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ON LEVENBERG-MARQUARDT-KACZMARZ ITERATIVE METHODS FOR SOLVING SYSTEMS OF NONLINEAR ILL-POSED EQUATIONS

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Abstract. In this article a modified Levenberg-Marquardt method coupled with a Kaczmarz strategy for obtaining stable solutions of nonlinear systems of ill-posed operator equations is investigated. We show that the proposed method is a convergent regularization method. Numerical tests are presented for a non-linear inverse doping problem based on a bipolar model.

1. **Introduction.** In this paper we propose a new method for obtaining regularized approximations of systems of nonlinear ill-posed operator equations.

The *inverse problem* we are interested in consists of determining an unknown quantity $x \in X$ from the set of data $(y_0, \dots, y_{N-1}) \in Y^N$, where X, Y are Hilbert spaces and $N \geq 1$ (the case $y_i \in Y_i$ with possibly different spaces $Y_0, \dots, Y_{N-1}$ can be treated analogously). In practical situations, we do not know the data exactly. Instead, we have only approximate measured data $y_i^\delta \in Y$ satisfying

$$\|y_i^\delta - y_i\| \leq \delta_i\,, \quad i = 0, \dots, N-1\,, \tag{1}$$

with $\delta_i > 0$ (noise level). We use the notation $\delta := (\delta_0, \dots, \delta_{N-1})$. The finite set of data above is obtained by indirect measurements of the parameter x, this process being described by the model

$$F_i(x) = y_i\,, \quad i = 0, \dots, N-1\,, \tag{2}$$

where $F_i : D_i \subset X \to Y$, and D_i are the corresponding domains of definition.

Standard methods for the solution of system (2) are based in the use of *Iterative type* regularization methods [1, 6, 11, 12, 13] or *Tikhonov type* regularization

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methods [6, 19, 23, 24, 25, 20] after rewriting (2) as a single equation $\mathbf{F}(x) = y$, where

$$\mathbf{F} := (F_0, \ldots, F_{N-1}) : \bigcap\nolimits_{i=0}^{N-1} D_i \to Y^N \tag{3}$$

and $y := (y_0, \ldots, y_{N-1})$.

The starting point of our approach is the Levenberg-Marquardt method [10, 17, 18, 5] for solving ill-posed problems, which is defined by

$$x_{k+1}^\delta = x_k^\delta - (\mathbf{F}'(x_k^\delta)^* \mathbf{F}'(x_k^\delta) + \alpha I)^{-1} \mathbf{F}'(x_k^\delta)^* (\mathbf{F}(x_k^\delta) - y^\delta) .$$

where $\mathbf{F}'(z)$ is the Frechet-derivative of $\mathbf{F}$ in z and $\mathbf{F}'(z)^*$ is its adjoint. Motivated by the ideas in [8, 4, 9, 3], we propose in this article a *loping Levenberg-Marquardt-Kaczmarz method* (L-LMK method) for solving (2). This iterative method is defined by

$$x_{k+1}^\delta = x_k^\delta + \omega_k \, h_k \, , \tag{4}$$

where

$$h_k := (F_{[k]}'(x_k^\delta)^* F_{[k]}'(x_k^\delta) + \alpha I)^{-1} F_{[k]}'(x_k^\delta)^* (y_{[k]}^\delta - F_{[k]}(x_k^\delta)) \tag{5}$$

and

$$\omega_k := \begin{cases} 1 & \text{if } \|F_{[k]}(x_k^\delta) - y_{[k]}^\delta\| \geq \tau \delta_{[k]} \\ 0 & \text{otherwise} \end{cases} . \tag{6}$$

Here $\alpha > 0$ is an appropriately chosen number (see (11) below), $[k] := (k \mod N) \in \{0, \ldots, N-1\}$, and $x_0^\delta = x_0 \in X$ is an initial guess, possibly incorporating some *a priori* knowledge about the exact solution, and $\tau > 1$ a fixed constant (see (11) below).

The L-LMK method consists in incorporating the Kaczmarz strategy into the Levenberg-Marquardt method. This procedure is analog to the one introduced in [8], [4], [9], and [3] regarding the Landweber-Kaczmarz (LK) iteration, the Steepest-Descent-Kaczmarz (SDK) iteration, the Expectation-Maximization-Kaczmarz (EMK) iteration, and the Iteratively Regularized Gauss-Newton-Kaczmarz (IRGNK) respectively. As usual in Kaczmarz type algorithms [22], a group of N subsequent steps (starting at some multiple k of N) is called a *cycle*. The L-LMK iteration should be terminated when, for the first time, all x_k^δ are equal within a cycle. That is, we stop the iteration at

$$k_*^\delta := \min\{lN \in \mathbb{N} : x_{lN}^\delta = x_{lN+1}^\delta = \cdots = x_{lN+N}^\delta\} , \tag{7}$$

Notice that k_*^δ is the smallest multiple of N such that

$$x_{k_*^\delta}^\delta = x_{k_*^\delta+1}^\delta = \cdots = x_{k_*^\delta+N}^\delta , \tag{8}$$

or equivalently (see Proposition 2 below) such that

$$\omega_{k_*^\delta-1} = \omega_{k_*^\delta} = \cdots = \omega_{k_*^\delta+N-1} = 0 .$$

For exact data ($\delta = 0$) we have $\omega_k = 1$ for each k and the L-LMK iteration reduces to the Levenberg-Marquardt-Kaczmarz (LMK) method. For noisy data however, the L-LMK method is fundamentally different from the LMK method: The bang-bang relaxation parameter ω_k effects that the iterates defined in (4), (5) become stationary if all components of the residual vector $\|F_i(x_k^\delta) - y_i^\delta\|$ fall below a pre-specified threshold. This characteristic renders (4), (5) a regularization method in the sense of [6] (see Section 4).

The article is outlined as follows. In Section 2 we formulate basic assumptions and derive some auxiliary estimates required for the analysis. In Section 3 we prove a convergence result for the LMK method. In Section 4 we prove a semiconvergence result for the L-LMK method. In Section 5 a numerical experiment for an inverse doping problem is presented. Section 6 is devoted to final remarks and conclusions.

2. **Assumptions and basic results.** We begin this section by introducing some assumptions, that are needed for the convergence analysis presented in the next sections. These assumptions derive from the classical assumptions used in the analysis of iterative regularization methods [6, 12, 20].

(A1) The operators F_i and their linearizations F_i' – see (A2) – are continuous and the corresponding domains of definition D_i have nonempty interior, i.e., there exists $x_0 \in X$, $\rho > 0$ such that $B_\rho(x_0) \subset \bigcap_{i=0}^{N-1} D_i$, where $B_\rho(x_0)$ is the ball of radius ρ around x_0. Moreover, we assume the existence of $C > 0$ such that

$$\|F_i'(x)\| \ \leq \ C\,, \qquad x \in B_\rho(x_0) \tag{9}$$

(notice that $x_0^\delta = x_0$ is used as starting point of the L-LMK iteration).

(A2) We assume that the *local tangential cone condition* [6, 12]

$$\|F_i(\bar{x}) - F_i(x) - F_i'(x)(\bar{x} - x)\|_Y \ \leq \ \eta \|F_i(\bar{x}) - F_i(x)\|_Y\,, \qquad \forall\, x, \bar{x} \in B_\rho(x_0) \tag{10}$$

holds for some $\eta < 1$. This is a uniform assumption on the nonlinearity of the operators F_i. Note that $F_i'(x)$ need not necessarily be the Fréchet derivative of F_i at x, but it should be a bounded linear operator that continuously depends on x, see (A1).

