in} \quad \Omega, \qquad v_b = g_1 \quad \text{at} \quad \Gamma_1, \qquad (v_b)_\nu = 0 \quad \text{at} \quad \Gamma_2. \tag{4}$$

From (3), we can define the linear operator

$$L\colon \varphi \mapsto (v_a)_\nu|_{\Gamma_1}, \tag{5}$$

and from (4) we define the function $z := (v_b)_\nu|_{\Gamma_1}$. Since $T(\varphi) = L\varphi + z$, the CP can be written in the form of the operator equation

$$L\varphi = g_2 - z, \tag{6}$$

where the constant term z depends only on g_1, f and P. Therefore, it can be computed *a priori*.

In order to derive our first level set approach (section 3) for solving (6), we have to formulate the CP in such a way that L is continuous with respect to the L^2-norm. For the second level set approach (section 4), we state the CP such that L is continuous with respect to the L^1-norm. In the following, we present these two possible formulations of the CP.

2.1. Framework for the first level set approach

We consider the CP in the form of equation (6). Further, we assume the Cauchy data to satisfy

$$(g_1, g_2) \in H^{1/2}(\Gamma_1) \times \left[H_{00}^{1/2}(\Gamma_1)\right]', \tag{7}$$

and the source term f to be a $L^2(\Omega)$ distribution.

From this choice of g_1 and f, it follows that the mixed BVP in (4) has a unique solution $v_b \in H^1(\Omega)$ [15, 23]. Therefore, $z := (v_b)_\nu|_{\Gamma_1} \in \left[H_{00}^{1/2}(\Gamma_1)\right]'$ and the distribution $g_2 - z$ on the right-hand side of (6) is in $\left[H_{00}^{1/2}(\Gamma_1)\right]'$.

Note that, if we choose $\varphi \in L^2(\Gamma_2) \subset \left[H_{00}^{1/2}(\Gamma_2)\right]'$, the mixed BVP in (3) has a unique solution $v_a \in H^1(\Omega)$ and the linear operator L in (5) is well defined from $L^2(\Gamma_2)$ into $\left[H_{00}^{1/2}(\Gamma_1)\right]'$. Indeed, this assertion follows from

Proposition 2.1. *Let the domain* $\Omega \subset \mathbb{R}^d$ *with* $d = 2, 3$ *and the operator* P *be defined as in section 1. Then, the linear operator defined in (5) is an injective bounded map* $L\colon L^2(\Gamma_2) \to \left[H_{00}^{1/2}(\Gamma_1)\right]'$.

Proof. Since $d = 2, 3$, the boundary part Γ_2 is either a 1D or a 2D Lipschitz manifold and the embedding $L^2(\Gamma_2) \subset \left[H_{00}^{1/2}(\Gamma_2)\right]'$ is continuous. Therefore, given $\varphi \in L^2(\Gamma_2)$, the mixed BVP in (3) has a unique solution $v_a \in H^1(\Omega)$ satisfying the *a priori* estimate

$$\|v_a\|_{H^1(\Omega)} \leqslant C_1 \|\varphi\|_{[H_{00}^{1/2}(\Gamma_2)]'},$$

for some positive constant C_1 (depending on P, Ω and Γ_2). Now, from the continuity of the Neumann trace operator $\gamma_{N,1}\colon H^1(\Omega) \ni v \mapsto v_\nu|_{\Gamma_1} \in \left[H_{00}^{1/2}(\Gamma_1)\right]'$ follows

$$\|L\varphi\|_{[H_{00}^{1/2}(\Gamma_1)]'} \leqslant C_2 \|v_a\|_{H^1(\Omega)} \leqslant C_3 \|\varphi\|_{L^2(\Gamma_2)},$$

and the continuity of L follows. It remains to prove the injectivity of L. Note that, if $L\varphi = 0$, the function v_a in (3) satisfies $Pv_a = 0$ in Ω, $v_a = (v_a)_\nu = 0$ at Γ_1. Then, $\varphi = 0$ follows from the uniqueness of the weak solution for the CP [12]. □

Summarizing, this setup allows us to state the CP in the form of the operator equation (6), where L is the linear continuous operator:

$$L\colon L^2(\Gamma_2) \to \left[H_{00}^{1/2}(\Gamma_1)\right]' \tag{8}$$

defined in (5).

2.2. Framework for the second level set approach

In the following we shall assume $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ and define yet another functional analytical framework to analyze (6). The Cauchy data are assumed to satisfy

$$(g_1, g_2) \in \left[H_{00}^{1/2}(\Gamma_1)\right]' \times \left[H_{00}^{3/2}(\Gamma_1)\right]' \tag{9}$$

and the source term f to be a $H^{-1}(\Omega)$ distribution.

Due to the choice of g_1 and f above, the elliptic theory allows us to conclude that the mixed BVP in (4) has a unique solution $v_b \in L^2(\Omega)$ [10, 15]. Therefore, $z := (v_b)_\nu|_{\Gamma_1} \in \left[H_{00}^{3/2}(\Gamma_1)\right]'$ and the term $g_2 - z$ on the right-hand side of (6) is a distribution in $\left[H_{00}^{3/2}(\Gamma_1)\right]'$.

In the next proposition, we prove that the linear operator L in (5) is well defined continuous and injective from $L^1(\Gamma_2)$ to $\left[H_{00}^{3/2}(\Gamma_1)\right]'$.

Proposition 2.2. *Let the domain* $\Omega \subset \mathbb{R}^d$ *with* $d = 2, 3$ *and the operator* P *be defined as in section 1. Then, the linear operator defined in (5) is an injective bounded map* $L\colon L^1(\Gamma_2) \to \left[H_{00}^{3/2}(\Gamma_1)\right]'$.

