

# Nonlinear regularization methods for ill-posed problems with piecewise constant or strongly varying solutions

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## Abstract

In this paper we consider nonlinear ill-posed problems with piecewise constant or strongly varying solutions. A class of nonlinear regularization methods is proposed, in which smooth approximations to the Heavyside function are used to reparameterize functions in the solution space by an auxiliary function of levelset type. The analysis of the resulting regularization methods is carried out in two steps: first, we interpret the algorithms as nonlinear regularization methods for recovering the auxiliary function. This allows us to apply standard results from regularization theory, and we prove convergence of regularized approximations for the auxiliary function; additionally, we obtain the convergence of the regularized solutions, which are obtained from the auxiliary function by the nonlinear transformation. Second, we analyze the proposed methods as approximations to the levelset regularization method analyzed in [Frühau F, Scherzer O and Leitão A 2005 Analysis of regularization methods for the solution of ill-posed problems involving discontinuous operators *SIAM J. Numer. Anal.* **43** 767–86], which follows as a limit case when the smooth functions used for the nonlinear transformations converge to the Heavyside function. For illustration, we consider the application of the proposed algorithms to elliptic Cauchy problems, which are known to be severely ill-posed, and typically allow only for limited reconstructions. Our numerical examples demonstrate that the proposed methods provide accurate reconstructions of piecewise constant solutions also for these severely ill-posed benchmark problems.

(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

We consider the solution of inverse ill-posed problems

$$F(x) = y^\delta, \quad (1)$$

where  $F : D(F) \subset \mathcal{X} \rightarrow \mathcal{Y}$  is a linear or nonlinear operator between Hilbert spaces  $\mathcal{X}$  and  $\mathcal{Y}$ , and the available data  $y^\delta$  are some approximation of the correct data  $y = F(x^\dagger)$  corresponding to the true solution  $x^\dagger$ . It is well known that ill-posed problems can be solved in a stable way only by regularization methods [14, 29], and that the quality of the regularized solutions depends not only on the quality of the data, e.g. a bound on the data noise  $\|y - y^\delta\| \leq \delta$ , but in particular also on the incorporation of available *a priori* information in the reconstruction methods. This is reflected in the dependence of reconstruction errors and convergence rates on so-called source conditions [14].

In this paper we propose nonlinear regularization methods for problem (1) that allow us to incorporate *a priori* information of the following form.

(A) The unknown solution  $x$  of (1) has a special structure, namely it can only attain certain values (piecewise constant) or can be assumed to have steep gradients between regions of almost constant value (strongly varying).

It is worth mentioning that assumption (A) is valid in several relevant applications such as mine detection [16], inverse scattering [11], reconstruction of doping profiles in semiconductors [23] or process monitoring via impedance tomography [20].

Standard regularization methods, such as Tikhonov regularization, are not appropriate for the reconstruction of solutions satisfying assumption (A), as they generate good approximations only for smooth solutions; this is reflected in the dependence of convergence rates on source conditions. For severely ill-posed problems, only very mild (logarithmic) source conditions are physically reasonable and, therefore, only poor reconstructions of piecewise constant or strongly varying solutions can be expected.

Motivated by the unsatisfactory performance of classical regularization methods, special nonlinear regularization methods, such as BV-regularization [1, 27] or levelset methods [7, 17, 24, 26, 28], have been designed for problems with non-smooth solutions satisfying assumption (A). The nonlinear regularization method outlined in the following falls into this group of methods.

In order to facilitate the stable reconstruction of solutions satisfying (A), we consider a parameterization of the unknown function  $x$  in the form

$$x = H_\varepsilon(\phi), \quad (2)$$

where the real function  $H_\varepsilon$  denotes a smooth, nonlinear, strictly monotonically increasing function (see section 2). For simplicity of presentation, let us assume that  $x$  is a piecewise constant function with values in  $\{0, 1\}$ , in which case we choose  $H_\varepsilon$  to be a smooth approximation of the Heaviside function  $H$ . If  $H_\varepsilon$  is strictly monotonically increasing, the transformation  $x = H_\varepsilon(\phi)$  establishes a one-to-one relation between the auxiliary function  $\phi$  and the solution  $x$ . The function  $\phi$  acts as a kind of levelset function, i.e.  $x$  attains values close to zero where  $\phi$  is negative, and values close to 1 where  $\phi$  is positive; the zero levelset of  $\phi$  is the region where the transition from zero to 1 occurs (in  $x$ ).

By the nonlinear transformation (2), the inverse problem of determining  $x$  under assumption (A) is transformed into the problem of finding an auxiliary function  $\phi$  solving

$$G(\phi) := F(H_\varepsilon(\phi)) = y^\delta \quad (3)$$

(note that, due to the choice of the function  $H_\varepsilon$ , problem (3) becomes nonlinear, even if the original inverse problem (1) is linear). For the stable solution of (3), we consider Tikhonov regularization, i.e. we define approximate solutions as minimizers of a regularized functional

$$\|G_\varepsilon(\phi) - y^\delta\|^2 + \alpha \|\phi - \phi^*\|^2 \quad (4)$$

for some  $\alpha > 0$  and some *a priori* guess  $\phi^*$ . The choice of the norm for the regularization term depends on the problem setting, e.g. on the mapping properties of the operator  $F$ ; see sections 2 and 3 for details. Since  $H_\varepsilon$  is monotonically increasing, minimizing (4) over some set  $D$  is equivalent to minimizing

$$\|F(x) - y^\delta\|^2 + \alpha \|H_\varepsilon^{-1}(x) - H_\varepsilon^{-1}(x^*)\|^2 \quad (5)$$

over  $H_\varepsilon^{-1}(D)$ , which amounts to Tikhonov regularization applied to the original problem (1) with a nonlinear regularization term. This is the reason why the resulting methods are called *nonlinear regularization methods*. While functional (5) may look unusual, regularization theory for (4) is straightforward (see section 2).

One of the main goals of this paper is to investigate the special case that  $H_\varepsilon$  approximates a step function  $H$ . In this case, the nonlinear regularization methods (4) or (5) can be interpreted as approximations to a levelset method investigated in [17]. In the limiting case, the auxiliary function  $\phi$  is a levelset function in the sense that

$$x = H(\phi) = \begin{cases} 1, & \text{for } \phi \geq 0, \\ 0, & \text{else.} \end{cases}$$

The approximations obtained by using the smooth parameterization by  $H_\varepsilon$  can be understood as a relaxation of the levelset method using the discontinuous function  $H$  for the transformation. As a matter of fact, similar relaxations are frequently used for the implementation of levelset methods, e.g. in the minimization of Mumford–Shah-like functionals in image processing [9, 10, 25].

In order to illustrate the benefits of our approach, we consider a benchmark example for severely ill-posed problems, namely the solution of elliptic Cauchy problems, which arise in several biomedical, engineering and industrial applications (e.g. the inverse problem in corrosion detection in [3, 8, 19]). Due to the severe ill-posedness of these test problems, the reconstruction of non-smooth solutions, e.g. the determination of contact and non-contact zones, or the localization of regions with or without activity, is particularly difficult. As our numerical test results demonstrate, the nonlinear regularization approach investigated in this manuscript significantly improves the quality of reconstructions in comparison to standard regularization methods.

The paper is organized as follows: in section 2, we introduce the parameterization by smooth functions  $H_\varepsilon$ , and we analyze the resulting nonlinear inverse problems (3) of determining the (levelset) function  $\phi$ , and their stable solution by Tikhonov regularization. Section 3 then deals with the special case that  $H_\varepsilon$  approximates a step function  $H$ , in which case the resulting methods can be analyzed within the framework of levelset methods presented in [17]. In section 4, we state and discuss our model problems in detail, and verify the conditions needed for our analysis. Numerical tests are then presented in section 5, and some conclusions are given in the final section.

## 2. Nonlinear regularization for inverse problems with strongly varying solutions

In this section we transform the original inverse problem (1) into a nonlinear problem (3) for determining the auxiliary function  $\phi$  which parameterizes the solution  $x = H_\varepsilon(\phi)$  of

(1). Before we formulate and analyze this approach in detail, let us summarize some basic assumptions on the original inverse problem.

### 2.1. Basic assumptions and parameterization

Let  $F : \mathcal{D}(F) \subset \mathcal{X} \rightarrow \mathcal{Y}$  be a continuous, compact operator between real Hilbert spaces  $\mathcal{X}$  and  $\mathcal{Y}$ . The space  $\mathcal{X}$  will be chosen in order to reflect the properties of the solution, so for the reconstruction of solutions with steep gradients, which we are most interested in this section, we consider the choice  $\mathcal{X} = H^1$  over some domain in  $\mathbb{R}^d$ . The results however carry over easily to  $\mathcal{X} = L^2$ , which will also be considered in the numerical examples in section 5. We further assume that a solution  $x^\dagger$  of the inverse problem with unperturbed data  $y$  exists, i.e.  $F(x^\dagger) = y$ , and that the solution has a certain structure. For illustration, we consider that

$$x^\dagger \in K := \{x \in \mathcal{X} : \underline{x} \leq x \leq \bar{x}\} \subset \mathcal{D}(F). \quad (6)$$

The perturbed data  $y^\delta$  in (1) are assumed to satisfy a bound

$$\|y - y^\delta\| \leq \delta, \quad (7)$$

for some noise level  $\delta \geq 0$ . When considering convergence rates results below, we will further require that  $F$  is Fréchet differentiable and the derivative satisfies a Lipschitz condition

$$\|F'(x_2) - F'(x_1)\| \leq L_F \|x_2 - x_1\|, \quad (8)$$

for some  $L_F > 0$  and all  $x_1, x_2 \in \mathcal{D}(F)$ .