(A3) There exists an element $x^* \in B_{\rho/2}(x_0)$ such that $\mathbf{F}(x^*) = y$, where $y = (y_0, \ldots, y_{N-1})$ are the exact data satisfying (1).

We are now in position to choose the positive constants α and τ in (5), (6). For the rest of this article we shall assume

$$\alpha \ > \ \frac{C^2 q}{1-q}\,, \qquad \tau \ > \ \frac{1+\eta}{1-\eta} > 1\,, \qquad \eta + (1+\eta)\tau^{-1} < q < 1 \tag{11}$$

for some $0 < q < 1$.

In the sequel we verify some basic facts that are helpful for the convergence analysis derived in the next two sections. The first result concerns some useful identities.

Lemma 2.1. *Let x_k^δ, h_k and α be defined by (4), (5) and (11) respectively. Moreover, assume that (A1) - (A3) hold true.*

a) For all $k \in \mathbb{N}$ we have

$$y_{[k]}^\delta - F_{[k]}(x_k^\delta) - F_{[k]}'(x_k^\delta)h_k \ = \ \alpha(F_{[k]}'(x_k^\delta)F_{[k]}'(x_k^\delta)^* + \alpha I)^{-1}(y_{[k]}^\delta - F_{[k]}(x_k^\delta))\,.$$

b) Moreover, if $\omega_k = 1$, we have

$$h_k \ = \ -\alpha^{-1}F_{[k]}'(x_k^\delta)^*\left[F_{[k]}'(x_k^\delta)(x_{k+1}^\delta - x_k^\delta) + F_{[k]}(x_k^\delta) - y_{[k]}^\delta\right],$$

$$F_{[k]}'(x_k^\delta)(x_{k+1}^\delta - x_k^\delta) + F_{[k]}(x_k^\delta) - y_{[k]}^\delta \ = \ \alpha(F_{[k]}'(x_k^\delta)F_{[k]}'(x_k^\delta)^* + \alpha I)^{-1}(F_{[k]}(x_k^\delta) - y_{[k]}^\delta)\,.$$

c) Define $B_k^\delta := \alpha(A_kA_k^* + \alpha I)^{-1}(F_{[k]}(x_k^\delta) - y_{[k]}^\delta) = F_{[k]}'(x_k^\delta)(x_{k+1}^\delta - x_k^\delta) + F_{[k]}(x_k^\delta) - y_{[k]}^\delta$ (we write $B_k = B_k^\delta$ for $\delta = 0$). Then

$$q\,\|F_{[k]}(x_k^\delta) - y_{[k]}^\delta\| \ \leq \ \|B_k^\delta\| \ \leq \ \|F_{[k]}(x_k^\delta) - y_{[k]}^\delta\|\,. \tag{12}$$

Proof. The proof of a) and b) is straightforward and will be omitted. To prove c) notice that

$$\|\alpha(A_kA_k^*+\alpha I)^{-1}(F_{[k]}(x_k^\delta)-y_{[k]}^\delta)\| \geq \frac{\alpha}{C^2+\alpha}\|F_{[k]}(x_k^\delta)-y_{[k]}^\delta\| \geq q\,\|F_{[k]}(x_k^\delta)-y_{[k]}^\delta\|$$

with $A_k := F'_{[k]}(x_k^\delta)$. On the other hand, we have $\|\alpha(A_kA_k^*+\alpha I)^{-1}(F_{[k]}(x_k^\delta)-y_{[k]}^\delta)\| \leq \|F_{[k]}(x_k^\delta)-y_{[k]}^\delta\|$. □

Remark 1. According to [10] the Levenberg-Marquardt-iteration should be implemented with variable α_k, which for the LMK would mean that α_k is chosen in such a way that $h_k = h_k(\alpha)$ in (5) satisfies

$$\|F'_{[k]}(x_k^\delta)h_k + F_{[k]}(x_k^\delta)-y_k^\delta\| \;=\; q\,\|F_{[k]}(x_k^\delta)-y_k^\delta\|\,, \tag{13}$$

for some $0<q<1$. From (12) and monotonicity of the mapping $\alpha \mapsto \|B_k^\delta\|$, (see, e.g., [6]), it follows that α as chosen in (11) is larger than the α_k's defined in [10] (see [7, Theorem 3.3.1]).

It is worth noticing that Lemma 2.2 as well as Proposition 1 remain valid with α_k chosen as in (13).

The following lemma is an important auxiliary result, which will be used to prove a monotonicity property of the L-LMK iteration.

Lemma 2.2. *Let x_k^δ, h_k, α and q be defined by (4), (5) and (11) respectively. Moreover, assume that (A1) - (A3) hold true. If $x_k^\delta \in B_\rho(x_0)$ for some $k\in\mathbb{N}$, then*

$$\begin{aligned}\|x_{k+1}^\delta - x^*\|^2 - \|x_k^\delta - x^*\|^2 \leq 2\frac{\omega_k}{\alpha}\|B_k^\delta\|\left[(\eta-q)\|F_{[k]}(x_k^\delta)-y_{[k]}^\delta\| + (1+\eta)\delta_{[k]}\right]\\ - \|x_{k+1}^\delta - x_k^\delta\|^2\,,\end{aligned} \tag{14}$$

where B_k^δ is defined as in Lemma 2.1.

Proof. Let $A_k := F'_{[k]}(x_k^\delta)$. If $\omega_k = 0$, (14) is obvious. If $\omega_k = 1$, it follows from (4), (5) and Lemma 2.1 that

$$\begin{aligned}&\|x_{k+1}^\delta - x^*\|^2 - \|x_k^\delta - x^*\|^2\\
&\quad= 2\,\langle x_{k+1}^\delta - x_k^\delta,\; x_{k+1}^\delta - x^*\rangle - \|x_{k+1}^\delta - x_k^\delta\|^2\\
&\quad= -2\alpha^{-1}\,\langle A_k(x_{k+1}^\delta - x_k^\delta) + F_{[k]}(x_k^\delta) - y_{[k]}^\delta,\\
&\qquad\qquad A_k(x_{k+1}^\delta - x^*) \pm A_kx_k^\delta \pm F_{[k]}(x_k^\delta) \pm y_{[k]}^\delta\rangle\\
&\qquad - \|x_{k+1}^\delta - x_k^\delta\|^2\\
&\quad= -2\alpha^{-1}\left[\|B_k^\delta\|^2 + \langle B_k^\delta,\; A_k(x_k^\delta - x^*) - F_{[k]}(x_k^\delta) + y_{[k]}^\delta \pm F_{[k]}(x^*)\rangle\right]\\
&\qquad - \|x_{k+1}^\delta - x_k^\delta\|^2\\
&\quad= 2\alpha^{-1}\left[-\|B_k^\delta\|^2 + \langle B_k^\delta,\; -F_{[k]}(x^*) + F_{[k]}(x_k^\delta) + A_k(x^* - x_k^\delta)\rangle\right.\\
&\qquad \left. + \langle B_k^\delta,\; F_{[k]}(x^*) - y_{[k]}^\delta\rangle\right] - \|x_{k+1}^\delta - x_k^\delta\|^2\,.\end{aligned}$$

Now, applying the Cauchy-Schwarz inequality, (1) and (10) with $\bar{x} = x^* \in B_{\rho/2}(x_0)$, $x = x_k^\delta \in B_\rho(x_0)$, leads to

$$\begin{aligned}
&\|x_{k+1}^\delta - x^*\|^2 - \|x_k^\delta - x^*\|^2 \\
&\quad \le 2\alpha^{-1}\left[-\|B_k^\delta\|^2 + \|B_k^\delta\|\,\eta\|F_{[k]}(x_k^\delta) - F_{[k]}(x^*) \pm y_{[k]}^\delta\| + \|B_k^\delta\|\delta_{[k]}\right] \\
&\qquad - \|x_{k+1}^\delta - x_k^\delta\|^2 \\
&\quad \le 2\alpha^{-1}\|B_k^\delta\|\left[-\|B_k^\delta\| + \eta\|F_{[k]}(x_k^\delta) - y_{[k]}^\delta\| + (\eta+1)\delta_{[k]}\right] - \|x_{k+1}^\delta - x_k^\delta\|^2 . \qquad (15)
\end{aligned}$$

The estimate (14) follows now plugging (12) into (15). □

Our next goal is to prove a monotonicity property, known to be satisfied by classical iterative regularization methods (e.g., Landweber [6], steepest descent [21]), and also by Kaczmarz type methods (e.g., loping Landweber-Kaczmarz [8], loping Steepest-Descent-Kaczmarz [4], loping Expectation-Maximization-Kaczmarz [9]).