Proof. If $d = 2$, the boundary part Γ_2 is a 1D manifold and the Sobolev embedding theorem [1, 15] implies $H_0^s(\Gamma_2) \subset L^\infty(\Gamma_2)$ for $s > 1/2$. If $d = 3$, the embedding above still holds, however only for $s > 1$. In either case, we can take $s = 3/2$ and conclude that $L^1(\Gamma_2) \subset \left[L^\infty(\Gamma_2)\right]' \subset H^{-3/2}(\Gamma_2)$. This result together with $H^{-3/2}(\Gamma_2) \subset \left[H_{00}^{3/2}(\Gamma_2)\right]'$ imply the continuity of the embedding $L^1(\Gamma_2) \subset \left[H_{00}^{3/2}(\Gamma_2)\right]'$. Thus, from the elliptic theory we conclude that given $\varphi \in L^1(\Gamma_2)$, the mixed BVP in (3) has a unique solution $v_a \in L^2(\Omega)$ satisfying the *a priori* estimate

$$\|v_a\|_{L^2(\Omega)} \leqslant C_1 \|\varphi\|_{[H_{00}^{3/2}(\Gamma_2)]'},$$

for some positive constant C_1 (depending on P, Ω and Γ_2). Now, from the continuity of the Neumann trace operator $\gamma_{N,1}\colon L^2(\Omega) \ni v \mapsto v_\nu|_{\Gamma_1} \in [H_{00}^{3/2}(\Gamma_1)]'$ follows

$$\|L\varphi\|_{[H_{00}^{3/2}(\Gamma_1)]'} \leqslant C_2 \|v_a\|_{L^2(\Omega)} \leqslant C_3 \|\varphi\|_{L^1(\Gamma_2)},$$

and the continuity of L is proven. In order to prove the injectivity of L, one argues analogously as in the last part of the proof of proposition 2.1. $\square$

Summarizing, this setup allows us to state the CP in the form of equation (6), where L is the linear continuous operator:

$$L\colon L^1(\Gamma_2) \to \left[H_{00}^{3/2}(\Gamma_1)\right]' \tag{10}$$

defined in (5).

2.3. *A remark on the dimension of* $\Omega \subset \mathbb{R}^d$

In the approach presented in subsection 2.1, the key argument to prove the desired regularity of the operator L (see (8)) was the continuity of the embedding $L^2(\Gamma_2) \subset \left[H_{00}^{1/2}(\Gamma_2)\right]'$. Since the continuity of this embedding does not depend on d, proposition 2.1 actually holds for every $d \in \mathbb{N}$.

In the case of the approach presented in subsection 2.2, the situation is different: proposition 2.2 only holds for $d = 2, 3$. Indeed, the desired regularity of L in (10) depends on the continuity of the embedding $L^1(\Gamma_2) \subset \left[H_{00}^{3/2}(\Gamma_2)\right]'$, which follows from the Sobolev embedding theorem:

$$H_0^s(\Gamma_2) \subset C^0(\overline{\Gamma}_2) \subset L^\infty(\Gamma_2) \qquad \text{for} \quad s > (d-1)/2$$

(note that $\Gamma_2 \in \mathbb{R}^{d-1}$). Therefore, if $\Omega \in \mathbb{R}^d$ for $d \geqslant 4$, we would have to choose $s \geqslant 2$ in the proof of proposition 2.2. The proof would no longer hold, since the mixed BVP in (3) would not have a solution v_a on a Sobolev space of non-negative index.

This is not actually a disadvantage of the second level set approach, since almost all real-life applications are related to either two- or three-dimensional domains Ω.

2.4. *A remark on noisy Cauchy data*

The second remark concerns the investigation of the CP for noisy data. If only corrupted noisy data $\left(g_1^\delta, g_2^\delta\right)$ are available, we assume the existence of a consistent pair of Cauchy data (g_1, g_2) such that

$$\left\|g_1 - g_1^\delta\right\|_{L^2(\Gamma_1)} + \left\|g_2 - g_2^\delta\right\|_{L^2(\Gamma_1)} \leqslant \delta. \tag{11}$$

For clarity of the presentation, we separately discuss the two frameworks introduced above in this section.

Noisy data within the framework of subsection 2.2. Let the noisy data be given as in (11) and the exact Cauchy data satisfy (9). Since z in (6) depends continuously on g_1 in the $\left[H_{00}^{1/2}(\Gamma_1)\right]'$

topology, we can solve the mixed BVP in (4) using g_1^δ as data and obtain a corresponding $z^\delta \in \left[H_{00}^{3/2}(\Gamma_1)\right]'$, such that

$$\left\|(g_2 - z) - \left(g_2^\delta - z^\delta\right)\right\|_{[H_{00}^{3/2}(\Gamma_1)]'} \leqslant C\delta, \tag{12}$$

where the constant C depends on Ω, P, Γ_1 and f. Summarizing, we have

Lemma 2.3. *Within the framework of subsection 2.2, the CP with noisy data satisfying (11) reduces to equation*

$$L\varphi = g_2^\delta - z^\delta,$$

where L satisfies (10) and the right-hand side $\left(g_2^\delta - z^\delta\right)$ satisfies (12).

Noisy data within the framework of subsection 2.1. Let the noisy data be given as in (11) and the exact Cauchy data satisfy (7). Since z in (6) depends continuously on g_1 in the $H^{1/2}(\Gamma_1)$ topology, a natural question arises:

Is it possible to obtain from measured data $\left(g_1^\delta, g_2^\delta\right)$ satisfying (11), a corresponding $z^\delta \in \left[H_{00}^{1/2}(\Gamma_1)\right]'$ such that $\|z - z^\delta\|_{[H_{00}^{1/2}(\Gamma_1)]'} \leqslant \delta$?

We claim that such z^δ can be obtained under the *a priori* assumption $g_1 \in H^s(\Gamma_1)$ for some $s > 1/2$. Indeed, under this assumption, Engl and Leitão [12, lemma 8] guarantee the existence of a smoothing operator S: $L^2(\Gamma_1) \to H^{1/2}(\Gamma_1)$, and of a positive function μ with $\lim_{t\downarrow 0}\mu(t) = 0$, such that for $\delta > 0$ and $g_1^\delta \in L^2(\Gamma_1)$ with $\left\|g_1 - g_1^\delta\right\|_{L^2(\Gamma_1)} \leqslant \delta$, we have $\left\|g_1 - S(g_1^\delta)\right\|_{H^{1/2}(\Gamma_1)} \leqslant \mu(\delta)$. Thus, after smoothing the noisy data g_1^δ, we obtain $\hat{g}_1^\delta := S\left(g_1^\delta\right) \in H^{1/2}(\Gamma_1)$. Next, we solve the mixed BVP in (4) using $\hat{g}_1^\delta$ as data, and obtain a corresponding $z^\delta \in \left[H_{00}^{1/2}(\Gamma_1)\right]'$ with $\|z - z^\delta\|_{[H_{00}^{1/2}(\Gamma_1)]'} \leqslant C\mu(\delta)$, with C as in (12). Once we are able to give an affirmative answer to the question, it follows from (11) that

$$\left\|(g_2 - z) - \left(g_2^\delta - z^\delta\right)\right\|_{[H_{00}^{1/2}(\Gamma_1)]'} \leqslant (1 + C)\max\{\delta, \mu(\delta)\}. \tag{13}$$

Summarizing, we have

Lemma 2.4. *Consider the framework of subsection 2.1. Assume the noisy Cauchy data satisfy (11), where $g_1 \in H^s(\Gamma_1)$ for some $s > 1/2$. Then the CP reduces to equation*

$$L\varphi = g_2^\delta - z^\delta,$$

where L satisfies (8) and the right-hand side $\left(g_2^\delta - z^\delta\right)$ satisfies (13).