In order to facilitate steep gradients in the solution  $x$  of (1), we parameterize the function in the form  $x = H_\varepsilon(\phi)$ . For this purpose, let  $H_\varepsilon$  be a smooth, strictly monotonically increasing function satisfying the following conditions:

$$(i) \quad H_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}, \quad (ii) \quad 0 < H'_\varepsilon(\cdot) \leq C'_\varepsilon, \quad (iii) \quad |H''_\varepsilon(\cdot)| \leq C''_\varepsilon, \quad (9)$$

for some positive constants  $C'_\varepsilon, C''_\varepsilon$ . Moreover, we assume that  $K \subset H_\varepsilon(\mathcal{X}) \subset \mathcal{D}(F)$ .

**Example 2.1.** Assume that  $\underline{x} = 0$  and  $\bar{x} = 1$  in the definition of  $K$ , and that  $\mathcal{D}(F)$  contains  $H^1$  functions with values in  $[-\varepsilon, 1 + \varepsilon]$ . Then the function

$$H_\varepsilon(x) = \frac{1 + 2\varepsilon}{2} (\operatorname{erf}(x/\varepsilon) + 1) - \varepsilon \quad (10)$$

satisfies conditions (i)–(iii) with  $C'_\varepsilon \sim \varepsilon^{-1}$  and  $C''_\varepsilon \sim \varepsilon^{-2}$ . Moreover,  $H'_\varepsilon$  is bounded away from zero uniformly on  $H_\varepsilon^{-1}([\underline{x}, \bar{x}])$ .

Note that, in the above example, for any  $x^\dagger \in K$  there exists a unique  $H_\varepsilon^{-1}(x^\dagger) =: \phi^\dagger \in \mathcal{X}$ . Using such a parameterization, we can rewrite the inverse problem (1) for  $x$  as a nonlinear inverse problem for the levelset function  $\phi$ , namely

Nonlinear problem: define  $G_\varepsilon(\phi) := F(H_\varepsilon(\phi))$ . Find  $\phi \in \mathcal{X}$  such that

$$G_\varepsilon(\phi) = y^\delta. \quad (11)$$

In what follows, we assume that a solution  $\phi^\dagger = H_\varepsilon^{-1}(x^\dagger)$  for unperturbed data exists in  $\mathcal{X}$ .

**Remark 2.2.** In case  $H'_\varepsilon(x)$  is bounded from below by some positive constant for the values of  $x$  that are attained by  $x^\dagger$ , the existence of  $\phi^\dagger$  is already implied by the attainability of the data required above. Therefore, the assumption that  $G_\varepsilon(\phi) = y$  has a solution is not restrictive. Note also that, under the given assumption on  $H_\varepsilon$  and  $\mathcal{D}(F)$ , we have  $\mathcal{D}(G_\varepsilon) = \mathcal{X}$ .

The nonlinear transformation also allows us to include box constraints on the solution into the formulation of the operator.

Let us shortly summarize the main properties of the nonlinear operator  $G_\varepsilon$ .

**Proposition 2.3.** *The operator  $G_\varepsilon : \mathcal{X} \rightarrow \mathcal{Y}$  defined by  $G_\varepsilon(\phi) = F(H_\varepsilon(\phi))$  is a compact continuous operator. If  $F$  is injective or Fréchet differentiable, then  $G_\varepsilon$  inherits these properties as well. Moreover, if (8) holds, then*

$$\|G'_\varepsilon(\phi_2) - G'_\varepsilon(\phi_1)\|_{\mathcal{X} \rightarrow \mathcal{Y}} \leq (L_F C_\varepsilon'^2 + C_\varepsilon'' \|F'(H_\varepsilon(\phi_2))\|_{\mathcal{X} \rightarrow \mathcal{Y}}) \|\phi_2 - \phi_1\|_{\mathcal{X}}; \quad (12)$$

thus  $G_\varepsilon$  is Lipschitz continuous with constant  $L_G := L_F C_\varepsilon'^2 + C_\varepsilon'' \sup_x \|F'(x)\|$ .

**Proof.** Definition  $F$  is the composition of a compact and a continuous operator, and thus compact. The Fréchet differentiability and the Lipschitz estimate on the derivative follow by applying the chain rule and the properties of  $H_\varepsilon$ . The injectivity is inherited, since  $H_\varepsilon$  is strictly monotonically increasing and hence injective.  $\square$

To summarize, under the given assumptions on  $H_\varepsilon$ , the nonlinear operator  $G_\varepsilon = F \circ H_\varepsilon$  inherits all properties of  $F$  that are relevant for the analysis of Tikhonov regularization methods [14].

## 2.2. Regularization

For the stable solution of (11) we consider Tikhonov regularization, i.e. approximate solutions  $\phi_\alpha^\delta$  are defined as minimizers of the functional

$$\mathcal{J}_{\alpha,\varepsilon}(\phi) := \frac{1}{2} \|G_\varepsilon(\phi) - y^\delta\|^2 + \frac{\alpha}{2} \|\phi - \phi^*\|^2, \quad (13)$$

where  $\alpha > 0$  is the regularization parameter and  $\phi^*$  is a reference function (e.g.  $\phi^* = H_\varepsilon^{-1}(x^*)$ , where  $x^*$  is an *a priori* guess for the solution  $x^\dagger$ ). The existence of minimizers, as well as convergence for vanishing data noise, now follow by standard arguments [14].

**Theorem 2.4.** *For  $\alpha > 0$ , functional (13) attains a minimizer  $\phi_\alpha^\delta \in \mathcal{X}$ .*

**Proof.** The operator  $G_\varepsilon$  is continuous and compact; thus, it is weakly continuous. Consequently, the functional  $\mathcal{J}_\alpha$  is weakly lower semi-continuous, coercive and bounded from below, which guarantees the existence of a minimizer.  $\square$

In order to guarantee the convergence of the regularized solutions  $\phi_\alpha^\delta$  with  $\delta \rightarrow 0$ , one has to provide an appropriate strategy for choosing the regularization parameter  $\alpha$  in dependence of the noise level. To simplify the statement of the following theorem, we assume that  $F$  is injective, and thus the solution of the inverse problem is unique (see [14, Chapter 10] for the general case).

**Theorem 2.5.** *Let  $F$  be injective, let  $x^\dagger$  denote the solution of  $F(x) = y$  and let  $\phi^\dagger = H_\varepsilon^{-1}(x^\dagger) \in \mathcal{X}$ . If  $\{y^{\delta_n}\}$  denotes a sequence of perturbed data satisfying  $\|y - y^{\delta_n}\| \leq \delta_n \rightarrow 0$ , and if  $\alpha_n$  is chosen such that  $\alpha_n \rightarrow 0$  and  $\delta_n^2/\alpha_n \rightarrow 0$ , then the regularized solutions  $\phi_{\alpha_n}^\delta$  converge to the true solution, i.e.*

$$\|\phi_{\alpha_n}^{\delta_n} - \phi^\dagger\| \rightarrow 0 \quad \text{and} \quad \|H_\varepsilon(\phi_{\alpha_n}^{\delta_n}) - x^\dagger\| \rightarrow 0.$$

**Proof.** Standard regularization theory for nonlinear inverse problems [14] guarantees the convergence of subsequences to a minimum norm solution. Since the solution of (11) is unique, all subsequences have the same limit.  $\square$

Theorem 2.5 is a qualitative statement and does not provide any quantitative information about the errors  $\|\phi_\alpha^\delta - \phi^\dagger\|$  for some given  $\delta$ . In fact, the convergence can be arbitrarily slow

in general [14]. In order to guarantee a rate of convergence, a source condition has to be satisfied, e.g. let  $x^\dagger = H_\varepsilon(\phi^\dagger)$  and assume that  $\phi^\dagger$  satisfies

$$\phi^\dagger = \phi^* + G'_\varepsilon(\phi^\dagger)^* w \quad \text{for some } w \in \mathcal{Y}. \quad (14)$$

Then the following quantitative result holds.