Proposition 1 (Monotonicity)**.** *Under the assumptions of Lemma 2.2, for all $k < k_*^\delta$ the iterates x_k^δ remain in $B_{\rho/2}(x^*) \subset B_\rho(x_0)$ and satisfy (14). Moreover,*

$$\|x_{k+1}^\delta - x^*\|^2 \le \|x_k^\delta - x^*\|^2 , \quad k < k_*^\delta . \qquad (16)$$

Proof. From (A3) it follows that $x_0 \in B_{\rho/2}(x^*)$. If $\omega_0 = 0$, then $x_1^\delta = x_0^\delta = x_0 \in B_{\rho/2}(x^*)$ and (16) is satisfied with equality for $k = 0$. If $\omega_0 = 1$, it follows from Lemma 2.2 that (14) holds for $k = 0$. Then we conclude from (6) and (14) that

$$\|x_{k+1}^\delta - x^*\|^2 - \|x_k^\delta - x^*\|^2 \le 2\alpha^{-1}\|B_k^\delta\|\|F_{[k]}(x_k^\delta) - y_{[k]}^\delta\|\left[\eta + (1+\eta)\tau^{-1} - q\right] .$$

Due to (11) the last term on the right hand side is non positive. Thus, (16) holds for $k = 0$. In particular we have $x_1^\delta \in B_{\rho/2}(x^*)$. The proof follows now using an inductive argument. □

In the next two sections we provide a complete convergence analysis for the L-LMK iteration, showing that it is a convergent regularization method in the sense of [6].

3. Convergence for exact data.

Unless otherwise stated, we assume in the sequel that (A1) - (A3) hold true and that x_k^δ, h_k, α, τ and q are defined by (4), (5) and (11). Our main goal in this section is to prove convergence in the case $\delta_i = 0$, $i = 0, \dots, N-1$. As already observed in Section 1 the L-LMK reduces in this case to the LMK iteration (i.e. $\omega_k = 1$ in (6)). For exact data $y = (y_0, \dots, y_{N-1})$, the iterates in (4) are denoted by x_k, in contrast to x_k^δ in the noisy data case.

Remark 2. It is worth noticing that there exists an x_0-minimal norm solution of (2) in $B_{\rho/2}(x_0)$, i.e., a solution $x^\dagger$ of (2) such that $\|x^\dagger - x_0\| = \inf\{\|x - x_0\| : x \in B_{\rho/2}(x_0) \text{ and } \mathbf{F}(x) = y\}$. Moreover, $x^\dagger$ is the only solution of (2) in $B_{\rho/2}(x_0) \cap \big(x_0 + \ker(F'(x^\dagger))^\perp\big)$. This assertion is a direct consequence of [11, Proposition 2.1]. For a detailed proof we refer the reader to [12].

In the sequel we derive some estimates that are helpful for the proof of the convergence result. From Proposition 1 it follows that (14) holds for all $k \in \mathbb{N}$. Since the data is exact, (14) can be rewritten as

$$\|x_{k+1} - x^*\|^2 - \|x_k - x^*\|^2 \le 2\alpha^{-1}(\eta - q)\|B_k\|\,\|F_{[k]}(x_k) - y_{[k]}\| - \|x_{k+1} - x_k\|^2 . \qquad (17)$$

Now inserting (12) into (17) and summing over all k, leads to

$$\sum_{k=0}^{\infty} \|F_{[k]}(x_k) - y_{[k]}\|^2 \;\leq\; \alpha[2q(q-\eta)]^{-1}\|x_0 - x^*\|^2 \;<\; \infty \tag{18a}$$

(notice that $q > \eta + (1+\eta)\tau^{-1} > \eta$), and also to

$$\sum_{k=0}^{\infty} \|B_k\|^2 \;\leq\; \alpha[2(q-\eta)]^{-1}\|x_0 - x^*\|^2 \;<\; \infty\,. \tag{18b}$$

On the other hand, neglecting the last term on the right hand side of (17) and summing over all k, leads to

$$\sum_{k=0}^{\infty} \|B_k\|\,\|F_{[k]}(x_k) - y_{[k]}\| \;\leq\; \alpha[2(q-\eta)]^{-1}\|x_0 - x^*\|^2 \;<\; \infty\,. \tag{18c}$$

Finally, neglecting the first term on the right hand side of (17) and summing over all k, leads to

$$\sum_{k=0}^{\infty} \|x_{k+1} - x_k\|^2 \;\leq\; \|x_0 - x^*\|^2 \;<\; \infty\,. \tag{18d}$$

Theorem 3.1 (Convergence for exact data)**.** *For exact data, the iteration x_k converges to a solution of (2), as $k \to \infty$. Moreover, if the kernel condition* [5]

$$\mathcal{N}(\mathbf{F}'(x^\dagger)) \subseteq \mathcal{N}(\mathbf{F}'(x)) \quad \text{for all } x \in B_\rho(x_0)\,, \tag{19}$$

is satisfied, where $\mathbf{F}$ *is defined as in (3), then $x_k \to x^\dagger$.*

Proof. We define $e_k := x^\dagger - x_k$. From Proposition 1 it follows that $\|e_k\|$ is monotone non-increasing. Therefore, $\|e_k\|$ converges to some $\epsilon \geq 0$. In the following we show that e_k is in fact a Cauchy sequence.

In order to show that e_k is a Cauchy sequence, it suffices to prove $|\langle e_n - e_k, e_n\rangle| \to 0$, $|\langle e_n - e_l, e_n\rangle| \to 0$ as $k, l \to \infty$ with $k \leq l$ for some $k \leq n \leq l$ [11, Theorem 2.3]. Let $k \leq l$ be arbitrary, $k = k_0 N + k_1$, $l = l_0 N + l_1$, $k_1, l_1 \in \{0, \ldots, N-1\}$, let $n_0 \in \{k_0, \ldots, l_0\}$ be such that

$$\begin{aligned} &\sum_{s=0}^{N-1} \left\{\|x_{n_0N+s+1} - x_{n_0N+s}\| + \|F_s(x_{n_0N+s}) - y_s\|\right\} \leq \\ &\leq \sum_{s=0}^{N-1} \left\{\|x_{i_0N+s+1} - x_{i_0N+s}\| + \|F_s(x_{i_0N+s}) - y_s\|\right\}, \quad \text{for all } i_0 \in \{k_0, \ldots, l_0\}\,, \end{aligned} \tag{20}$$