3. A first level set approach

In this section, we investigate a level set method for the CP based on the approach introduced in [6]. For this purpose, we consider the functional analytical framework derived in subsection 2.1 and summarized in lemma 2.4.

The starting point of this approach is the assumption that the solution $\overline{\varphi}$ of (6) is the characteristic function χ_D of a subdomain $D \subset\subset \Gamma_2$. A function ϕ: $\Gamma_2 \times \mathbb{R}^+ \to \mathbb{R}$ is introduced, allowing the definition of the level sets $D(t) = \{\phi(\cdot, t) \geqslant 0\}$. The function ϕ should be chosen such that

$$\varphi(\cdot, t) := \chi_{D(t)} \to \chi_D = \overline{\varphi} \tag{14}$$

as $t \to \infty$. The level set method corresponds to a continuous evolution for an artificial time t, where the *level set function* ϕ is defined by a Hamilton–Jacobi equation of the form

$$\frac{\partial \phi}{\partial t} + V \cdot \nabla\phi = 0, \tag{15}$$

with the initial value $\phi(x, 0) = \phi_0(x)$, where ϕ_0 is an appropriate indicator function of a measurable set $D_0 \subset\subset \Gamma_2$. The function $V: \mathbb{R}^{d-1} \times \mathbb{R}^+ \to \mathbb{R}^{d-1}$ describes the velocity of the level sets of ϕ. Following [26], V is chosen in the normal direction of the level set curves of ϕ, i.e. $V = v\nabla\phi/|\nabla\phi|$, for $v: \mathbb{R}^{d-1} \times \mathbb{R}^+ \to \mathbb{R}$.

Here, the guideline for the choice of the *velocity function* V is a property of the asymptotic regularization method [27]. In this method, the approximations $\varphi(\cdot, t)$ for the solution $\overline{\varphi}$ of (6) satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(\cdot, t) - \overline{\varphi}\|^2 = -2\|L(\varphi(\cdot, t)) - (g_2 - z)\|^2. \tag{16}$$

As we shall see, a level set method satisfying (16) can be analyzed in a similar way to asymptotic regularization[3]. Our first goal is to determine how to choose the velocity V such that if ϕ solves (15), then the function φ defined in (14) satisfies (16).

Proposition 3.1. *Let the function h be defined by*

$$h(x, t) = (-1 + 2\varphi(x, t)) \cdot \operatorname{div} V(x, t), x \in \Gamma_2, t \in \mathbb{R}^+,$$

for some $V \in L^\infty(0, T, L^2(\mathbb{R}^{d-1}))^{d-1}$ *with* $\operatorname{div} V \in L^1(0, T, L^\infty(\mathbb{R}^{d-1})) \cap L^\infty(0, T, L^2(\mathbb{R}^{d-1}))$. *Moreover, let the level set function ϕ satisfy (15) and φ be defined by (14). Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(\cdot, t) - \overline{\varphi}\|^2_{L^2(\Gamma_2)} = \int_{\Gamma_2} (\varphi(\cdot, t) - \overline{\varphi}) \cdot h(x, t)\,\mathrm{d}x, \tag{17}$$

for all $t \in (0, T)$.

Proof. See [6, proposition 3.3]. □

Remark 3.2. If V satisfies the assumptions of proposition 3.1 for each $T > 0$, then (17) is satisfied for all $t \in \mathbb{R}^+$.

Since we are assuming the operator L to satisfy (8), we conclude from proposition 3.1 that relation (16) is satisfied for the velocity V satisfying

$$-\operatorname{div} V = 2(-1 + 2\varphi)^{-1} L^*(L\varphi - (g_2 - z)) \text{ in } \Gamma_2 \times \mathbb{R}^+. \tag{18}$$

Observe that if the normal derivative $V_\nu(\cdot, t)$ vanishes on $\partial\Gamma_2$ for $t \in \mathbb{R}^+$, then the support of $\varphi(\cdot, t)$ remains a subset of $\overline{\Omega}$ during the evolution, which is a desirable property. It is worth noting that a solution V of (18) with a homogeneous Neumann boundary condition on $\partial\Gamma_2$ always exists. Indeed, it is enough to choose $V = \nabla\psi$, where ψ solves

$$-\Delta\psi = 2(-1 + 2\varphi)^{-1} L^*(L\varphi - (g_2 - z)) \text{ on } \Gamma_2, \qquad \psi = 0 \text{ at } \partial\Gamma_2.$$

In the following we derive a convergence analysis for the level set method defined by (14) and (15), with the choice of velocity in (18). We consider noisy Cauchy data as in lemma 2.4. Moreover, we define the *stopping time* $T(\delta, g_1^\delta, g_2^\delta)$ by the generalized discrepancy principle [11]

$$T\left(\delta, g_1^\delta, g_2^\delta\right) := \inf\left\{t \in \mathbb{R}^+; \left\|L(\varphi(\cdot, t)) - \left(g_2^\delta - z^\delta\right)\right\| \leqslant \tau\delta\right\}, \tag{19}$$

for some $\tau > 1$.

The next theorem summarizes the main convergence and stability results for this level set method

Theorem 3.3 (convergence analysis). *Let V satisfy (18) and φ, ϕ be defined by (14) and (15), respectively.*

[3] The method of asymptotic regularization cannot be used directly to construct piecewise approximations for $\overline{\varphi}$, since it uses the time derivative of $\varphi(\cdot, t)$ which is not defined in $L^2(\Gamma_2)$.