**Theorem 2.6.** *Let the assumptions of theorem 2.5 hold. Moreover, assume that  $F$  is Fréchet differentiable with Lipschitz continuous derivative (8) and that  $\phi^\dagger$  satisfies the source condition (14) for some  $w$  with norm  $\|w\| < 1/L_G$  where  $L_G$  is given in proposition 2.3. Then for the parameter choice  $\alpha \sim \delta$  there holds*

$$\|\phi_{\alpha_n}^{\delta_n} - \phi^\dagger\| = O(\sqrt{\delta_n}), \quad \|H_\varepsilon(\phi_{\alpha_n}^{\delta_n}) - x^\dagger\| = O(\sqrt{\delta_n}).$$

**Proof.** The convergence rate for  $\phi_\alpha^\delta$  follows from standard regularization theory [14] and proposition 2.3. The result for  $x_\alpha^\delta = H_\varepsilon(\phi_\alpha^\delta)$  then follows from the fact that  $H'_\varepsilon$  is bounded.  $\square$

**Remark 2.7.** Let us consider the source condition (14) in more detail. If  $\mathcal{X} = L^2$ , then by the chain rule, the source condition can be rewritten as

$$\phi^\dagger - \phi^* = H'_\varepsilon(\phi^\dagger)F'^*(x^\dagger)w, \quad \text{for some } w \in \mathcal{Y}.$$

Since  $H_\varepsilon$  is invertible, this condition can always be interpreted as a condition on  $x^\dagger$ , namely

$$x^\dagger = H_\varepsilon(\phi^\dagger) = H_\varepsilon(H'_\varepsilon[H_\varepsilon^{-1}(x^\dagger)]F'(x^\dagger)^*w)$$

(for simplicity we have assumed  $\phi^* = 0$ ). Note that the source condition depends nonlinearly on the solution  $x^\dagger$ , even if  $F$  is linear. If  $H_\varepsilon$  is a simple scaling, i.e.  $H_\varepsilon(\phi) = \varepsilon^{-1}\phi$ , we obtain with  $H'_\varepsilon = \varepsilon^{-1}$  that

$$x^\dagger = \varepsilon^{-1}\phi^\dagger = \varepsilon^{-1}\varepsilon^{-1}F'(x^\dagger)^*w = F'(x^\dagger)^*\varepsilon^{-2}w = F'(x^\dagger)\tilde{w},$$

which amounts to the standard source condition for the inverse problem (1).

Throughout this section, we considered parameterization described by smooth functions  $H_\varepsilon$  for approximating strongly varying solutions of the inverse problem (1). For the approximation of piecewise continuous functions, it might be advantageous to use a parameterization by a non-smooth function. The analytical results discussed in this section, however, no longer apply in that case, and we have to adopt a different analysis technique.

### 3. Nonlinear regularization for inverse problems with piecewise constant solutions

In this section we consider solving the inverse problem (1) under the assumption that the solution  $x^\dagger$  is piecewise constant and binary valued. We concentrate on the case that  $x^\dagger$  can be represented as the characteristic function of a sufficiently regular set. Nevertheless, possible extensions are indicated at the end of this section.

#### 3.1. Basic assumptions

Let  $\Omega \subset \mathbb{R}^d$  be a bounded domain with Lipschitz boundary, and assume that  $x^\dagger$  can be represented as the characteristic function of a sufficiently regular set, i.e.

$$x^\dagger \in K := \{x : x = \chi_D \text{ where } D \subset \Omega \text{ is measurable and } \mathcal{H}^{d-1}(\partial D) < \infty\},$$

where  $\mathcal{H}^{d-1}(\partial D)$  denotes the  $(d-1)$ -dimensional Hausdorff measure of the boundary  $\partial D$ . It can be shown that the signed distance function of  $\partial D$  is in  $H^1(\Omega)$ , which implies that there exists a levelset function  $\phi^\dagger \in H^1(\Omega)$  such that

$$x^\dagger = H(\phi^\dagger), \quad (15)$$

where  $H : \mathbb{R} \rightarrow \{0, 1\}$  denotes the Heaviside function. Note that  $H$  is the pointwise limit of  $H_\varepsilon$  defined in example 2.1, as  $\varepsilon \rightarrow 0$ ; so (15) can be understood as the limit case of (2) (see section 3.4 below). Since  $H$  is a discontinuous function, the analysis of section 2 cannot be applied directly.

For the remainder of this section we further assume that  $\{x \in L^\infty(\Gamma) : -\varepsilon \leq x \leq 1 + \varepsilon\} \subset \mathcal{D}(F)$  for some  $\varepsilon > 0$ , and that  $F$  is a continuous operator with respect to the  $L^p$ -topology for some  $1 \leq p < d/(d-1)$ , i.e.

$$\|F(x_2) - F(x_1)\|_Y \rightarrow 0, \quad \text{as } \|x_2 - x_1\|_{L^p} \rightarrow 0.$$

Our goal in this section is to derive a nonlinear regularization method based on the discontinuous parameterization (15), in a similar way as we did in section 2 using (2). In order to make the connection with the results of the previous section, we will utilize the approximation of the discontinuous Heaviside function  $H$  by the smooth strictly increasing functions  $H_\varepsilon$  (10).

The fact that  $H_\varepsilon$  can attain values in a larger interval  $[-\varepsilon, 1 + \varepsilon]$  ensures that the true solution  $x^\dagger$ , which is assumed to be binary valued, can in fact be parameterized by a levelset function. However, other choices of approximations are possible, e.g. in [17], piecewise linear continuous (but not continuously differentiable) approximations have been used.

### 3.2. A Tikhonov method with BV- $H^1$ regularization

For defining regularized solutions, we consider the following Tikhonov-type functional:

$$\mathcal{F}_\alpha(\phi) := \frac{1}{2} \|F(H(\phi)) - y^\delta\|_Y^2 + \alpha [\beta |H(\phi)|_{BV} + \frac{1}{2} \|\phi - \phi^*\|_{H^1}^2]. \quad (16)$$

Here  $\alpha > 0$  plays the role of a regularization parameter,  $\beta > 0$  is a scaling factor,  $BV$  denotes the space of functions of bounded variation [4, 18] and  $|\cdot|_{BV}$  is the bounded variation seminorm. The Tikhonov functional (16) amounts to the functional  $\mathcal{J}_{\alpha,\varepsilon}$  of the previous section with  $H_\varepsilon$  replaced by  $H$  and an additional regularization term added (this latter term will allow us to consider the limit  $\varepsilon \rightarrow 0$ ).

Since  $H$  is discontinuous, we are not able to prove directly that functional (16) attains a minimizer. However, utilizing the framework of [17], we are able to guarantee the existence of generalized minimizers.

**Definition 3.1.** Let  $H_\varepsilon$  be defined as above and  $1 \leq p < d/(d-1)$ .

(i) A pair of functions  $(x, \phi) \in L^\infty \times H^1$  is called admissible if there exists a sequence  $\{\phi_k\}_{k \in \mathbb{N}}$  in  $H^1$  such that  $\phi_k \rightarrow \phi$  in  $L^2$ , and there exists a sequence  $\{\varepsilon_k\}_{k \in \mathbb{N}}$  of positive numbers converging to zero such that  $H_{\varepsilon_k}(\phi_k) \rightarrow x$  in  $L^p$ . The set of admissible pairs is denoted by  $Ad := \{(x, \phi) \text{ admissible}\}$ .

(ii) The functional  $\mathcal{F}_\alpha(x, \phi)$  is defined on  $Ad$  by

$$\mathcal{F}_\alpha(x, \phi) := \|F(x) - y^\delta\|_Y^2 + \alpha \rho(x, \phi), \quad (17)$$

where  $\rho(x, \phi) := \inf \liminf_{k \rightarrow \infty} \{\beta |H_{\varepsilon_k}(\phi_k)|_{BV} + \frac{1}{2} \|\phi_k - \phi^*\|_{H^1}^2\}$ , the infimum being taken with respect to all sequences  $\{\phi_k\}_{k \in \mathbb{N}}$  and  $\{\varepsilon_k\}_{k \in \mathbb{N}}$  characterizing  $(x, \phi)$  as an element of  $Ad$ .

(iii) A generalized minimizer of  $\mathcal{F}_\alpha(\phi)$  is a minimizer of  $\mathcal{F}_\alpha(x, \phi)$  on  $Ad$ .

(iv) A generalized solution of (1) is a pair  $(x, \phi) \in Ad$  such that  $F(x) = y$ .

**Remark 3.2.** The above definitions allow us to consider  $\mathcal{F}_\alpha$  not only as a functional on  $H^1$  but also as a functional defined on the w-closure of the graph of  $H$ , contained in  $L^\infty \times H^1$ . In order to express the relation of the two functionals, we use the same symbol for both. Note



that for sufficiently regular  $\phi$ , the definitions coincide, i.e.  $\mathcal{F}_\alpha(H(\phi), \phi) = \mathcal{F}_\alpha(\phi)$ . Similarly, the regularization term in (16) is now interpreted as a functional  $\rho : Ad \rightarrow \mathbb{R}^+$ .

### 3.3. Convergence analysis

In order to prove some relevant regularity properties of the regularization functional  $\rho$  in (17) we require the following auxiliary lemma.

**Lemma 3.3.** *The following assertions hold true:*

- (i) *The semi-norm  $|\cdot|_{BV}$  is weakly lower semi-continuous with respect to  $L^p$ -convergence, i.e. let  $x_k \in BV$  and  $x_k \rightarrow x$  in  $L^p$ , then  $|x|_{BV} \leq \liminf_{k \rightarrow \infty} |x_k|_{BV}$ .*
- (ii)  *$BV$  is compactly embedded in  $L^p$  for  $1 \leq p < d/(d-1)$ . Any bounded sequence  $x_k \in BV(\Gamma)$  has a subsequence  $x_{k_j}$  converging to some  $x$  in  $L^p$ .*

**Proof.** The results follow from the continuous embedding of  $L^p$  into  $L^1$  and from [4, section 2.2.3].  $\square$

We are now ready to prove the existence of a generalized minimizer  $(\bar{x}_\alpha, \bar{\phi}_\alpha)$  of  $\mathcal{F}_\alpha$  in  $Ad$ .