and set $n = n_0 N + N - 1$. Therefore

$$\begin{aligned}
&|\langle e_n - e_k, e_n\rangle| \\
&\quad = \Big|\sum_{i=k}^{n-1} \langle (x_{i+1} - x_i), (x_n - x^\dagger)\rangle\Big| \\
&\quad = \Big|\sum_{i=k}^{n-1} \alpha^{-1} \langle F'_{[i]}(x_i)(x_{i+1} - x_i) + F_{[i]}(x_i) - y_{[i]}, F'_{[i]}(x_i)(x_n - x^\dagger)\rangle\Big| \\
&\quad = \Big|\sum_{i=k}^{n-1} \alpha^{-1} \langle F'_{[i]}(x_i)(x_{i+1} - x_i) + F_{[i]}(x_i) - y_{[i]}, \\
&\qquad\qquad F'_{[i]}(x_i)(x_n - x_i) + F'_{[i]}(x_i)(x_i - x^\dagger)\rangle\Big| \\
&\quad \le \sum_{i=k}^{n-1} \alpha^{-1} \|B_i\| \left[(1+\eta)\,\|F_{[i]}(x_n) - F_{[i]}(x_i)\| + (1+\eta)\,\|F_{[i]}(x_i) - F_{[i]}(x^\dagger)\|\right] \\
&\quad \le \alpha^{-1}(1+\eta) \sum_{i=k}^{n-1} \|B_i\| \left[\|F_{[i]}(x_n) - y_{[i]}\| + 2\|F_{[i]}(x_i) - y_{[i]}\|\right]
\end{aligned} \tag{21}$$

With $i = i_0 N + i_1$ we get

$$\begin{aligned}
&\|F_{[i]}(x_n) - y_{[i]}\| \\
&\quad = \|F_{i_1}(x_{n_0 N + N - 1}) - y_{i_1}\| \\
&\quad \le \|F_{i_1}(x_{n_0 N + i_1}) - y_{i_1}\| + \sum_{s=i_1}^{N-2} \|F_{i_1}(x_{n_0 N + s + 1}) - F_{i_1}(x_{n_0 N + s})\| \\
&\quad \le \|F_{i_1}(x_{n_0 N + i_1}) - y_{i_1}\| + \frac{1}{1-\eta} \sum_{s=i_1}^{N-2} \|F'_{i_1}(x_{n_0 N + s})(x_{n_0 N + s + 1} - x_{n_0 N + s})\| \\
&\quad \le \|F_{i_1}(x_{n_0 N + i_1}) - y_{i_1}\| + \frac{C}{1-\eta} \sum_{s=i_1}^{N-2} \|x_{n_0 N + s + 1} - x_{n_0 N + s}\| \\
&\quad \le \Big(1 + \frac{C}{1-\eta}\Big) \sum_{s=i_1}^{N-2} \left\{\|x_{n_0 N + s + 1} - x_{n_0 N + s}\| + \|F_s(x_{n_0 N + s}) - y_s\|\right\}.
\end{aligned}$$

Hence, by minimality (20) we get

$$\|F_{[i]}(x_n) - y_{[i]}\| \;\le\; \Big(1 + \frac{C}{1-\eta}\Big) \sum_{s=0}^{N-1} \left\{\|x_{i_0 N + s + 1} - x_{i_0 N + s}\| + \|F_s(x_{i_0 N + s}) - y_s\|\right\}.$$

Inserting this into (21) we obtain

$$\begin{aligned}
&|\langle e_n - e_k, e_n\rangle| \\
&\quad \le 2\alpha^{-1}(1+\eta) \sum_{i=k}^{n-1} \|B_i\|\, \|F_{[i]}(x_i) - y_{[i]}\| + \Big(1 + \frac{C}{1-\eta}\Big)(1+\eta)\ \mathrm{sum}_2
\end{aligned} \tag{22}$$

with

$$\begin{aligned}
\mathrm{sum}_2 &= \sum_{i_0=k_0}^{n_0-1}\sum_{i_1=0}^{N-1} \alpha^{-1}\|F'_{i_1}(x_{i_0N+i_1})(x_{i_0N+i_1+1}-x_{i_0N+i_1}) \\
&\qquad +F_{i_1}(x_{i_0N+i_1})-y_{i_1}\| \\
&\qquad \sum_{s=0}^{N-1}\big\{\|x_{i_0N+s+1}-x_{i_0N+s}\|+\|F_s(x_{i_0N+s})-y_s\|\big\} \\
&\le N\sum_{i_0=k_0}^{n_0-1}\sum_{i_1=0}^{N-1}\Big\{\frac{1}{2}\Big(\alpha^{-1}\|F'_{i_1}(x_{i_0N+i_1})(x_{i_0N+i_1+1}-x_{i_0N+i_1}) \\
&\qquad +F_{i_1}(x_{i_0N+i_1})-y_{i_1}\|\Big)^2 \\
&\qquad + \|x_{i_0N+i_1+1}-x_{i_0N+i_1}\|^2+\|F_{i_1}(x_{i_0N+i_1})-y_{i_1}\|^2\Big\} \\
&\le \frac{N}{\min\{2\alpha,1\}}\sum_{i=k_0}^{n}\Big\{\alpha^{-1}\|B_i\|^2+\|x_{i_0N+i_1+1}-x_{i_0N+i_1}\|^2 \\
&\qquad +\|F_{i_1}(x_{i_0N+i_1})-y_{i_1}\|^2\Big\}.
\end{aligned}$$

Hence by (18a), (18c), (18d), (18b), both terms on the right hand side of (22) go to zero as $k,l\to\infty$. Analogously one shows that $\langle e_n-e_l,e_n\rangle\to 0$ as $l\to\infty$.

Thus, e_k is a Cauchy sequence and $x_k=x^\dagger-e_k$ converges to some element $x^+\in X$. Since the residuals $\|F_{[k]}(x_k)-y_{[k]}\|$ converge to zero, x^+ is a solution of (2).

Now assume $\mathcal{N}(\mathbf{F}'(x^\dagger))\subseteq\mathcal{N}(\mathbf{F}'(x))$, for $x\in B_\rho(x_0)$. Since $x_{k+1}-x_k$ is either zero or h_k, it follows from Lemma 2.1 b) that $x_{k+1}-x_k\in\mathcal{R}(F'_{[k]}(x_k)^*)\subset\mathcal{N}(F'_{[k]}(x_k))^\perp\subset\mathcal{N}(\mathbf{F}'(x_k))^\perp\subset\mathcal{N}(\mathbf{F}'(x^\dagger))^\perp$. An inductive argument shows that all iterates x_k are elements of $x_0+\mathcal{N}(\mathbf{F}'(x^\dagger))^\perp$. Therefore, $x^*\in x_0+\mathcal{N}(\mathbf{F}'(x^\dagger))^\perp$. By Remark 2, $x^\dagger$ is the only solution of (2) in $B_{\rho/2}(x_0)\cap(x_0+\mathcal{N}(\mathbf{F}'(x^\dagger))^\perp)$, and so the second assertion follows. □

Remark 3. In order to consider the variable choice of α according to (13) let us consider for the moment a condition which is slightly stronger than the tangential cone condition, namely the *range invariance condition*

(A2') There exist linear bounded operators $R_i(\bar x,x)$ satisfying

$$F'_i(\bar x) = R_i(\bar x,x)\,F'_i(x)\,,\quad \|R_i(\bar x,x)-I\|\le c_R\,,\qquad x,\bar x\in B_\rho(x_0)\,, \tag{23}$$

for some $0<c_R<1$.
Notice that from $F'_i(x)(\bar x-x)=\big(\int_0^1 R_i(x+\theta(\bar x-x),x)\,d\theta\big)^{-1}(F_i(\bar x)-F_i(x))$, it follows that (A2') implies (A2) with $\eta=c_R(1-c_R)^{-1}$.