(i) *Monotonicity. For noisy Cauchy data and* $\tau > 1$*, the iteration error is strictly monotone decreasing, i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(\cdot,t) - \overline{\varphi}\|^2 < 0,$$

for all $t > 0$ *with* $\left\|L(\varphi(\cdot,t)) - \left(g_2^\delta - z^\delta\right)\right\| > \tau\delta$*. Moreover, for exact Cauchy data (i.e.* $\delta = 0$*), we have the inequality*

$$\int_0^\infty \|L(\varphi(\cdot,t)) - L(\overline{\varphi})\|^2\,\mathrm{d}t < \infty.$$

(ii) *Convergence. If the Cauchy data are exact, then* $\varphi(\cdot,t) \to \overline{\varphi}$ *in* $L^2(\Gamma_2)$ *as* $t \to \infty$*, where* $\overline{\varphi}$ *is the solution of (6) corresponding to the consistent data* (g_1, g_2)*.*

(iii) *Stability. For noisy Cauchy data, the stopping time* $T\left(\delta, g_1^\delta, g_2^\delta\right) =: T_\delta$ *defined by (19) with* $\tau > 1$ *is finite. Moreover, given a sequence* $\delta_k \to 0$ *and* $\left\{\left(g_1^{\delta_k}, g_2^{\delta_k}\right)\right\}_k$ *corresponding noisy data satisfying (11) for some consistent data pair* (g_1, g_2)*, then the approximations* $\varphi(\cdot, T_{\delta_k})$ *converge to* $\overline{\varphi}$ *in* $L^2(\Gamma_2)$ *as* $\delta_k \to 0$*, where* $\overline{\varphi}$ *is the solution of (6) corresponding to* (g_1, g_2)*.*

Proof. Item (i): the monotonicity result is a consequence of (16). The inequality in the second statement is a well-known property of the asymptotic regularization and its proof is analogous to [6, proposition 4.1].

Item (ii): this statement concerning convergence for exact data is also known to hold for the Landweber iteration as well as for the asymptotic regularization. The proof follows the lines of [6, theorem 4.4].

Item (iii): this convergence result for noisy data has a counterpart in the asymptotic regularization. The proof carries over from [27, theorem 4]. □

Remark 3.4. Under the assumptions of theorem 3.3 one can prove, analogously as in item (i), that the residual function $t \mapsto \left\|L(\varphi(\cdot,t)) - \left(g_2^\delta - z^\delta\right)\right\|$ is monotonically non-increasing.

4. A second level set approach

In the following, we investigate a second level set method for the CP. Our approach is based on [14]. In what follows, we shall consider the functional analytical framework for the CP discussed in subsection 2.2 and summarized in lemma 2.3.

Let functions φ and ϕ be defined as in section 3. For simplicity, we adopt the notation $Y := H_{00}^{3/2}(\Gamma_1)'$. If we denote by H the Heaviside projector[4], then the Cauchy problem (6) can be written in the form of the constrained optimization problem

$$\min \left\|L\varphi - \left(g_2^\delta - z^\delta\right)\right\|_Y^2, \qquad \text{s.t.} \quad \varphi = H(\phi).$$

Alternatively, we can minimize

$$\min \left\|L(H(\phi)) - \left(g_2^\delta - z^\delta\right)\right\|_Y^2, \tag{20}$$

[4] The projector H and its approximation H_ε are defined by

$$H(t) := \begin{cases} 0 & \text{for} \quad t < 0, \\ 1 & \text{for} \quad t \geqslant 0. \end{cases} \qquad H_\varepsilon(t) := \begin{cases} 0 & \text{for} \quad t < -\varepsilon, \\ 1 + t/\varepsilon & \text{for} \quad -\varepsilon \leqslant t \leqslant 0, \\ 1 & \text{for} \quad t \geqslant 0. \end{cases}$$

over $\phi \in H^1(\Gamma_2)$. The Tikhonov regularization for (20) using $TV - H^1$ penalization consists of the minimization of the cost functional

$$\mathcal{F}_\alpha(\phi) := \left\| L(H(\phi)) - \left(g_2^\delta - z^\delta\right)\right\|_Y^2 + \alpha\left[\beta|H(\phi)|_{BV} + \|\phi - \phi_0\|_{H^1}^2\right], \tag{21}$$

where $\alpha > 0$ plays the rule of a regularization parameter and $\beta > 0$ is a scaling factor [14, 24]. Since H is a discontinuous operator, one cannot prove that the Tikhonov functional in (21) attains a minimizer.

In order to guarantee the existence of a minimizer of $\mathcal{F}_\alpha$, we use the concept of generalized minimizers in [14, lemma 2.2]. $\mathcal{F}_\alpha$ is no longer considered as a functional on H^1, but as a functional defined on the w-closure of the graph of H, contained in $BV \times H^1(\Gamma_2)$. A generalized minimizer of $\mathcal{F}_\alpha(\phi)$ is defined as a minimizer of

$$\mathcal{F}_\alpha(\xi, \phi) := \left\| L(H(\phi)) - \left(g_2^\delta - z^\delta\right)\right\|_Y^2 + \alpha\rho(\xi, \phi) \tag{22}$$

on the set of admissible pairs:

$$Ad := \{(\xi, \phi) \in L^\infty(\Gamma_2) \times H^1(\Gamma_2);\ \exists\{\phi_k\} \in H^1 \text{ and } \{\varepsilon_k\} \in \mathbb{R}^+ \text{s.t.}$$
$$\lim_{k\to\infty} \varepsilon_k = 0,\ \lim_{k\to\infty} \|\phi_k - \phi\|_{L^2} = 0,\ \lim_{k\to\infty} \|H_{\varepsilon_k}(\phi_k) - \xi\|_{L^1} = 0\},$$

where $\rho(\xi, \phi) := \inf_{\{\phi_k\},\{\varepsilon_k\}} \liminf_{k\to\infty} \left\{2\beta|H_{\varepsilon_k}(\phi_k)|_{BV} + \|\phi_k - \phi_0\|_{H^1}^2\right\}$.

As a consequence of this definition, the penalization term in (21) can be interpreted as a functional ρ: $Ad \to \mathbb{R}^+$. In order to prove coerciveness and weak lower semi-continuity of ρ, the assumption that L is a continuous operator on a L^1 space is crucial (see proposition 2.2). These properties of ρ are the main arguments needed to prove the existence of a generalized minimizer $(\overline{\xi}_\alpha, \overline{\phi}_\alpha)$ of $\mathcal{F}_\alpha$ in Ad [14, theorem 2.9].

The classical analysis of Tikhonov-type regularization methods [11] do apply to the functional $\mathcal{F}_\alpha$, as we shall see next.