**Theorem 3.4.** *Let the functionals  $\rho, \mathcal{F}_\alpha$  and the set  $Ad$  be defined as above. Moreover, let  $F$  be continuous with respect to the  $L^p$  topology for some  $1 \leq s < d/(d-1)$ . Then the following assertions hold true:*

- (i) *the functional  $\rho(x, \phi)$  is weakly lower semi-continuous and coercive on  $Ad$ ;*
- (ii) *the functional  $\mathcal{F}_\alpha(x, \phi)$  attains a minimizer on  $Ad$ .*

**Proof.** (i) Let  $(x, \phi) \in Ad$ . Then, there exist sequences  $\{\phi_k\}_{k \in \mathbb{N}}$  and  $\{\varepsilon_k\}_{k \in \mathbb{N}}$  as in definition 3.1 (i). Thus, by the weak lower semi-continuity of the  $H^1$  norm,  $\|\phi - \phi_0\|_{H^1}^2 \leq \liminf_k \|\phi_k - \phi_0\|_{H^1}^2$ . Moreover, lemma 3.3 implies  $|x|_{BV} \leq \liminf_k |H_{\varepsilon_k}(\phi_k)|_{BV}$ . Therefore,

$$\beta|x|_{BV} + \frac{1}{2}\|\phi - \phi_0\|_{H^1}^2 \leq \rho(x, \phi).$$

The weak lower semi-continuity of  $\rho$  follows with similar arguments.

(ii) Since  $(0, -1) \in Ad$ , we have  $Ad \neq \emptyset$ , and moreover  $\inf \mathcal{F}_\alpha \leq \mathcal{F}_\alpha(0, -1) < \infty$ . Let  $(x_k, \phi_k) \in Ad$  be a minimizing sequence for  $\mathcal{F}_\alpha$ , i.e.  $\mathcal{F}_\alpha(x_k, \phi_k) \rightarrow \inf \mathcal{F}_\alpha$  as  $k \rightarrow \infty$ . Then  $\rho(x_k, \phi_k)$  is bounded. Item (i) above implies the boundedness of both sequences  $\|\phi_k - \phi_0\|_{H^1}$  and  $|x_k|_{BV}$ , which by lemma 3.3 allows us to extract subsequences (again denoted by  $\{x_k\}$  and  $\{\phi_k\}$ ) such that

$$x_k \rightharpoonup^* x \text{ in } BV, \quad x_k \rightarrow x \text{ in } L^p, \quad \text{and} \quad \phi_k \rightharpoonup \phi \text{ in } H^1, \quad \phi_k \rightarrow \phi \text{ in } L^2$$

for some  $(x, \phi) \in BV \times H^1$ . Now, arguing with the continuity of  $F : L^p \rightarrow \mathcal{Y}$  and item (ii) above, one obtains

$$\begin{aligned} \inf \mathcal{F}_\alpha &= \lim_{k \rightarrow \infty} \mathcal{F}_\alpha(x_k, \phi_k) = \lim_{k \rightarrow \infty} \{\|F(x_k) - h^\delta\|_{\mathcal{Y}}^2 + \alpha\rho(x_k, \phi_k)\} \\ &\geq \liminf_{k \rightarrow \infty} \{\|F(x_k) - h^\delta\|_{\mathcal{Y}}^2\} + \liminf_{k \rightarrow \infty} \{\alpha\rho(x_k, \phi_k)\} \\ &\geq \|F(x) - h^\delta\|_{\mathcal{Y}}^2 + \alpha\rho(x, \phi) = \mathcal{F}_\alpha(x, \phi). \end{aligned}$$

It remains to prove that  $(x, \phi) \in Ad$ . This is done analogously as in the final part of the proof of [17, theorem 2.9].  $\square$

The classical analysis of Tikhonov-type regularization methods [14] can now be applied to the functional  $\mathcal{F}_\alpha$ .



**Theorem 3.5** (Convergence). *Let  $x^\dagger$  denote the solution of the inverse problem (1), and let  $\{y^{\delta_n}\}$  denote a sequence of noisy data satisfying (7) with  $\delta_n \rightarrow 0$ . Moreover, let  $F$  be continuous with respect to the  $L^p$  topology for some  $1 \leq p < d/(d-1)$ . If the parameter choice  $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  satisfies  $\lim_{\delta \rightarrow 0} \alpha(\delta) = 0$  and  $\lim_{\delta \rightarrow 0} \delta^2 \alpha^{-1}(\delta) = 0$ , then the generalized minimizers  $(x_n, \phi_n)$  of  $\mathcal{F}_{\alpha(\delta_n)}$  converge (up to subsequences) in  $L^p \times L^2$  to a generalized solution  $(\bar{x}, \bar{\phi}) \in Ad$  of (11). If, moreover,  $F$  is injective, then  $\bar{x} = x^\dagger$ .*

The proof uses the standard arguments and is thus omitted. For details, we refer to [17].

### 3.4. Stabilized approximation

We conclude this section by establishing a connection between the convergence results in this section with the ones for the case  $\varepsilon > 0$  presented in section 2. Namely, we prove that generalized minimizers of the functional  $\mathcal{F}_\alpha$  defined in (17) can be approximated by minimizers of smoothed functionals

$$\mathcal{F}_{\alpha,\varepsilon}(\phi) := \frac{1}{2} \|F(H_\varepsilon(\phi)) - y^\delta\|^2 + \alpha [\beta |H_\varepsilon(\phi)|_{BV} + \frac{1}{2} \|\phi - \phi^*\|_{H^1}^2]. \quad (18)$$

The existence of minimizers  $\phi_\alpha^\delta$  of  $\mathcal{F}_{\alpha,\varepsilon}$  in  $H^1$  is established in the following Lemma.

**Lemma 3.6.** *For any  $\phi^* \in H^1$ ,  $\varepsilon > 0$ ,  $\alpha > 0$  and  $\beta \geq 0$ , the functional  $\mathcal{F}_{\alpha,\varepsilon}$  in (18) attains a minimizer.*

**Proof.** For  $\beta > 0$ , the statement follows from theorem 3.4, with  $H$  replaced by  $H_\varepsilon$ . Note that in the case  $\varepsilon > 0$ , there is a unique relation between  $\phi$  and  $x := H_\varepsilon(\phi)$ . The case  $\beta = 0$  was implicitly analyzed in theorem 3.4 (see also theorem 2.4).  $\square$

**Remark 3.7** (strongly varying solutions, case  $\varepsilon > 0$ ). With a similar analysis as in theorem 2.5 for the case  $\beta = 0$ , respectively theorem 3.5 for  $\beta > 0$ , it follows that for fixed  $\varepsilon > 0$  (at least subsequences of) the minimizers  $\phi_\alpha^\delta$  of (18) converge to a (generalized) solution  $\phi^\dagger$  of the nonlinear problem (11), if  $\alpha(\delta)$  is such that  $\alpha(\delta) \rightarrow 0$  and  $\delta^2/\alpha \rightarrow 0$  with  $\delta \rightarrow 0$ . In particular,  $x_\alpha^\delta = H_\varepsilon(\phi_\alpha^\delta)$  converges in  $L^p$  to the solution  $x^\dagger$  of the original problem (1) if the solution is assumed to be unique and satisfy  $x^\dagger = H_\varepsilon(\phi^\dagger)$ .

In the sequel, we show in which sense the minimizers of the smoothed functional (18) approximate the generalized minimizers of functional (16).

**Theorem 3.8.** *Let  $F$  be continuous with respect to the  $L^p$  topology for some  $1 \leq p < d/(d-1)$ . For each  $\alpha > 0$  and  $\varepsilon > 0$  denote by  $\phi_{\alpha,\varepsilon}^\delta$  a minimizer of  $\mathcal{F}_{\alpha,\varepsilon}$ . Given  $\alpha > 0$  and a sequence  $\varepsilon_k \rightarrow 0^+$ , there exists a subsequence  $(H(\phi_{\alpha,\varepsilon_k}^\delta), \phi_{\alpha,\varepsilon_k}^\delta)$  converging in  $L^p(\Gamma) \times L^2(\Gamma)$  and the limit is a generalized minimizer of  $\mathcal{F}_\alpha$  in  $Ad$ .*

**Proof.** The minimizers  $\phi_{\alpha,\varepsilon_k}$  of  $\mathcal{F}_{\alpha,\varepsilon_k}$  are uniformly bounded in  $H^1$ . Moreover,  $H_{\varepsilon_k}(\phi_{\alpha,\varepsilon_k})$  is uniformly bounded in  $BV$ . Then these sequences converge strongly in  $L^p \times L^2$  to a limit  $(\tilde{x}, \tilde{\phi}) \in L^\infty \times L^2$ , and consequently  $(\tilde{x}, \tilde{\phi}) \in Ad$ . In order to prove that  $(\tilde{x}, \tilde{\phi})$  minimizes  $\mathcal{F}_\alpha$ , one argues with the continuity of  $F : L^p \rightarrow \mathcal{Y}$  and theorem 3.4.  $\square$

**Remark 3.9.** Let us further clarify the relation to the nonlinear regularization methods discussed in section 2. For this purpose, consider the stabilized functional (18), and assume that  $\varepsilon > 0$  is fixed, which will be the typical setting in a numerical realization. Then

$$|H_\varepsilon(\phi)|_{BV} = \int_{\Omega} |\nabla H_\varepsilon(\phi)| \leq \sqrt{|\Omega|} \|H'_\varepsilon\|_{L^\infty} \|\nabla \phi\|_{L^2} \leq \sqrt{|\Omega|} C'_\varepsilon \|\phi - \phi^*\|_{H^1}$$

if conditions (9) hold and  $\phi^*$  is constant. Thus for  $\varepsilon$  fixed, the  $BV$ -regularization term can be omitted, and the stabilized functional (18) can be replaced by the Tikhonov functional (13) of section 2. This is the form, we will actually use in our numerical experiments.