If (A2) is substituted by (A2') in Theorem 3.1, the estimates can be improved to

$$
\begin{aligned}
&|\langle e_n - e_k, e_n\rangle| \\
&\quad\le \frac{c_R}{2(1-c_R)} \sum_{i_0=k_0}^{n_0-1}\sum_{i_1=0}^{N-1} \\
&\qquad \Big\{ \frac{\omega_{i_0N+i_1}}{\alpha_{i_0N+i_1}} \|F'_{i_1}(x_i)(x_{i_0N+i_1+1} - x_{i_0N+i_1} + F_{i_1}(x_{i_0N+i_1}) - y_{i_1}\|^2 \\
&\qquad + \tfrac{1}{q^2}\tfrac{\omega_{i_0N+i_1}}{\alpha_{i_0N+i_1}} \|F'_{i_1}(x_i)(x_{n_0N+i_1+1} - x_{n_0N+i_1}) + F_{i_1}(x_{n_0N+i_1}) - y_{i_1}\|^2 \Big\} \\
&\qquad + \frac{N}{2q^2} \sum_{i_0=k_0}^{n_0-1}\sum_{i_1=0}^{N-1} \\
&\qquad \Big\{ \frac{\omega_{i_0N+i_1}}{\alpha_{i_0N+i_1}} \|F'_{i_1}(x_i)(x_{i_0N+i_1+1} - x_{i_0N+i_1} + F_{i_1}(x_{i_0N+i_1}) - y_{i_1}\|^2 \\
&\qquad + \tfrac{\omega_{n_0N+i_1}}{\alpha_{n_0N+i_1}} \|F'_{i_1}(x_i)(x_{n_0N+i_1+1} - x_{n_0N+i_1}) + F_{i_1}(x_{n_0N+i_1}) - y_{i_1}\|^2 \Big\} \,.
\end{aligned} \tag{24}
$$

This suggests that if, instead of (20) the index n_0 is chosen from the more natural requirement

$$
\begin{aligned}
&\sum_{s=0}^{N-1} \tfrac{\omega_{n_0N+s}}{\alpha_{n_0N+s}} \|F'_s(x_{n_0N+s})(x_{n_0N+s+1} - x_{n_0N+s}) + F_s(x_{n_0N+s}) - y_s\|^2 \le \\
&\qquad \sum_{s=0}^{N-1} \tfrac{\omega_{i_0N+s}}{\alpha_{i_0N+s}} \|F'_s(x_{i_0N+s})(x_{i_0N+s+1} - x_{i_0N+s}) + F_s(x_{i_0N+s}) - y_s\|^2 \\
&\qquad\qquad \text{for all } i_0 \in \{k_0, \dots, l_0\} \,,
\end{aligned}
$$

then the analysis might extend to the variable parameter choice (13). Note however, that due to the factor $\frac{\omega_{i_0N+i_1}}{\alpha_{i_0N+i_1}}$ (instead of an in this context desirable i_0-independent factor $\frac{\omega_{n_0N+i_1}}{\alpha_{n_0N+i_1}}$) in (24), this is unfortunately not the case.

4. **Convergence for noisy data.** Throughout this section, we assume that (A1) - (A3) hold true and that x_k^δ, h_k, α, τ and q are defined by (4), (5) and (11).

Our main goal in this section is to prove that $x_{k_*^\delta}^\delta$ converges to a solution of (2) as $\delta \to 0$, where k_*^δ is defined in (7). The first step is to verify that, for noisy data, the stopping index k_*^δ is well defined.

Proposition 2. *Assume $\delta_{\min} := \min\{\delta_0, \dots \delta_{N-1}\} > 0$. Then k_*^δ in (7) is finite, and the estimate $k_*^\delta = O(\delta_{\min}^{-2})$ holds true. Moreover,*

$$
\|F_i(x_{k_*^\delta}^\delta) - y_i^\delta\| \;<\; \tau\delta_i \,, \qquad i = 0, \dots, N-1 \,. \tag{25}
$$

Proof. Assume by contradiction that for every $l \in \mathbb{N}$, there exists a $i(l) \in \{0, \dots, N-1\}$ such that $\|F_{i(l)}(x_{lN+i(l)}) - y_{i(l)}^\delta\| \ge \tau\delta_{i(l)}$. From Proposition 1 it follows that (14) holds for $k = 1, \dots, lN$. Summing over k, leads to

$$
-\|x_0 - x^*\|^2 \le \sum_{k=1}^{lN-1} 2\frac{\omega_k}{\alpha}\|B_k^\delta\| \left[(\eta - q)\,\|F_{[k]}(x_{k+1}^\delta) - y_{[k]}^\delta\| + (1+\eta)\delta_{[k]}\right] , \quad l \in \mathbb{N} \,.
$$

Using the fact that either $\omega_k = 0$ or $\|F_{[k]}(x_k^\delta) - y_{[k]}^\delta\| > \tau\delta_{[k]}$, we obtain

$$
\|x_0 - x^*\|^2 \;\ge\; \sum_{k=1}^{lN-1} 2\frac{\omega_k}{\alpha}\|B_k^\delta\|\,\delta_{[k]} \left[\tau(q-\eta) - (1+\eta)\right] . \tag{26}
$$

From equations (12), (26) and the fact that $\omega_{l'N+i(l')} = 1$ for all $l' \in \mathbb{N}$, we obtain

$$\|x_0 - x^*\|^2 \;\geq\; \left[\tau(q-\eta) - (1+\eta)\right] 2l\, \frac{q\tau\delta_{\min}^2}{\alpha}\,, \quad l \in \mathbb{N}\,. \tag{27}$$

Due to (11), the right hand side of (27) tends to $+\infty$ as $l \to \infty$, which gives a contradiction. Consequently, the minimum in (7) takes a finite value.

To prove $k_*^\delta = O(\delta_{\min}^{-2})$, it is enough to take $l = k_*^\delta/N \in \mathbb{N}$ in (27) and estimate

$$k_*^\delta \;\leq\; \alpha N \,\|x_0 - x^*\|^2 \left(2q\tau\left[\tau(q-\eta) - (1+\eta)\right]\delta_{\min}^2\right)^{-1}.$$

It remains to prove (25). Assume to the contrary that $\|F_i(x^\delta_{k_*^\delta}) - y_i^\delta\| \geq \tau\delta_i$ for some $i \in \{0,\dots,N-1\}$. From (6) and (7) it follows that $\omega_{k_*^\delta+i} = 1$ and $x^\delta_{k_*^\delta+i+1} = x^\delta_{k_*^\delta+i}$ respectively. Thus, from (14) and (12) it follows that

$$\begin{aligned} 0 \;&\leq\; 2\alpha^{-1}\|B^\delta_{k_*^\delta+i}\|\left[(\eta-q)\|F_i(x^\delta_{k_*^\delta+i}) - y_i^\delta\| + (1+\eta)\delta_i\right] \\ &\leq\; 2\alpha^{-1}\,\|F_i(x^\delta_{k_*^\delta}) - y_i^\delta\|^2\left[(\eta-q) + (1+\eta)\tau^{-1}\right]. \end{aligned}$$

However, since $\|F_i(x^\delta_{k_*^\delta}) - y_i^\delta\| \geq \tau\delta_{min} > 0$, the inequality above leads to $[(\eta-q)+(1+\eta)\tau^{-1}] \geq 0$. This contradicts (11), completing the proof of (25). □

Lemma 4.1. *Let $\delta_j = (\delta_{j,0},\dots,\delta_{j,N-1}) \in (0,\infty)^N$ be given with $\lim_{j\to\infty}\delta_j = 0$. Moreover, let $y^{\delta_j} = (y_0^{\delta_j},\dots,y_{N-1}^{\delta_j}) \in Y^N$ be a corresponding sequence of noisy data satisfying $\|y_i^{\delta_j} - y_i\| \leq \delta_{j,i}$, $i = 0,\dots,N-1$, $j \in \mathbb{N}$. Then, for each fixed $k \in \mathbb{N}$ we have $\lim_{j\to\infty} x_k^{\delta_j} = x_k$.*