Theorem 4.1 (convergence)**.** *Let α: $\mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying $\lim_{\delta\to 0} \alpha(\delta) = 0$ and $\lim_{\delta\to 0} \delta^2\alpha^{-1}(\delta) = 0$. Given a sequence $\delta_k \to 0$ and $\left\{\left(g_1^{\delta_k}, g_2^{\delta_k}\right)\right\}_k$ corresponding noisy data satisfying (11) for some consistent data pair (g_1, g_2), then the minimizers (ξ_k, ϕ_k) of $\mathcal{F}_{\alpha(\delta_k)}$ converge in $L^1(\Gamma_2) \times L^2(\Gamma_2)$ to a minimizer $(\overline{\xi}_\alpha, \overline{\phi}_\alpha)$ of $\mathcal{F}_\alpha$ in (22).*

Proof. See [14, section 2.3]. □

4.1. Numerical realization

As concerns numerical approximations to the solution of (6), the functional $\mathcal{F}_\alpha$ in (21) has an interesting property. Namely, its generalized minimizers can be approximated by minimizers of the stabilized functional[2]

$$\mathcal{F}_{\alpha,\varepsilon}(\phi) := \|L(H_\varepsilon(\phi)) - (g_2 - z)\|_Y^2 + \alpha\left[\beta|H_\varepsilon(\phi)|_{BV} + \|\phi - \phi_0\|_{H^1}^2\right]. \tag{23}$$

In other words, let $\phi_{\alpha,\varepsilon}$ be a minimizer of $\mathcal{F}_{\alpha,\varepsilon}$; given a sequence $\varepsilon_k \to 0^+$, we can find a subsequence $(H(\phi_{\alpha,\varepsilon}), \phi_{\alpha,\varepsilon})$ converging in $L^1(\Gamma_2) \times L^2(\Gamma_2)$ and the limit minimizes $\mathcal{F}_{\alpha,}$ in Ad.

The existence of minimizers of $\mathcal{F}_{\alpha,\varepsilon}$ in $H^1(\Gamma_2)$ still has to be cleared. Since H_ε is continuous and the operator L is linear, continuous, satisfying (10), the existence of minimizers for $\mathcal{F}_{\alpha,\varepsilon}$ follows directly from [14, lemma 3.1].

This relation between the minimizers of $\mathcal{F}_\alpha$ and $\mathcal{F}_{\alpha,\varepsilon}$ is the starting point for the derivation of a numerical method. We can formally write the optimality condition for $\mathcal{F}_{\alpha,\varepsilon}$ as

$$\alpha(I - \Delta)(\phi - \phi_0) = \mathcal{R}_{\alpha,\varepsilon}(\phi),$$

where

$$\mathcal{R}_{\alpha,\varepsilon}(\varphi) := -H_\varepsilon'(\phi)L'(H_\varepsilon(\phi))^*\big[L(H_\varepsilon(\phi)) - \big(g_2^\delta - z^\delta\big)\big] \\ + \beta\alpha H_\varepsilon'(\phi)\nabla\cdot(\nabla H_\varepsilon(\phi)/|\nabla H_\varepsilon(\phi)|).$$

Identifying $\alpha = 1/\Delta t$, $\phi(0) = \phi_0$, $\phi(\Delta t) = \phi$, we find

$$(I - \Delta)\left(\frac{\phi(\Delta t) - \phi(0)}{\Delta t}\right) = \mathcal{R}_{1/\Delta t,\varepsilon}(\phi(\Delta t)).$$

Considering Δt as a time discretization, we find that (in a formal sense) the iterative regularized solution $\phi(\Delta t)$ is a solution of an implicit time step for the dynamic system

$$(I - \Delta)\left(\frac{\partial\phi(t)}{\partial t}\right) = R_{1/\Delta t,\varepsilon}(\phi(t)). \tag{24}$$

Our second level set method is based on the solution of the dynamic system (24). In an algorithmic form, we have

(1) Choose $\phi_0 \in H^1(\Gamma_2)$ and set $k = 0$; compute $z^\delta = u_\nu|_{\Gamma_1}$, where

$$Pu = f \quad \text{in} \quad \Omega, \qquad u|_{\Gamma_1} = g_1^\delta, \qquad u_\nu|_{\Gamma_2} = 0.$$

(2) Evaluate the residual $r_k := L(H_\varepsilon(\phi_k)) - \big(g_2^\delta - z^\delta\big)$. Note that $L(H_\varepsilon(\phi_k)) = (u_k)_\nu|_{\Gamma_1}$, where

$$Pu_k = 0 \quad \text{in} \quad \Omega, \qquad u_k|_{\Gamma_1} = 0, \qquad (u_k)_\nu|_{\Gamma_2} = H_\varepsilon(\phi_k).$$

(3) Evaluate $V_k := L'(H_\varepsilon(\phi_k))^*(r_k)$. Note that $(L')^*(r_k) = -v_k|_{\Gamma_2}$, where

$$Pv_k = 0 \quad \text{in} \quad \Omega, \qquad v_k|_{\Gamma_1} = r_k, \qquad (v_k)_\nu|_{\Gamma_2} = 0.$$

(4) Evaluate the velocity w_k by solving

$$(I - \Delta)w_k = H_\varepsilon'(\phi_k)\left[-v_k + \beta\nabla\cdot\left(\frac{\nabla H_\varepsilon(\phi_k)}{|\nabla H_\varepsilon(\phi_k)|}\right)\right] \text{in } \Gamma_2, \qquad (w_k)_\nu|_{\partial\Gamma_2} = 0.$$

(5) Update the level set function $\phi_{k+1} = \phi_k + \frac{1}{\alpha}w_k$.

Note that steps 1 and 2 above involve the solution of a mixed BVP in $\Omega \subset \mathbb{R}^d$. On the third step, the computation of the velocity function for the level set method requires a solution of a Neumann BVP at Γ_2.

In the following section we present some numerical experiments, which were implemented using the above algorithm.

5. Numerical experiments

In what follows, we present three numerical experiments for the level set method analyzed in section 4. In subsection 5.1 we consider a problem with exact Cauchy data, where the solution is the characteristic function of a non-connected set. In subsection 5.2, we investigate how the degree of ill-posedness of an elliptic Cauchy problem affects the performance of the level set method. In subsection 5.3, we consider a problem with noisy Cauchy data and test the stability of our method.

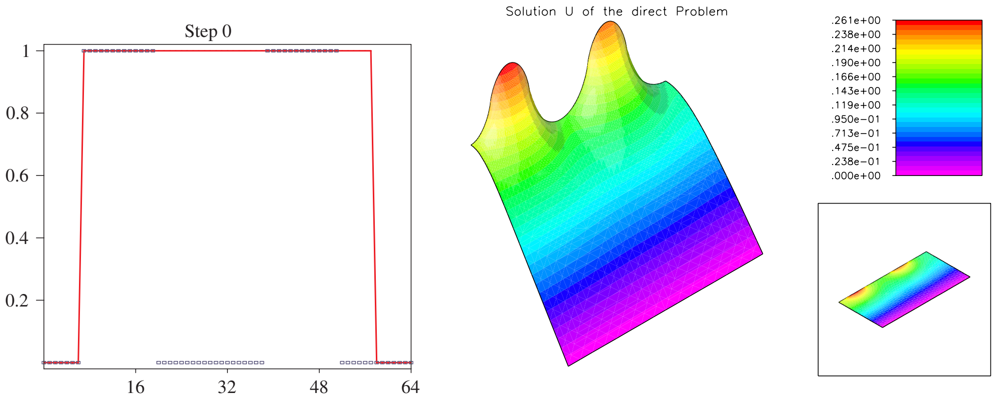

Figure 1. Framework for the first numerical experiment. On the left-hand side are the exact solution $\overline{\varphi}$ of the Cauchy problem (dotted blue line) and the initial guess for the level set method (solid red line). The solution u of the elliptic BVP corresponding to (25) is depicted on the right-hand side. Note that $u_\nu = g_2$ and $u = 0$ at Γ_1 (lower edge). Moreover, $u_\nu = \overline{\varphi}$ at Γ_2 (top edge).