### 3.5. Possible extensions

Provided  $F$  has the correct mapping properties, the results of section 2 hold also for the choice  $\mathcal{X} = L^2$ . Let us show now that it is possible to choose  $L^2$  spaces for the levelset function, even in the setting of this section.

**Remark 3.10.** Let  $Ad$  be defined as the set of all pairs  $(x, \phi) \in L^\infty \times L^2$  which can be approximated by a sequence of functions  $\phi_k \in L^2$  and  $\varepsilon_k > 0$  such that  $H_{\varepsilon_k}(\phi_k) \in BV$ ,  $H_{\varepsilon_k}(\phi_k) \rightarrow x$  in  $L^p$  and  $\phi_k \rightarrow \phi$  in  $H^{-1}$ . Obviously, this set is larger than the previous set of admissible pairs, so it is not empty. Moreover, the set is closed under weak convergence, i.e. the convergence of  $H_{\varepsilon_k}(\phi_k)$  in  $L^p$  and  $\phi_k$  in  $H^{-1}$ , and the results of this section carry over almost verbatim, if the  $H^1$  regularization is replaced by the term  $\|\phi - \phi^*\|_{L^2}$ .

**Remark 3.11.** Another possible extension is to relax also the  $BV$  regularization norm, e.g. to utilize  $\|H(\phi)\|_{L^2}^2 + \|\phi - \phi^*\|_{L^2}^2$  as a regularization term. In this case, the set of admissible parameters could be defined as pairs  $(x, \phi) \in L^\infty \times L^2$  which can be approximated by sequences  $\phi_k \in L^2$  and  $\varepsilon_k > 0$  in the sense that  $\phi_k \rightarrow \phi$  in  $H^{-1}$  and  $H_{\varepsilon_k}(\phi_k) \rightarrow x$  in  $H^{-s}$  for some  $s > 0$ . In this case, we would have to require that  $F$  is a continuous operator from  $H^{-s}$  to  $\mathcal{Y}$ , which is in fact the case for our model problem investigated in the next section. The advantage of this approach would be that any binary valued solution  $x^\dagger \in L^\infty$  would be admissible (the corresponding admissible pair being  $(x^\dagger, 0)$ ).

## 4. A model problem for severely ill-posed problems

In this section, we introduce a model problem for severely ill-posed problems, and verify the conditions needed to apply the theoretical results of the previous sections.

### 4.1. A Cauchy problem for the Poisson equation

Let  $\Omega \subset \mathbb{R}^3$  be an open bounded set with piecewise Lipschitz boundary  $\partial\Omega$ . We assume that  $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$ , where  $\Gamma_i$  are two open disjoint parts of  $\partial\Omega$ . In the sequel, we consider the elliptic Cauchy problem: find  $u \in H^1(\Omega)$  such that

$$-\Delta u = f \text{ in } \Omega, \quad \text{and} \quad u = g, \quad u_\nu + u = h \quad \text{at } \Gamma_1, \quad (19)$$

where  $u_\nu := \frac{du}{dn}$  denotes the normal derivative of  $u$ , the pair of functions  $(g, h) \in H^{1/2}(\Gamma_1) \times H_0^{1/2}(\Gamma_1)'$  are the *Cauchy data* and  $f \in L^2(\Omega)$  is a known source term in the model.

We call  $u \in H^1(\Omega)$  a solution of the Cauchy problem (19), if it satisfies  $-\Delta u = f$  in the weak sense and the boundary conditions  $u = g, u_\nu + u = h$  on  $\Gamma_1$  in the sense of traces.

A solution of (19) also satisfies the mixed boundary value problem

$$-\Delta u = f \text{ in } \Omega, \quad u_\nu + u = h \text{ at } \Gamma_1, \quad u_\nu = x \text{ at } \Gamma_2. \quad (20)$$

If the function  $x$  is known, the solution  $u$  of the Cauchy problem can be computed stably by solving the mixed boundary value problem (20). We would like to mention that in general  $x \in H_0^{1/2}(\Gamma_2)'$ , and that for any such  $x$  problem (20) has a unique solution in  $H^1(\Omega)$ . The Cauchy problem (19) can thus be rephrased as finding the unknown Neumann data  $x = u_\nu|_{\Gamma_2}$ .

#### 4.2. Formulation as an operator equation

We will now rewrite (19), (20) in the form of an operator equation in Hilbert spaces. To this end, let  $u$  denote the solution of (20) and let  $F$  be defined by

$$F : x \mapsto u|_{\Gamma_1}. \quad (21)$$

It is straightforward to check that  $u$  is the solution of (19)–(20) if, and only if,  $x$  is a solution of the following problem.

**Inverse problem:** Let  $y = u|_{\Gamma_1}$ , with  $u$  defined as in (20). Find a function  $x$  such that

$$F(x) = y. \quad (22)$$

For convenience, we will in the sequel consider  $F$  as an operator on  $L^2(\Gamma_2)$ , i.e. we tacitly assume that a solution  $x$  is in  $L^2(\Gamma_2)$  rather than only in  $H_{00}^{1/2}(\Gamma_2)'$ . Since we are interested in the determination of piecewise constant solutions, this assumption is no further restriction.

The next result summarizes the main mapping properties of the operator  $F$ .

**Proposition 4.1.** *The mapping  $F : L^2(\Gamma_2) \rightarrow L^2(\Gamma_1)$ ,  $x \mapsto y = u|_{\Gamma_1}$ , with  $u$  defined as in (20), is an injective, affine linear, bounded and compact operator.*

**Proof.** The boundary value problem (20) has a unique solution in  $H^1(\Omega)$ , which depends continuously on the data, i.e.

$$\|u\|_{H^1(\Omega)} \leq B(\|f\|_{L^2(\Omega)} + \|h\|_{H^{-1/2}(\Gamma_1)} + \|x\|_{H^{-1/2}(\Gamma_2)}),$$

for some  $B \in \mathbb{R}$ . By the trace theorem,  $u|_{\Gamma_1}$  is in  $H^{1/2}(\Gamma_1)$ , which implies continuity of the operator  $F$ , and compactness follows from compact embedding of  $H^{1/2}(\Gamma_1) \hookrightarrow L^2(\Gamma_1)$ . The affine linearity of  $F$  is obvious, and the injectivity follows from the unique solvability of the boundary value problem (20).  $\square$

**Remark 4.2.** Since  $F$  is affine linear, it can be written in the form  $F(x) = Lx + v|_{\Gamma_1}$ , where  $v$  is defined by

$$-\Delta v = f \text{ in } \Omega, \quad v_\nu + v = h \text{ on } \Gamma_1, \quad v_\nu = 0 \text{ on } \Gamma_2,$$

and  $L$  is a linear operator. In the case  $f = 0$  and  $h = 0$ , which will be considered in the numerical tests in the next section, we have  $v = 0$ , and thus  $F = L$  becomes linear.

While the results of section 2 can be applied without further assumptions, we require a slightly more accurate assessment of the mapping properties of  $F$  in order to be able to apply the results of section 3.

**Corollary 4.3.** *The operator  $F$  defined in (21) is continuous from  $L^{3/2}(\Gamma_2)$  to  $L^2(\Gamma_1)$  as well as from  $H_{00(\Gamma_2)}^{-1/2}$  to  $L^2(\Gamma_1)$ .*

**Proof.** By the Sobolev embedding theorem [2],  $H^p(\Gamma_2)$  is compactly embedded in  $L^p(\Gamma_2)$  for  $p < 2(1-s)^{-1}$ . Since  $\Gamma_2 \subset \mathbb{R}^2$ , we have in particular  $H^{1/2}(\Gamma_2) \hookrightarrow L^p(\Gamma_2)$  for  $p < 4$ . This implies

$$H_{00}^{1/2} \subset H^{1/2} \subset L^3 \quad \text{and} \quad L^{3/2} = [L^3]' \subset H^{-1/2} \subset [H_{00}^{1/2}]'.$$

Hence  $x$  is an admissible Robin datum for (20), and the rest of the proof follows the lines of the previous result.  $\square$

**Corollary 4.4.** *The Cauchy Problem (22) is ill-posed.*

The Ill-posedness follows directly from the compactness and affine linearity of the forward operator  $F$ . According to [6], the Cauchy problem is in general even severely ill-posed; see also the example presented in section 5.