Proof. The proof of Lemma 4.1 uses an inductive argument in k. First assume $k = 0$ and notice that $x_0^{\delta_j} = x_0$ for $j \in \mathbb{N}$. Now, take $k > 0$ and assume that for all $k' \leq k$ we have $\lim_{j\to\infty} x_{k'}^{\delta_j} = x_{k'}$. Two cases must be considered: If $\omega_k = 1$ we estimate

$$\begin{aligned} &\|x_{k+1}^{\delta_j} - x_{k+1}\|^2 \\ &\leq \|x_k^{\delta_j} - x_k\| + \|h_k^{\delta_j} - h_k\| \\ &\leq \|x_k^{\delta_j} - x_k\| \\ &\quad + \|(F'_{[k]}(x_k^\delta)^*F'_{[k]}(x_k^\delta) + \alpha I)^{-1}F'_{[k]}(x_k^\delta)^* - (F'_{[k]}(x_k)^*F'_{[k]}(x_k) + \alpha I)^{-1}F'_{[k]}(x_k)^*\| \\ &\qquad \cdot \|y_{[k]}^\delta - F_{[k]}(x_k^\delta)\| \\ &\quad + \|(F'_{[k]}(x_k)^*F'_{[k]}(x_k) + \alpha I)^{-1}F'_{[k]}(x_k)^*\| \cdot \|y_{[k]}^\delta - y_{[k]} - F_{[k]}(x_k^\delta) + F_{[k]}(x_k)\| \\ &\leq \|x_k^{\delta_j} - x_k\| \\ &\quad + 9/4\,\alpha^{-1}\|F'_{[k]}(x_k^\delta) - F'_{[k]}(x_k)\|\|y_{[k]}^\delta - F_{[k]}(x_k^\delta)\| \\ &\quad + 1/2\,\alpha^{-1/2}\left\{\delta_{[k]} + \|F_{[k]}(x_k^\delta) - F_{[k]}(x_k)\|\right\}, \end{aligned} \tag{28}$$

where we have used the identity

$$\begin{aligned} &(A^*A + \alpha I)^{-1}A^* - (B^*B + \alpha I)^{-1}B^* \\ &= (A^*A + \alpha I)^{-1}(A^* - B^*) + (A^*A + \alpha I)^{-1}(B^*B - A^*A)(B^*B + \alpha I)^{-1}B^* \\ &= (A^*A + \alpha I)^{-1}(A - B)^* \\ &\quad + (A^*A + \alpha I)^{-1}(A^*(B - A) + (B^* - A^*)B)(B^*B + \alpha I)^{-1}B^* \end{aligned}$$

and the estimates

$$\|(A^*A+\alpha I)^{-1}\| \le \alpha^{-1}\,,$$
$$\|(A^*A+\alpha I)^{-1}A^*\| \le 1/2\,\alpha^{-1/2}\,,$$
$$\|A(A^*A+\alpha I)^{-1}A^*\| \le 1\,.$$

for linear operators A, B. Otherwise, if $\omega_k = 0$ we have $x_{k+1}^{\delta_j} = x_k^{\delta_j}$ and $\|F_{[k]}(x_k^{\delta_j}) - y_{[k]}^{\delta_j}\| \le \tau\delta_{k,j}$. Therefore,

$$\begin{aligned}\|x_{k+1}^{\delta_j} - x_{k+1}\| &= \|x_k^{\delta_j} - (x_k+h_k)\| \le \|x_k^{\delta_j} - x_k\| + 1/2\,\alpha^{-1/2}\|F_{[k]}(x_k) - y_{[k]}\| \\ &\le \|x_k^{\delta_j} - x_k\| \\ &\quad + 1/2\,\alpha^{-1/2}\big\{\|F_{[k]}(x_k) - F_{[k]}(x_k^{\delta_j})\| + \|F_{[k]}(x_k^{\delta_j}) - y_{[k]}^{\delta_j}\| + \|y_{[k]}^{\delta_j} - y_{[k]}\|\big\} \\ &\le \|x_k^{\delta_j} - x_k\| + 1/2\,\alpha^{-1/2}\big\{\|F_{[k]}(x_k) - F_{[k]}(x_k^{\delta_j})\| + (\tau+1)\delta_{k,j}\big\}\,. \qquad (29)\end{aligned}$$

Thus, it follows from (28), (29), the continuity of $F_{[k]}$, $F'_{[k]}$, and the induction hypothesis that $\lim\limits_{j\to\infty} x_{k+1}^{\delta_j} = x_{k+1}$. □

Now we are ready to state a semi-convergence result for the loping Levenberg-Marquardt-Kaczmarz iteration. For the proof, both Proposition 2 and Lemma 4.1 are required.

Theorem 4.2 (Convergence for noisy data)**.** *Let $\delta_j = (\delta_{j,0}, \ldots, \delta_{j,N-1})$ be a given sequence in $(0,\infty)^N$ with $\lim_{j\to\infty}\delta_j = 0$, and let $y^{\delta_j} = (y_0^{\delta_j}, \ldots, y_{N-1}^{\delta_j}) \in Y^N$ be a corresponding sequence of noisy data satisfying*

$$\|y_i^{\delta_j} - y_i\| \le \delta_{j,i}\,, \quad i = 0,\ldots,N-1\,,\ j\in\mathbb{N}\,.$$

Denote by $k_*^j := k_*(\delta_j, y^{\delta_j})$ the corresponding stopping index defined in (7). Then $x_{k_*^j}^{\delta_j}$ converges to a solution x^+ of (2). Moreover, if (19) holds, then $x_{k_*^j}^{\delta_j}$ converges to $x^\dagger$.

Proof. The proof follows the lines of [4, Theor.3.6] (see also [11]) and is divided into two cases. For the first case Proposition 2 and Lemma 4.1 are needed. For the second case, we need Proposition 1, Theorem 3.1 and Lemma 4.1. □

5. Numerical experiment.

5.1. **Description of the model problem.** In this section we introduce a model which plays a key rule in inverse doping problems related to measurements of the current flow, namely the *linearized stationary bipolar case close to equilibrium.*

This model is obtained from the drift diffusion equations by linearizing the Voltage-Current (VC) map at $U \equiv 0$ [14, 2], where the function $U = U(x)$ denotes the applied potential to the semiconductor device. This simplification is motivated by the fact that, due to hysteresis effects for large applied voltage, the VC-map can only be defined as a single-valued function in a neighborhood of $U = 0$. Additionally, we assume that the electron mobility $\mu_n(x) = \mu_n > 0$ and hole mobility $\mu_p(x) = \mu_p > 0$ are constant and that no recombination-generation rate is present [16, 15].