5.1. A Cauchy problem with a non-connected solution

One of the main advantages of the level set approach is the fact that no *a priori* assumption on the topology of the solution set is needed [26]. In this first numerical experiment, we exploit this feature of the method by solving a Cauchy problem with a non-connected solution. Let $\Omega = (0,1) \times (0,0.5)$ with $\Gamma_1 := (0,1) \times \{0\}$, $\Gamma_2 := (0,1) \times \{0.5\}$ and $\Gamma_3 := \partial\Omega / \{\Gamma_1 \cup \Gamma_2\}$. Consider the Cauchy problem

$$\Delta u = 0 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{at} \quad \Gamma_1, \qquad u_\nu = g_2 \quad \text{at} \quad \Gamma_1, \qquad u_\nu = 0 \quad \text{at} \quad \Gamma_3. \tag{25}$$

The Cauchy data $(0, g_2)$ are chosen in such a way that the solution $\overline{\varphi}$ of (25) is the indicator function of a non-connected subset of Γ_2 (see figure 1).

In figure 2 we show the level set iterations for the Cauchy problem (25) and in figure 3 the corresponding evolution of the level set function. Note the large number of steps required to obtain a precise approximation. As already observed when applying level set methods to other models [14, 26], the splitting of the level sets happens only after a large number of iterations (over 10 000 in this experiment). Nevertheless, the *a priori* required precision could always be reached (in this experiment, $\|\varphi_k - \overline{\varphi}\|_{L^2(\Gamma_2)} < 10^{-2}$ was reached after 15 000 steps). It is worth noting that, in our experiments, the total number of steps needed to reach a pre-specified precision does not depend on the initial guess for the level set method.

5.2. Degree of ill-posedness affecting convergence rates

In this experiment, we analyze the effect of the ill-posedness degree of the Cauchy problem on the performance of the level set method introduced in section 4. For this purpose, we introduce the domains $\Omega_1 := (0,1) \times (0,1)$ and $\Omega_2 := (0,1) \times (0,0.5)$. The boundary part $\Gamma_1 := (0,1) \times \{0\}$ is the same for both domains. Moreover, we define $\Gamma_{2,1} := (0,1) \times \{1\}$, $\Gamma_{2,2} := (0,1) \times \{0.5\}$ and $\Gamma_{3,i} := \partial\Omega_i / \{\Gamma_1 \cup \Gamma_{2,i}\}$, $i = 1, 2$.

We compare the performance of our method for the Cauchy problems $(\mathrm{CP})_i$, $i = 1, 2$, defined by

$$\Delta u_i = 0 \quad \text{in} \quad \Omega_i, \qquad u_i = 0 \quad \text{at} \quad \Gamma_1, \qquad (u_i)_\nu = g_{2,i} \quad \text{at} \quad \Gamma_1, \qquad (u_i)_\nu = 0 \quad \text{at} \quad \Gamma_3.$$

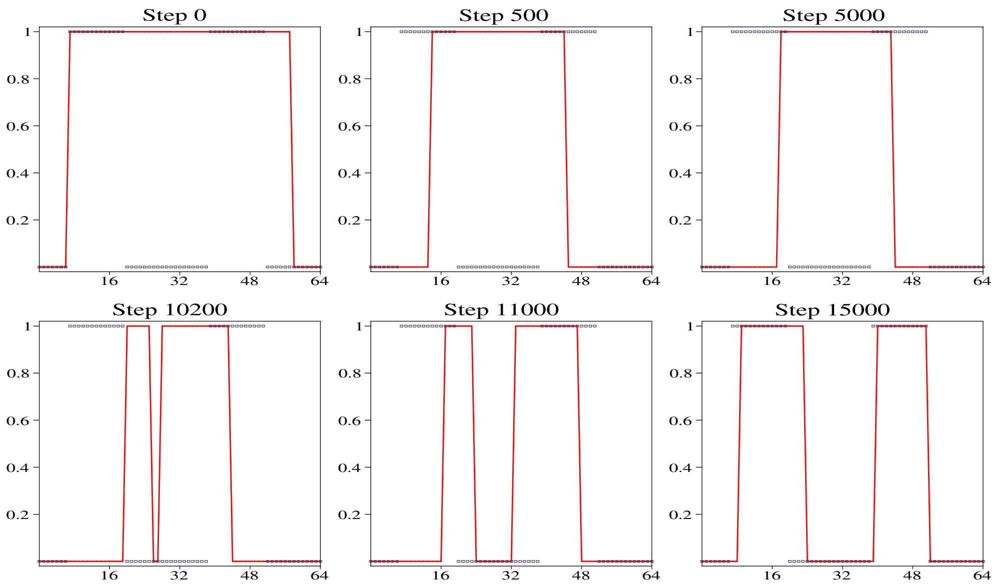

Figure 2. First numerical experiment: on the top left is the initial guess for the level set function (solid red line). The other pictures show the evolution of the level set method (solid red line) after 500, 5000, 10 200, 11 000 and 15 000 iterative steps. The dotted blue line represents the exact solution.