### 4.3. Remarks on noisy Cauchy data

In practice, only perturbed data  $(g^\delta, h^\delta)$  are available for problem (19). In this case, we assume the existence of a consistent Cauchy data pair  $(g, h) \in H^{1/2}(\Gamma_1) \times H_{00}^{1/2}(\Gamma_1)'$  such that

$$\|g - g^\delta\|_{H^{1/2}(\Gamma_1)} + \|h - h^\delta\|_{H^{-1/2}(\Gamma_1)} \leq \tilde{\delta}. \quad (23)$$

The Cauchy problem with noisy data is then defined by the operator equation

$$F^\delta(x) = y^\delta,$$

where  $F^\delta(x) := u^\delta|_{\Gamma_1}$  with  $u^\delta$  being defined as the solution of (20) with  $h$  replaced by  $h^\delta$ . From (23) and the continuous dependence of  $u$  on  $h$  one immediately obtains  $\|F^\delta(x) - F(x)\|_{H^{1/2}(\Gamma_1)} \leq C\tilde{\delta}$ , and  $\|y - y^\delta\|_{H^{1/2}(\Gamma_1)} = \|g - g^\delta\| \leq \tilde{\delta}$ . Since  $F$  is affine linear, perturbations in the operator can be related to perturbations in the data, and it again suffices to consider the unperturbed problem for the analysis, see [14]. Since we consider  $F$  as an operator mapping into  $L^2$ , we can also relax the assumptions on the data noise, i.e. we only require a bound on the data noise in the form

$$\|y^\delta - y\|_{L^2(\Gamma_1)} \leq \delta.$$

Summarizing, the Cauchy problem  $F(x) = y^\delta$  conforms to the standard conditions of inverse ill-posed problems with compact operators, and the basic results of regularization theory apply. In particular, for the choice

$$\mathcal{X} = L^2(\Gamma_2) \text{ or } \mathcal{X} = H^1(\Gamma_2) \quad \text{and} \quad \mathcal{Y} = L^2(\Gamma_1),$$

the operator  $F$  satisfies all the assumptions made in the previous sections.

## 5. Numerical realization and experiments

In this section, we illustrate the advantages of the levelset-type approaches discussed in the previous sections by numerical experiments. After introducing the discretization of our model problem, we sketch algorithms for minimizing the Tikhonov functional  $\mathcal{J}_{\alpha,\varepsilon}$ . The section is concluded with the presentation of some numerical tests.

### 5.1. A test problem and its ill-posedness

Let  $a > 0$  and define  $\Omega := (0, 1) \times (0, 1) \times (0, a)$ . We split the boundary  $\partial\Omega$  into three parts, i.e.  $\partial\Omega = \overline{\Gamma_M} \cup \overline{\Gamma_L} \cup \overline{\Gamma_a}$  with

$$\Gamma_M := (0, 1)^2 \times \{0\}, \quad \Gamma_a := (0, 1)^2 \times \{a\} \quad \text{and} \quad \Gamma_L := \partial\Omega \setminus \overline{\Gamma_0} \cup \overline{\Gamma_a}.$$

We assume that measurements can be made at  $\Gamma_M$ ; hence, Cauchy data are given there. The lateral boundary  $\Gamma_L$  is isolated, and the third part  $\Gamma_a$  is assumed to be inaccessible. The aim of solving the Cauchy problem is to determine the local flux distribution  $x$  at this inaccessible part of the boundary. The forward problem hence is governed by the mixed boundary value problem

$$-\Delta u = 0 \text{ in } \Omega, \quad u_\nu + u = 0 \text{ on } \Gamma_M, \quad u_\nu = 0 \text{ on } \Gamma_L, \quad u_\nu = x \text{ on } \Gamma_a, \quad (24)$$

and the inverse problem can be written as the operator equation

$$Lx = y \quad (25)$$

where the operator  $L$  is defined by  $Lx = u|_{\Gamma_M}$  and  $u$  denotes the solution of (24). Thus, the inverse problem consists in determining the Neumann trace  $x$  at the inaccessible part  $\Gamma_a$  of the boundary from measurements  $u|_{\Gamma_M}$ .

For a solution of the forward problem, we consider the following method based on Fourier series: let  $x_{m,n}$  denote the Fourier coefficients of a function  $x$  with respect to the expansion

$$x(s, t) = \sum_{m,n} x_{m,n} \cos(m\pi s) \cos(n\pi t).$$

The forward operator  $L$  then has the Fourier series representation

$$(Lx)(s, t) = \sum_{m,n} x_{m,n} A_{m,n}^{-1} \cos(m\pi s) \cos(n\pi t)$$

where the amplification factors  $A_{m,n}$  are given by

$$A_{m,n} = w_{m,n}\pi \sinh(w_{m,n}\pi a) + \cosh(w_{m,n}\pi a) \quad \text{with} \quad w_{m,n} := \sqrt{m^2 + n^2}.$$

A direct inversion of  $L$  leads to amplification of the  $(m, n)$ th Fourier component of the data perturbation by the factor  $A_{m,n}$ , which shows that the Cauchy problem (24) is exponentially ill-posed.

### 5.2. Implementation of the nonlinear regularization method

For the stable solution of the Cauchy problem (25), we consider the nonlinear regularization methods of section 2 with either  $L^2$  or  $H^1$  regularization. Recall that these methods can be considered as approximations to the levelset methods investigated in section 3, cf theorem 3.8 and remark 3.9. Let us shortly discuss how minimizers of the Tikhonov functional can be found numerically.

We start from the necessary first-order conditions for a minimum, which read

$$0 = H'_\varepsilon(\phi)L^*[LH_\varepsilon(\phi) - y^\delta] + \alpha[I - \gamma\Delta](\phi - \phi^*) =: \mathcal{R}_{\alpha,\varepsilon}(\phi), \quad (26)$$

where  $\gamma = 0$  in the case of  $L^2$  regularization and  $\gamma = 1$  if we employ  $H^1$  regularization. In both cases  $L^*$  denotes the adjoint of the operator  $L$  with respect to the  $L^2$  spaces.

For finding a solution to (26) we use a Gauß–Newton strategy, i.e. we start from the initial guess  $\phi_0 = \phi^*$ , and define the update  $\Delta\phi_k = \phi_{k+1} - \phi_k$  by

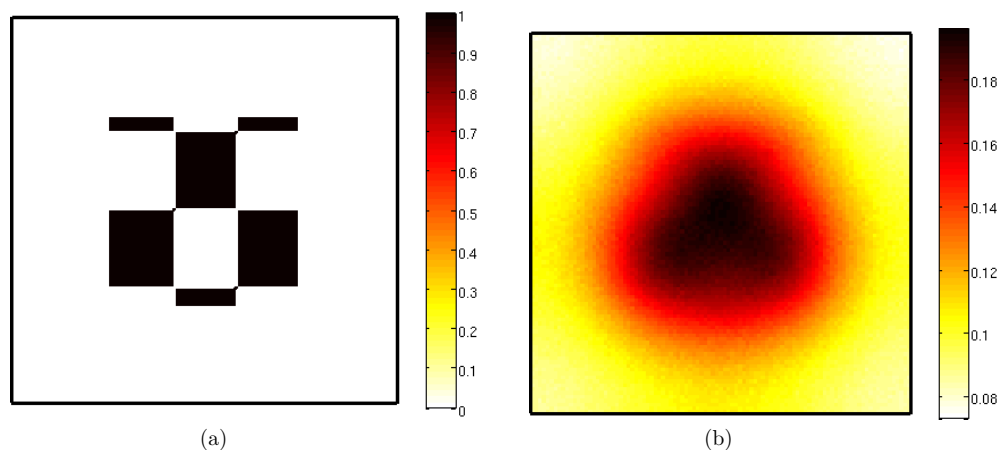
$$[H'_\varepsilon(\phi_k)L^*LH'_\varepsilon(\phi_k) + \alpha(I - \gamma\Delta)]\Delta\phi_k = -\mathcal{R}_{\alpha,\varepsilon}(\phi_k), \quad (27)$$

where  $H'_\varepsilon(\phi_k)$  has to be understood as pointwise multiplication. The discretized linear systems (27) are symmetric and can be solved by the conjugate gradient method.

The iteration (27) is stopped as soon as the norm of  $\mathcal{R}_{\alpha,\varepsilon}$  is sufficiently small. Instead of applying the iteration (27) with fixed  $\alpha = \alpha(\delta)$ , we choose a sequence of regularization parameters  $\alpha_k = \max\{\alpha(\delta), \alpha_0 q^k\}$  for some  $0 < q < 1$ , and stop the outer Newton iteration, as soon as the discrepancy  $\|G_\varepsilon(\phi_k) - y^\delta\| \leq \tau\delta$  for some  $\tau > 1$ . For our numerical tests, we choose  $\alpha(\delta) = \delta^{-1.9}$ , and we stop the outer Newton iteration as soon as the  $\alpha_k = \alpha(\delta)$  or the discrepancy  $\|G_\varepsilon(\phi_k) - y^\delta\| \leq \tau\delta$  for some  $\tau > 1$ . Thus, we effectively use an iteratively regularized Gauß–Newton method [5, 21].