Under these assumptions the Gateaux derivative of the VC-map Σ_C at the point $U = 0$ in the direction $h \in H^{3/2}(\partial\Omega_D)$ is given by the expression

$$\Sigma_C'(0)h = \int_{\Gamma_1} \left(\mu_n\, e^{V_{\rm bi}} \hat{u}_\nu - \mu_p\, e^{-V_{\rm bi}} \hat{v}_\nu\right) ds, \tag{30}$$

where the concentrations of electrons and holes $(\hat{u}, \hat{v})$ (written in terms of the Slotboom variables) solve

$$\begin{aligned}
\operatorname{div}(\mu_n e^{V^0} \nabla \hat{u}) &= 0 && \text{in } \Omega && \text{(31a)}\\
\operatorname{div}(\mu_p e^{-V^0} \nabla \hat{v}) &= 0 && \text{in } \Omega && \text{(31b)}\\
\hat{u} = -\hat{v} &= -h && \text{on } \partial\Omega_D && \text{(31c)}\\
\nabla \hat{u} \cdot \nu = \nabla \hat{v} \cdot \nu &= 0 && \text{on } \partial\Omega_N && \text{(31d)}
\end{aligned}$$

and the potential V^0 is the solution of the thermal equilibrium problem

$$\begin{aligned}
\lambda^2 \Delta V^0 &= e^{V^0} - e^{-V^0} - C(x) && \text{in } \Omega && \text{(32a)}\\
V^0 &= V_{\rm bi}(x) && \text{on } \partial\Omega_D && \text{(32b)}\\
\nabla V^0 \cdot \nu &= 0 && \text{on } \partial\Omega_N\,. && \text{(32c)}
\end{aligned}$$

Here $\Omega \subset \mathbb{R}^d$ is a domain representing the semiconductor device; the boundary $\partial\Omega$ of Ω is divided into two nonempty disjoint parts: $\partial\Omega = \overline{\partial\Omega_N} \cup \overline{\partial\Omega_D}$. The Dirichlet part of the boundary $\partial\Omega_D$ models the Ohmic contacts, where the potential V as well as the concentrations $\hat{u}$ and $\hat{v}$ are prescribed; the Neumann part $\partial\Omega_N$ of the boundary corresponds to insulating surfaces, thus a zero current flow and a zero electric field in the normal direction are prescribed; the Dirichlet part of the boundary splits into $\partial\Omega_D = \Gamma_0 \cup \Gamma_1$, where the disjoint boundary parts Γ_i, $i = 0, 1$, correspond to distinct contacts (differences in $U(x)$ between different segments of $\partial\Omega_D$ correspond to the applied bias between these two contacts).

The function $C(x)$ is the *doping profile* and models a preconcentration of ions in the crystal, so $C(x) = C_+(x) - C_-(x)$ holds, where C_+ and C_- are concentrations of negative and positive ions respectively. In those subregions of Ω in which the preconcentration of negative ions predominate (P-regions), we have $C(x) < 0$. Analogously, we define the N-regions, where $C(x) > 0$ holds. The boundaries between the P-regions and N-regions (where C changes sign) are called *pn-junctions*. Moreover, $V_{\rm bi}$ is a given logarithmic function [2].

5.2. **Inverse doping problem.** The inverse problem we are concerned with consists in determining the doping profile function C in (32) from measurements of the linearized VC-map $\Sigma_C'(0)$ in (30). Notice that we can split the inverse problem in two parts: The first step is to define the function $\gamma(x) := e^{V^0(x)}$, $x \in \Omega$, and solve the parameter identification problem

$$\begin{aligned}
\operatorname{div}(\mu_n \gamma \nabla \hat{u}) = 0 \quad &\text{in } \Omega & \qquad \operatorname{div}(\mu_p \gamma^{-1} \nabla \hat{v}) = 0 \quad &\text{in } \Omega\\
\hat{u} = -U(x) \quad &\text{on } \partial\Omega_D & \qquad \hat{v} = U(x) \quad &\text{on } \partial\Omega_D\\
\nabla \hat{u} \cdot \nu = 0 \quad &\text{on } \partial\Omega_N & \qquad \nabla \hat{v} \cdot \nu = 0 \quad &\text{on } \partial\Omega_N
\end{aligned} \tag{33}$$

for γ, from measurements of $[\Sigma_C'(0)](U) = \int_{\Gamma_1} (\mu_n \gamma \hat{u}_\nu - \mu_p \gamma^{-1} \hat{v}_\nu)\, ds$. The second step consists in the determination of the doping profile in $C(x) = \gamma(x) - \gamma^{-1}(x) - \lambda^2 \Delta(\ln \gamma(x))$, $x \in \Omega$. Since the evaluation of C from γ can be explicitly performed in a stable way, we shall focus on the problem of identifying the function parameter γ in (33).

Summarizing, the inverse doping profile problem in the linearized stationary bipolar model (close to equilibrium) for pointwise measurements of the current density reduces to the identification of the parameter γ in (33) from measurements of the DN map

$$\begin{array}{rccl} \Lambda_\gamma : & H^{3/2}(\partial\Omega_D) & \to & \mathbb{R}\,. \\ & U & \mapsto & \int_{\Gamma_1}(\mu_n\gamma\hat{u}_\nu - \mu_p\gamma^{-1}\hat{v}_\nu)\,ds \end{array}$$

In the formulation of the inverse problem we shall take into account some restrictions imposed by the practical experiments, namely

i) The voltage profile $U \in H^{3/2}(\partial\Omega_D)$ must satisfy $U|_{\Gamma_1} = 0$ (in practice, U is chosen to be piecewise constant on the contact Γ_1 and to vanish on Γ_0);

ii) The identification of γ has to be performed from a finite number $N \in \mathbb{N}$ of measurements, i.e. from the data $\left\{(U_i, \Lambda_\gamma(U_i))\right\}_{i=0}^{N-1} \in \left[H^{3/2}(\Gamma_0)\times\mathbb{R}\right]^N$.

Therefore, we can write this particular inverse doping profile problem in the abstract formulation of system (2), namely

$$F_i(\gamma) \;=\; \Lambda_\gamma(U_i) \;=:\; y_i\,,\; i = 0,\ldots,N-1\,, \tag{34}$$

where U_i are fixed voltage profiles chosen as above; $X := L^2(\Omega) \supset D(F_i) := \{\gamma \in L^\infty(\Omega);\, 0 < \gamma_m \le \gamma(x) \le \gamma_M$, a.e. in $\Omega\}$; $Y := \mathbb{R}$.

To the best of our knowledge, assumptions (A1) - (A3) are not satisfied for the Dirichlet-to-Neumann operator Λ_γ. Therefore, although the operators F_i are continuous [2], the analytical convergence results of the previous sections do not apply for system (34).

In the following numerical experiment we assume that nine measurements are available, i.e. $N = 9$, in (34). The domain $\Omega \subset \mathbb{R}^2$ is the unit square, and the boundary parts are defined as follows

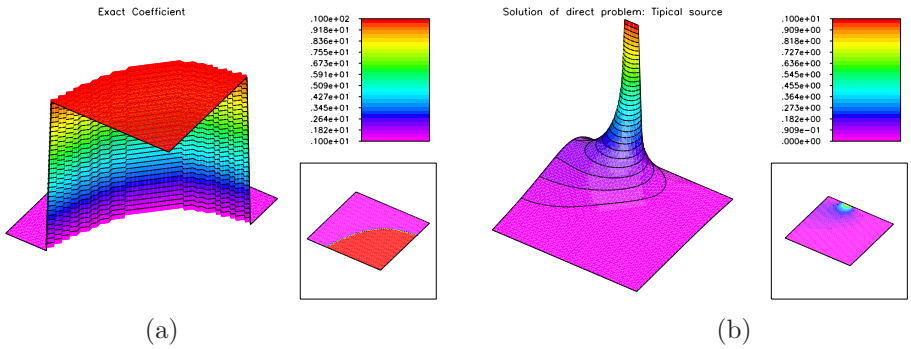
$$\begin{array}{c} \Gamma_1 \;:=\; \{(x,1)\,;\; x\in(0,1)\}\,,\quad \Gamma_0 \;:=\; \{(x,0)\,;\; x\in(0,1)\}\,, \\ \partial\Omega_N \;:=\; \{(0,y)\,;\; y\in(0,1)\}\cup\{(1,y)\,;\; y\in(0,1)\}\,. \end{array}$$

The fixed inputs U_i, are chosen to be piecewise constant functions supported in Γ_0

$$U_i(x) \;:=\; \begin{cases} 1, & |x - x_i| \le 2^{-4} \\ 0, & \text{else} \end{cases} \qquad i = 0,\ldots,N-1\,,$$

where the points x_i are uniformly spaced in $[0,1]$. The parameter γ to be identified is shown in Figure 1 (a) (notice that $\gamma(x) \in [0,10]$ a.e. in Ω). In Figure 1 (b) a typical voltage source U_i (applied at Γ_0) and the corresponding solution $\hat{u}$ of (33) are shown. In these two pictures, as well as in the forthcoming ones, Γ_1 is the lower left edge and Γ_0 is the top right edge (the origin corresponds to the upper right corner).