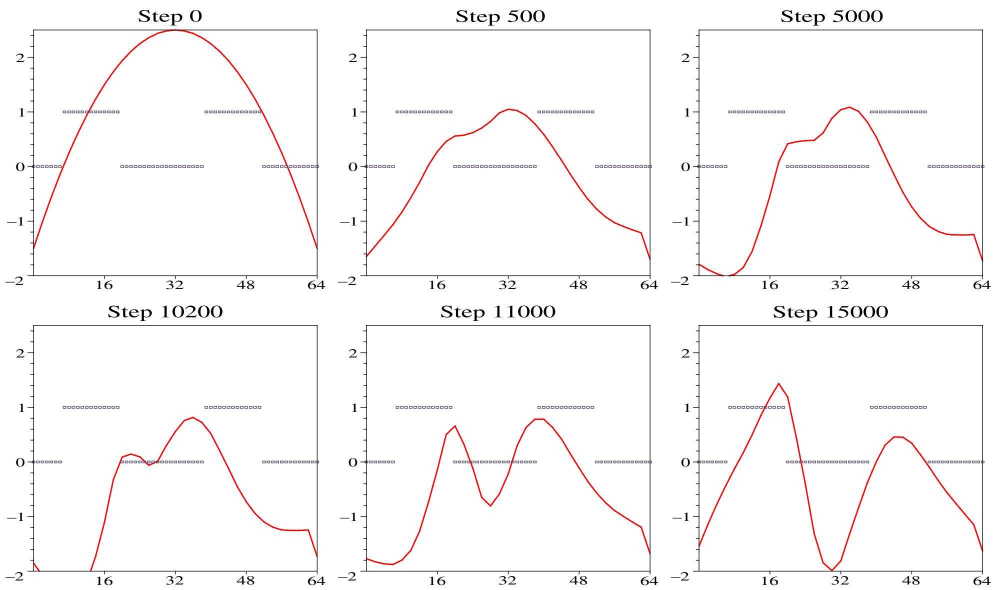

Figure 3. First numerical experiment: on the top left is the initial guess for the level set function (solid red line). The other pictures show the evolution of the level set function (solid red line) after 500, 5000, 10 200, 11 000 and 15 000 iterative steps. In all pictures, the exact solution $\overline{\varphi}$ of the Cauchy problem is represented by the dotted blue line.

The Cauchy data $(0, g_{2,i})$ are chosen such that problems $(\mathrm{CP})_i$ have the same solution, i.e. $(u_1)_\nu|_{\Gamma_{2,1}} = (u_2)_\nu|_{\Gamma_{2,2}}$ (see figures 4 and 5).

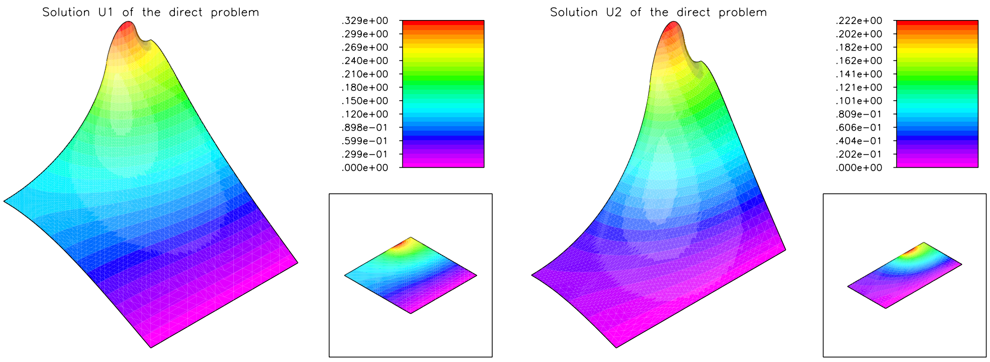

Figure 4. Framework for the second numerical experiment: the solutions u_i of the elliptic BVPs corresponding to $(\mathrm{CP})_i$, $i = 1, 2$, are depicted on the left- and on the right-hand sides respectively. The Cauchy data $g_{2,i}$ correspond to $(u_i)_\nu$ at Γ_1 (lower edge). Also note that the Dirichlet boundary condition $u_1 = u_2 = 0$ at Γ_1 holds. Both Cauchy problems share the same solution $(u_1)_\nu|_{\Gamma_{2,1}} = \overline{\varphi} = (u_2)_\nu|_{\Gamma_{2,2}}$ (top edge).

It is known from the literature that the ill-posedness degree of the CP increases with the *distance* between the boundary parts Γ_1 and Γ_2 [23]. Since we are considering simple domains Ω_i, it is possible to compute the eigenvalues $\{\lambda_{i,j}\}_j$ of the operators L_i in (5) corresponding to $(\mathrm{CP})_i$:

$$\lambda_{i,j} = \sinh(j/i)^{-1}, \qquad j = 1, 2, \ldots, \qquad i = 1, 2.$$

This gives us a measure how ill-posed $(\mathrm{CP})_1$ is when compared with $(\mathrm{CP})_2$. In the following, we compare the performance of the level set method for both problems (exact Cauchy data are used). In figures 5 and 6, we see the level set iterations for problems $(\mathrm{CP})_1$ and $(\mathrm{CP})_2$ respectively. Due to the difference between the ill-posedness degree of both problems, the method converges faster for problem $(\mathrm{CP})_2$. Nevertheless, the same accuracy can be reached for both problems if one iterates long enough. We conclude that the degree of ill-posedness only affects the amount of computational effort needed to obtain an approximate solution with *a priori* defined precision, and not the quality of the final approximation. Our experiments indicate that the number of iteration steps needed by the level set method (to reach a desired accuracy) increases exponentially with the distance between Γ_1 and Γ_2, the same way the degree of ill-posedness also does.

5.3. An experiment with noisy Cauchy data

For the next numerical experiment, we consider once more problem $(\mathrm{CP})_2$ in subsection 5.2. This time, however, we perturbed the Cauchy data $g_{2,2}$ with 10% random noise (figure 7). The initial guess for the level set method is the same as used in the second experiment (see figure 6).

The performance of the level set method for exact and corrupted data can be compared in figures 6 and 7. Note that for the noisy data, the best possible approximation is obtained after 300 steps. We iterated further (until 500 steps) and observed that, although the level set function oscillates, the corresponding level sets remain almost unchanged.