### 5.3. Numerical tests

In our numerical tests, we choose different values for the thickness  $a$  of the domain for the model problem of subsection 5.1, and try to reconstruct a binary valued coefficient (the unknown Neumann data) depicted in figure 1(a). The choice of  $a$  affects the amplification factors  $A_{m,n}$  and thus the severity of ill-posedness of the inverse problem, see table 3. The Cauchy data at the measurement boundary  $\Gamma_M$  are given by  $h = 0$  and  $g = y$ . Here,  $h$  is used in the definition of the forward problem (24), and  $g = y$  is used as data for the inverse problem (25).



**Figure 1.** Setup of the first numerical experiment: (a) true solution (Neumann data at  $\Gamma_a$ ); (b) measured Dirichlet data at  $\Gamma_M$  for a domain with thickness  $a = 0.1$ .

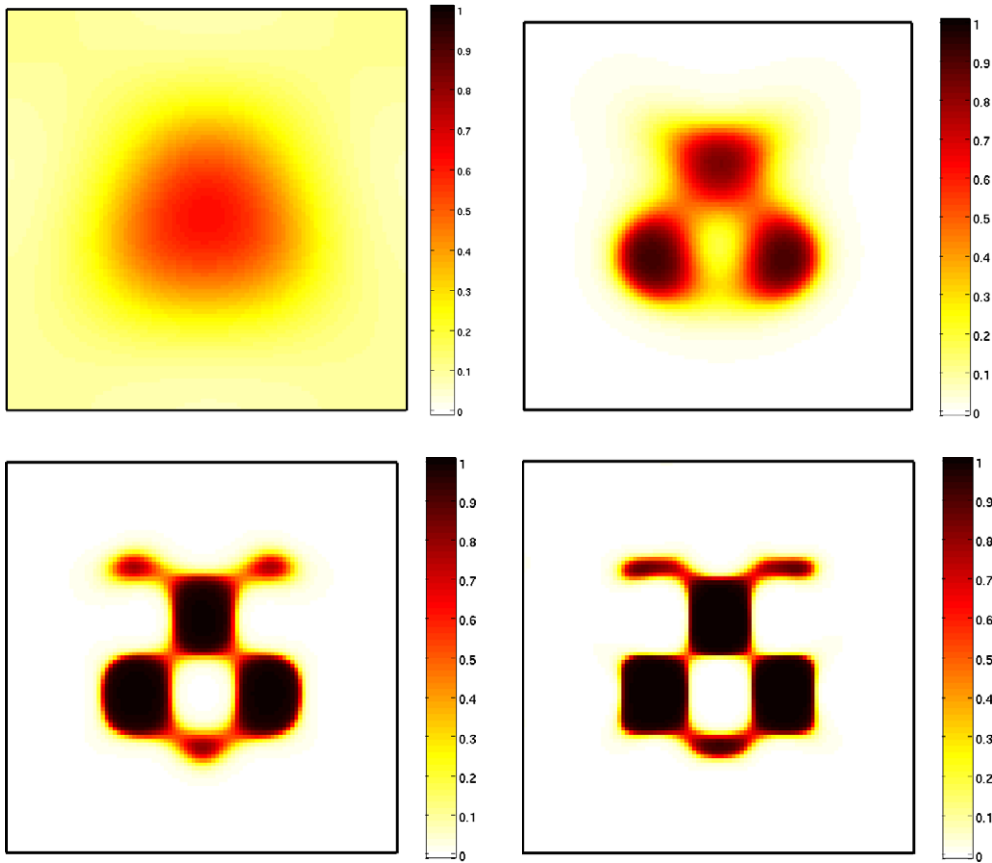
The data are generated by solving the mixed boundary value problem (24) with  $x$  as depicted in figure 1(a) by a finite difference method on a  $100 \times 100 \times 100$  grid and the data  $y$  corresponding to (25) are additionally perturbed by random noise of size  $\delta$  in the  $L^2(\Gamma_a)$  norm.

For the reconstruction, the forward problems are discretized by the Fourier expansion discussed in subsection 5.1 using  $100 \times 100$  Fourier modes. The discretizations for generating the data and for solving the inverse problem are chosen very fine in order to minimize the perturbations due to discretization errors.

For a solution of the inverse problem, we consider the nonlinear regularization method of section 2, and we utilize the Gauß–Newton method outlined in section 5.2 for minimizing the Tikhonov functional (13). Throughout our numerical experiments we use  $\varepsilon = 0.01$ , and as the initial levelset function we choose the constant function  $\phi_0 = 0$ , which is also used as *a priori* guess  $\phi^*$  and corresponds to an *a priori* guess  $x^* = 0.5$ .

*Test case 1:* In the first example, we set  $a = 0.1$ . The corresponding data  $y$ , and some iterates obtained with algorithm (27) for a noise level  $\delta = 0.01\%$  are displayed in figure 2. Figure 3 displays the reconstructions obtained for larger noise levels  $\delta = 1\%$  and  $0.1\%$ . In table 1 we compare the iteration numbers and reconstruction errors for the levelset-type method (13) with  $L^2$  and  $H^1$  regularization. In both cases, we utilize the Gauß–Newton methods (27) for the minimization of the Tikhonov functionals. While the reconstructions obtained for different regularization norms are rather similar, the iteration numbers increase significantly when regularizing in the stronger norm. This effect has been analyzed in [12, 13] for iterative regularization of nonlinear and linear problems.

*Test case 2:* In order to illustrate the advantages of the nonlinear regularization method (13) over standard regularization methods, we choose the trivial transformation  $H_\varepsilon(x) := x$ , in which case (13) amounts to standard Tikhonov regularization applied to the solution of the linear inverse problem (25). For a numerical realization, we again use algorithm (27), which now amounts to Tikhonov regularization with an iterative choice of regularization parameter.



**Figure 2.** Levelset reconstruction  $x_k = H_\varepsilon(\phi_k)$  for iterations  $k = 1, 6, 11, 18$  of method (27) with  $\gamma = 0$  and noise level  $\delta = 0.01\%$ .

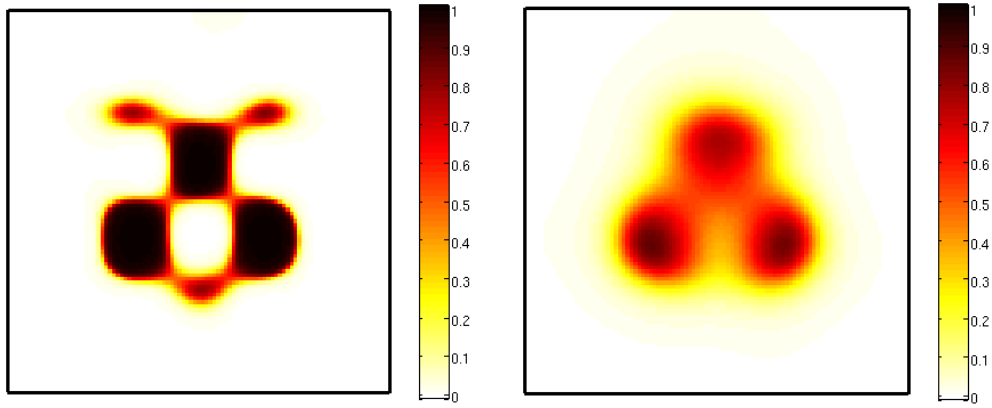
**Table 1.** Reconstruction errors and iteration numbers (Newton steps and total number of inner iterations) obtained with the nonlinear levelset-type regularization method (13) with  $L^2$  respectively  $H^1$  regularization. Both functionals are minimized numerically by the Gauß-Newton method (27) with parameter  $\varepsilon = 0.01$  in the nonlinear transformation.

$\delta$	$\ x^{L^2} - x^\dagger\ _{L^2}$	$N(n)$	$\ x^{H^1} - x^\dagger\ _{L^2}$	$N(n)$
10%	0.3258	1 (1)	0.3276	1 (1)
1%	0.2608	5 (13)	0.2661	6 (10)
0.1%	0.1556	11 (52)	0.1604	16 (189)
0.01%	0.0835	17 (241)	0.0953	25 (996)

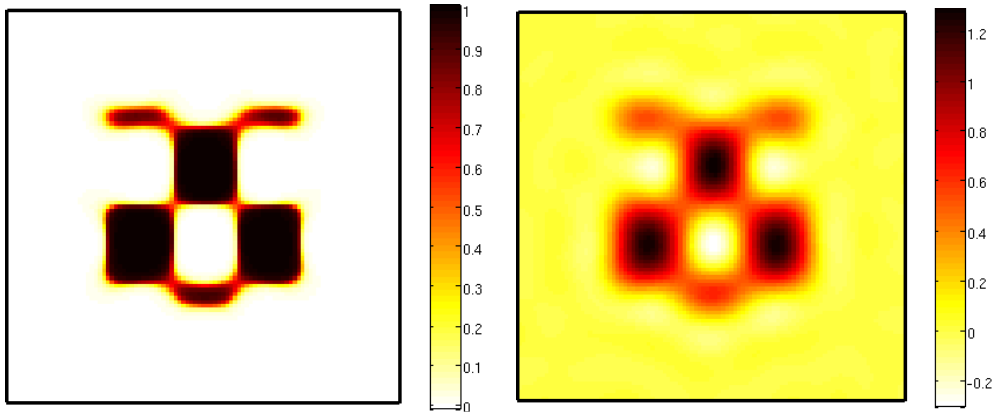
Figure 4 displays the solutions obtained with the nonlinear regularization method ( $H_\varepsilon$  as in (10)) and standard Tikhonov regularization ( $H_\varepsilon = id$ ) for a noise level of  $\delta = 0.01\%$  and thickness  $a = 0.1$ .