For comparison purposes we implemented both the L-LMK and the L-LK iteration. The initial condition for both methods is presented in Figure 2 (c). The linear system in the L-LMK is solved inexactly by three CG steps, so the numerical effort for one step of the L-LMK is three times the one for one step of the L-LK. In the computations it turned out that the performance of the L-LMK is not very sensitive to the value of α. The "exact" data y_i, $i = 0,\ldots,8$, were obtained by solving the direct problems (33) using a finite element type method and adaptive mesh refinement (approx 8000 elements). Artificially generated (random) noise of 5% was introduced to y_i in order to generate the noisy data y_i^δ for the inverse problem. In order to avoid inverse crimes, a coarser grid (with approx 2000 elements) was used in the finite element method to implement the L-LMK and L-LK iterations.



(a) (b)

FIGURE 1. In picture (a) the parameter γ to be identified is shown. In picture (b) a typical voltage source U_i (boundary condition) and the corresponding solution $\hat{u}$ of (33) are shown.

For both iterative methods the same stopping rule (7) was used. We assumed exact knowledge of the noise level and chose $\tau = 2$. In Figure 2 (e) we plot, for each one of the iterations, the number of non-loped inner steps in each cycle. For the L-LMK iteration (solid red line) the stopping criterion is achieved after 24 cycles, while the L-LK iteration (dashed blue line) is stopped after 205 cycles. In this picture one also observes that the computational effort to perform the L-LMK cycles decreases much faster than in the L-LK iteration.

The quality of the final result obtained with the L-LMK method can be seen at Figure 2 (b), where the iteration error for the approximation obtained after 24 cycles is depicted. In Figure 2 (a) we present the iteration error for the L-LK iteration after 205 cycles, when the stopping criterion is reached.

Since the same noisy data and the same stopping rule were used for both iterations, the quality of the final results in Figures 2 (a) and (b) is similar. However, the L-LMK iteration needed a much smaller number of cycles to reach the stopping criterion than the L-LK iteration (starting from the same initial guess). Moreover, the number of actually performed inner steps per cycle is much smaller for the L-LMK iteration. All these observations lead us to conclude that the L-LMK iteration is numerically much more efficient than the L-LK iteration.

6. **Conclusions.** In this article we propose a new iterative method for inverse problems of the form (2), namely the L-LMK iteration. In the case of exact data this method reduces to the LMK iteration.

In the L-LMK iteration we omit an update of the LMK iteration (within one cycle) if the corresponding i-th residual is below some threshold. Consequently, the L-LMK method is not stopped until all residuals are below the specified threshold. We provide a complete convergence analysis for the L-LMK iteration, proving that it is a convergent regularization method in the sense of [6]. Moreover, we provide a numerical experiment for a nonlinear inverse doping problem and observe that the L-LMK iteration generates results that are comparable with other Kaczmarz type iterations. The specific example considered in Section 5.2 indicates that the L-LMK iteration is numerically more efficient than the L-LK iteration.

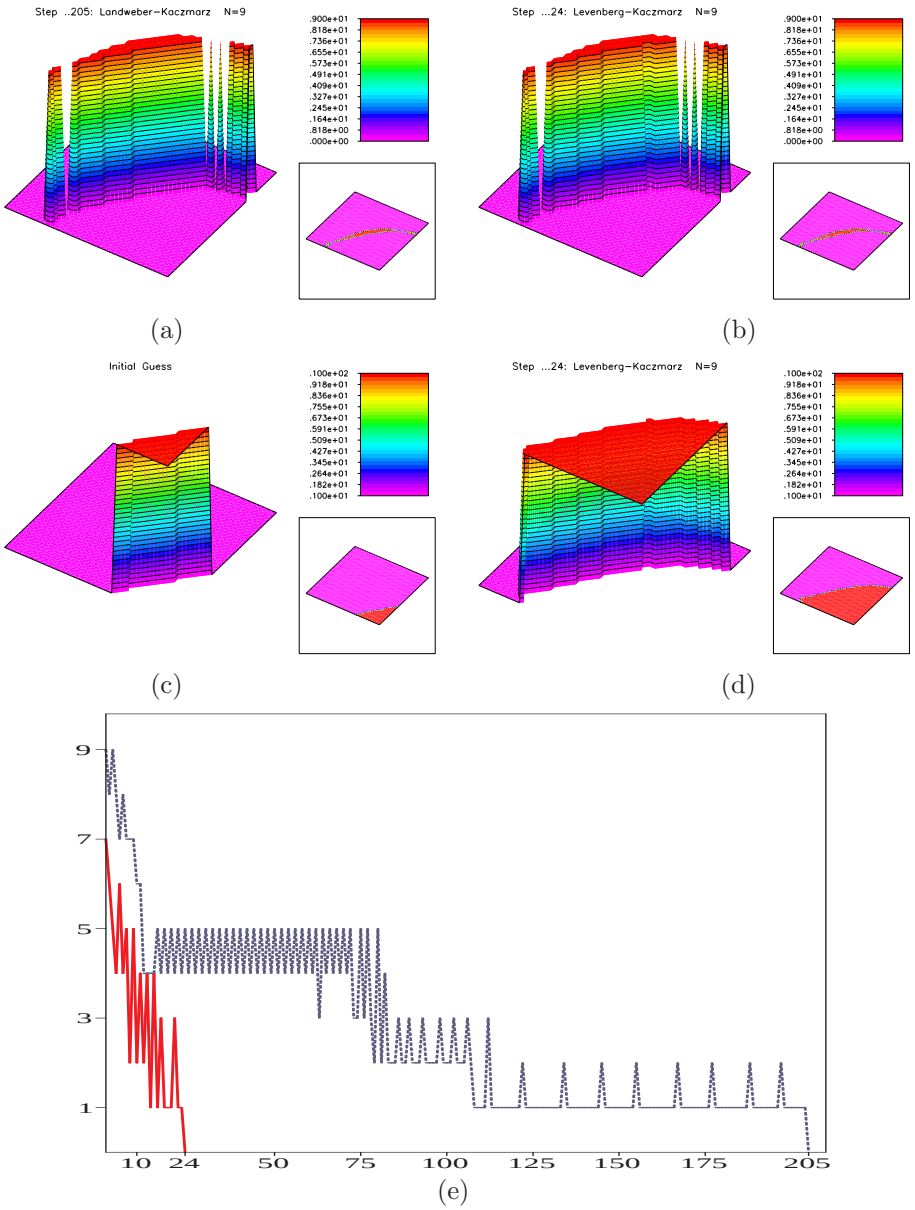


FIGURE 2. Numerical experiment with noisy data: On picture (a) the iterative error obtained with the L-LK method after 205 cycles. On picture (b) the iterative error obtained with the L-LMK method after 24 cycles. On picture (c) the initial condition for both iterative methods. On picture (d) the approximate solution obtained with the L-LMK method after 24 cycles. On picture (e) the number of non-loped inner steps in each cycle for L-LMK (solid red) and L-LK (dashed blue), respectively.

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