Since we assume the Cauchy data to satisfy $g_1 = 0$ in our experiments, it seems natural not to introduce noise in this component of the data. We conjecture that this is the main reason for obtaining stable reconstruction results for this exponentially ill-posed problem in

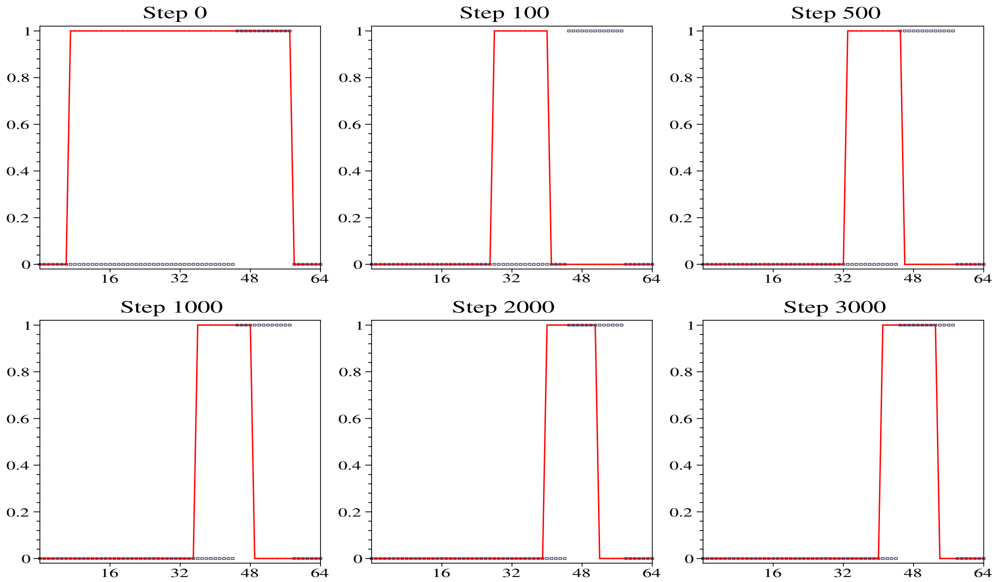

Figure 5. Second numerical experiment: on the top left are the exact solution $\overline{\varphi}$ of the Cauchy problems (dotted blue line) and the initial guess for the level set method (solid red line). The other pictures show the evolution of the level set method for problem $(\mathrm{CP})_1$ after 100, 500, 1000, 2000 and 3000 iterative steps. The solid (red) line represents the iteration and the dotted (blue) line the exact solution.

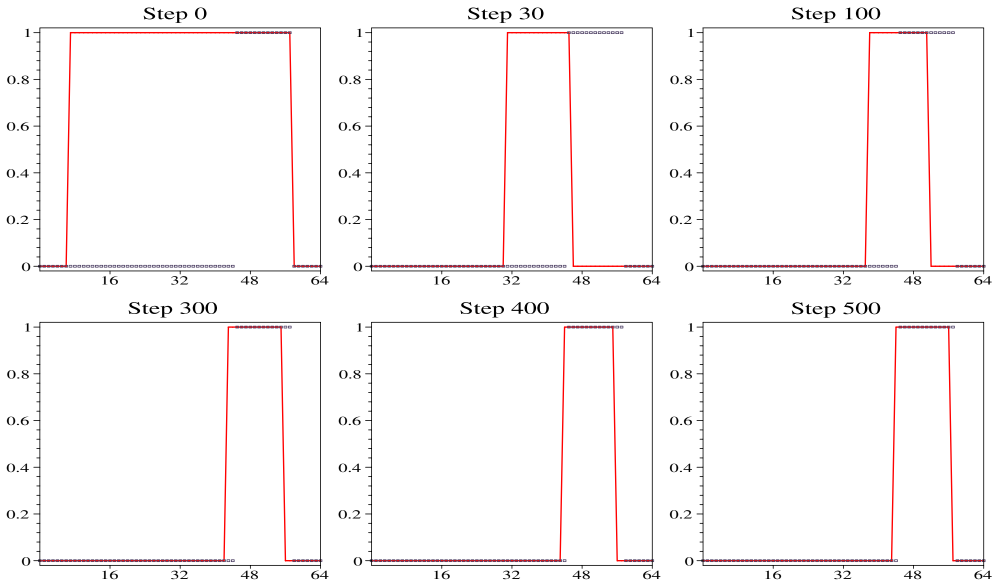

Figure 6. Second numerical experiment: evolution of the level set method for problem $(\mathrm{CP})_2$. Plots of the initial guess and after 30, 100, 300, 400 and 500 iterative steps. The solid (red) line represents the iteration and the dotted (blue) line the exact solution.

the presence of high levels of noise (in [12] only 5% noise is used; moreover it is not white noise but corresponds to an eigenfunction of L).

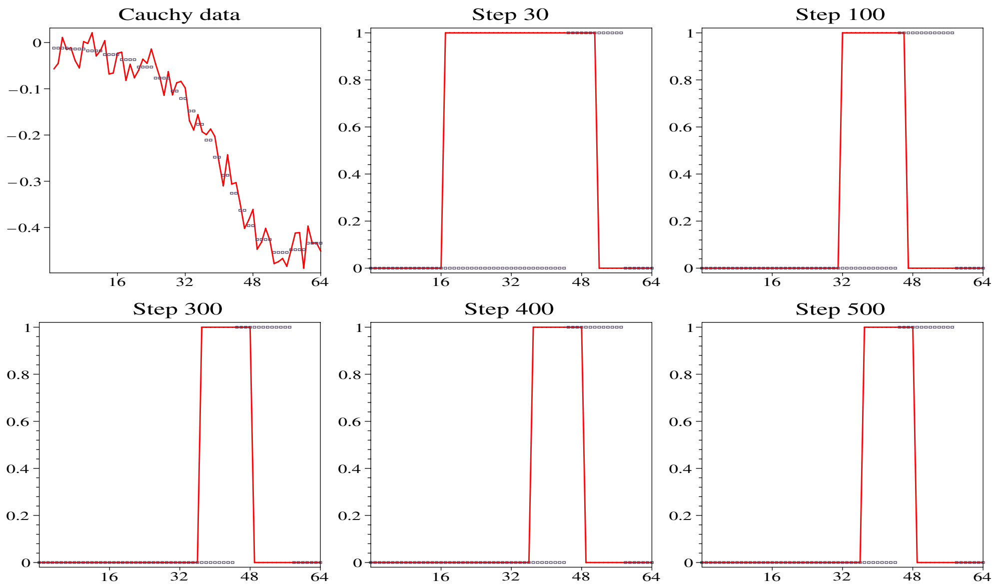

Figure 7. Third numerical experiment: on the top left are the exact Cauchy data $g_{2,2}$ (dotted blue line) and the data corrupted with 10% random noise (solid red line). The other pictures show the evolution of the level set method for problem $(\mathrm{CP})_2$ with noisy data after 30, 200, 300, 400 and 500 iterative steps. The solid (red) line represents the iteration and the dotted (blue) line the exact solution.

6. Conclusions

In this paper, two possible level set approaches for solving elliptic Cauchy problems are considered. For each of them, a corresponding framework is established and a convergence analysis is provided (monotony, convergence, stability results). Further, we discuss the numerical realization of the second level set approach. Three different numerical experiments illustrate relevant features of this level set method: rates of convergence, adaptability to identify non-connected inclusions, and robustness with respect to noise.

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