The reconstruction errors of this comparison are listed in table 2. Note that in particular for small noise levels, the reconstructions obtained by the nonlinear regularization methods are much better, e.g. in order to obtain a reconstruction comparable to the one of the nonlinear





**Figure 3.** Levelset reconstructions  $x_k^* = H_\varepsilon(\phi_k^*)$  using  $L^2$  regularization ( $\gamma = 0$ ) for thickness  $a = 0.1$  and noise levels  $\delta = 0.1\%$  (left) and  $\delta = 1\%$  (right).



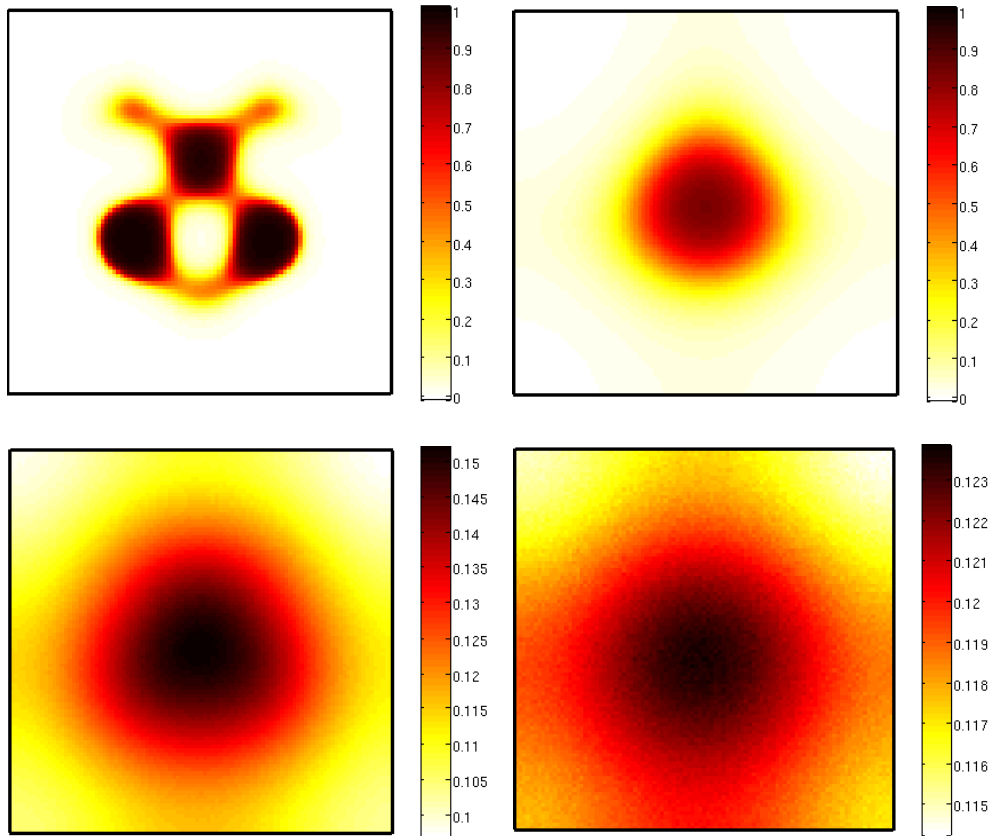
**Figure 4.** Comparison of the reconstructions obtained for a noise level  $\delta = 0.01\%$  by the nonlinear regularization method (13) (left) and standard Tikhonov regularization (right).  $\gamma = 0$  in both cases, and the functionals are minimized numerically by the Gauß–Newton method (27).

**Table 2.** Reconstruction errors and iteration numbers (Newton steps and total number of inner iterations) obtained with the nonlinear levelset-type regularization method (NL) and standard Tikhonov regularization (TIK). Both functionals are minimized numerically by the Gauß–Newton method (27).

$\delta$	$\ x^{NL} - x^\dagger\ _{L^2}$	$N(n)$	$\ x^{TIK} - x^\dagger\ _{L^2}$	$N(n)$
10%	0.3258	1 (1)	0.3213	2 (2)
1%	0.2608	5 (13)	0.2708	8 (9)
0.1%	0.1556	11 (52)	0.2012	14 (44)
0.01%	0.0835	17 (241)	0.1416	18 (112)

regularization method with a noise level of  $\delta = 0.1\%$ , data with only  $\delta = 0.01\%$  noise have to be used for the standard Tikhonov regularization.

*Test case 3:* In a final test case, we study the influence of ill-posedness on the quality of the reconstructions by varying the thickness parameter  $a$ .



**Figure 5.** Levelset reconstructions  $x_k^* = H_\varepsilon(\phi_k^*)$  using  $L^2$  regularization ( $\gamma = 0$ ) for noise levels  $\delta = 0.01\%$  and domain thickness  $a = 0.2$  (left) and  $a = 0.5$  (right). The second row displays the corresponding data  $y^\delta$ . Note that the data are almost constant for the thick domain  $a = 0.5$ .

**Table 3.** Number of Fourier components that can be stably reconstructed for varying thickness of the computational domain according to the condition  $A_{m,n}^{-1} \geq \delta$ .

$\delta$	10%	1%	0.1%	0.01%
$a = 0.1$	7	42	134	289
$a = 0.2$	3	18	47	91
$a = 0.5$	2	5	12	21

Figure 5 displays the reconstructions obtained with the nonlinear regularization methods discussed in this paper and the corresponding data for different choices of the thickness parameter  $a$ .

The results obtained for  $a = 0.5$  are not satisfactory, although a small noise level  $\delta = 0.01\%$  has been used. Let us shortly highlight why this is the case: for stability reasons, only Fourier components corresponding to amplification factors with  $A_{m,n}^{-1} \geq \delta$  should be used for stable reconstructions; the other Fourier components are damped out by the regularization procedure. In table 3 we list the number of Fourier components that actually satisfy this condition.

Since the non-smooth solution  $x^\dagger$  cannot be represented well by only few Fourier components, only relatively bad reconstructions are obtained for thick domains. The presence of only few relevant Fourier components is also reflected in the data  $y^\delta$ , which are almost constant for  $a = 0.5$  (see figure 5).

## 6. Final remarks and conclusions

In this paper the stable solution of inverse problems with piecewise constant or strongly varying solutions has been considered. These problems have been approached by parameterizing the unknown function via some auxiliary levelset function.

The nonlinear inverse problems arising from the parameterization of the solution by operators  $H_\varepsilon$  ( $\varepsilon > 0$ ) have been analyzed in the framework of Tikhonov regularization for nonlinear inverse problems. The limit case  $\varepsilon \rightarrow 0$ , which corresponds to a parameterization of the solution by the Heaviside operator  $H$ , has also been considered. The resulting discontinuous, nonlinear inverse problem is analyzed in the framework of a levelset approach introduced in [17]. The connection between this levelset approach and the nonlinear regularization methods above has been discussed in detail.

For the limit case  $\varepsilon \rightarrow 0$ , we considered different regularization terms,  $BV \times L^2$  and  $L^2 \times L^2$ , as alternatives to the  $BV \times H^1$  regularization functional proposed in [17]. Moreover, for  $\varepsilon > 0$ , the  $BV$  component in the  $BV$ - $H^1$  regularization case is dominated by the  $H^1$  term in the penalization term, which justifies to omit the  $BV$  term for the numerical realization in section 5; see also [17, 30, 31].

This motivates the use of Newton-type methods for the solution of the optimality systems for the Tikhonov functionals, which together with an iterative solution of the linearized (Newton) systems makes the considered approach very efficient, compared e.g. with fixed-point algorithms considered previously [15, 22]).

The nonlinear regularization methods have then been applied for solving an elliptic Cauchy problem with strongly varying solution (a classical example for severely ill-posed problems). A comparison with classical Tikhonov regularization applied to the linear inverse problem illustrates that the quality of the reconstructions can be improved considerably by the use of nonlinear regularization methods. We also tested and compared  $L^2$  and  $H^1$  penalization of the levelset function, and observed that the minimizer of the Tikhonov functional with  $L^2$  penalization can be obtained using a much smaller number of steps of the Newton-type method, in accordance with results on regularization in Hilbert Scales [13, 12].

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