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Level-set approaches of L_2-type for recovering shape and contrast in ill-posed problems

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We investigate level-set-type approaches for solving ill-posed inverse problems, under the assumption that the solution is a piecewise constant function. Our goal is to identify the level sets as well as the level values of the unknown parameter function. Two distinct level-set frameworks are proposed for solving the inverse problem. Among both of them, the level-set function is assumed to be in L_2. Corresponding Tikhonov regularization approaches are derived and analysed. Existence of minimizers for the Tikhonov functionals is proven. Moreover, convergence and stability results of the variational approaches are established, characterizing the Tikhonov approaches as regularization methods.

Keywords: ill-posed problems; level set methods; piecewise constant level set; regularization; Tikhonov functionals

1. Introduction

Several inverse problems of interest consist of identifying an unknown physical quantity $u \in X$, that can be represented by a piecewise constant real function over a bounded given domain Ω, from the set of data $y \in Y$, where X, Y are Hilbert spaces. The relation between the unknown parameter function and the problem data is described by the model

$$F(u) = y, \tag{1}$$

where F: $D(F) \subset X \to Y$, what corresponds to the fact that the set of data is obtained by indirect measurements of the parameter. In practical applications the exact data $y \in Y$ is, in general, not known. One is given only approximate measured data $y^\delta \in Y$, corrupted by noise of level $\delta > 0$ and satisfying

$$\|y^\delta - y\|_Y \leq \delta. \tag{2}$$

Level-set approaches in the case where the unknown function u is piecewise constant distinguishing between two given values, were considered in [1–9]. In this case, since the level values of u are known, one needs only to identify the level sets of u, i.e. the inverse problem reduces to a shape identification problem. In the case where the unknown

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function u is a piecewise constant function distinguishing between several given values, multiple level set approaches were considered in [6,10,11]. For numerical implementations of level-set-type methods for solving inverse problems, we refer the reader to [12,13].

If the level values of u are also unknown, the inverse problem becomes harder, since one has to identify both the level sets as well as the level values of the unknown parameter u. In this case, the dimension of the parameter space increases by the number of unknown level values.

Our starting point in this article is the assumption that the parameter function u in (1) is a piecewise constant function assuming two distinct unknown values, i.e. $u(x) \in \{c^1, c^2\}$; a.e. in $\Omega \subset \mathbb{R}^d$. In this case one can assume the existence of an open measurable set $D \subset\subset \Omega$ s.t. $u(x) = c^1$, $x \in D =: D_1$ and $u(x) = c^2$, $x \in \Omega / D =: D_2$.

In this article we propose two level-set approaches to represent the unknown parameter u:

(1) **Standard level set approach (sLS):** This approach consists in introducing the level set function ϕ, in $L_2(\Omega)$, which acts as a regularization on the parameter space. We use the Heaviside projector, H, to represent a solution of (1) in the form

$$u = c^1 H(\phi) + c^2(1 - H(\phi)) =: P_s(\phi, c^j). \tag{3}$$

Notice that $u(x) = c^i$, $x \in D_i$, where the sets D_i are defined by $D_1 = \{x \in \Omega;\ \phi(x) \geq 0\}$ and $D_2 = \{x \in \Omega;\ \phi(x) < 0\}$. Thus, the operator P_s establishes a straightforward relation between the level sets of ϕ and the sets D_i representing our *a priori* knowledge about the solution u.
Within this sLS framewok, the inverse problem in (1), with data given as in (2), can be written in the form of the operator equation

$$F(P_s(\phi, c^j)) = y^\delta. \tag{4}$$

In order to obtain approximate solutions to (4), we propose the minimization of the Tikhonov functional

$$\mathcal{G}_{\alpha,s}(\phi, c^j) := \|F(P_s(\phi, c^j)) - y^\delta\|_Y^2 + \alpha\Big\{\beta_1 |H(\phi)|_{\mathrm{BV}} + \beta_2 \|\phi\|_{L_2(\Omega)}^2 + \beta_3 \|c^j\|_{\mathbb{R}^2}^2\Big\} \tag{5}$$

based on TV–L_2 penalization. Here $|H(\phi)|_{BV}$ is the functional defined by $|\psi|_{BV} := \sup\{\int_\Omega \psi \nabla \cdot \theta \, \mathrm{d}x;\quad \theta \in C_c^1(\Omega; \mathbb{R}^n), \|\theta\|_{L^\infty(\Omega)} \leq 1\}$. Concurrent approaches were proposed in [5,6,14] (using TV penalization) and [3,11] (using TV-H^1 penalization).

(2) **Piecewise constant level set approach (pcLS):** In the sequel, we introduce the *piecewise constant* level set function $\phi \in L_2(\Omega)$ such that $\phi(x) = i$, $x \in D_i$, $i = 1, 2$. Then, defining the auxiliary functions $\psi_1(t) := 2 - t$ and $\psi_2(t) := t - 1$, we represent the characteristic functions of the subdomains D_i in the form $\chi_{D_i}(x) = \psi_i(\phi(x))$. Consequently, a solution of (1) can be written in the form

$$u = c^1 \psi_1(\phi) + c^2 \psi_2(\phi) =: P_{pc}(\phi, c^j). \tag{6}$$

Notice that the piecewise constant assumption on ϕ corresponds to the constraint $\mathcal{K}(\phi) = 0$, where $\mathcal{K}(\phi) := (\phi - 1)(\phi - 2)$ is a smooth non-linear operator.

Within this framework, the inverse problem in (1), with data given as in (2), can be written in the form of the abstract operator equation

$$\begin{cases} F(P_{pc}(\phi, c^j)) = y^\delta, \\ \text{s.t. } \phi \in \{L_2(\Omega);\ \ \mathcal{K}(\phi) = 0\}. \end{cases} \tag{7}$$

Approximate solutions to (7) can be obtained by minimizing the Tikhonov functional

$$\mathcal{G}_{\alpha,pc}(\phi, c^j) := \|F(P_{pc}(\phi, c^j)) - y^\delta\|_Y^2 + \beta_3\|\mathcal{K}(\phi)\|_{L_1} + \alpha\Big\{\beta_1|P_{pc}(\phi, c^j)|_{\mathrm{BV}} + \beta_2\|c^j\|_{\mathbb{R}^2}^2\Big\}. \tag{8}$$

Notice that the minimization of the functional $\mathcal{G}_{\alpha,\,pc}$ furnishes a regularized solution to the system of operator equations:

$$\begin{bmatrix} F(P_{pc}(\phi, c^j)) \\ \mathcal{K}(\phi) \end{bmatrix} = \begin{bmatrix} y^\delta \\ 0 \end{bmatrix}.$$

The penalization term in (8) corresponds basically to a *TV* regularization strategy.

One should notice that, in the limit case $\alpha \to 0$ (Recall that in the presence of noise, $\delta > 0$, the regularization parameter α is a function of the noisy level, i.e., $\alpha = \alpha(\delta)$; see Theorem 7.) the minimizers $(\psi_\alpha, c_\alpha^j)$ of $\mathcal{G}_{\alpha,pc}$ converge to some limit $(\overline{\psi}, \overline{c}^j)$ satisfying $F(P_{pc}(\overline{\psi}, \overline{c})) = y$ and $\mathcal{K}(\overline{\psi}) = 0$. Thus, the limit level-set function $\overline{\psi}$ is indeed piecewise constant (as suggested by the name of the approach).

It is worth noticing that α is the unique regularization parameter in the Tikhonov functionals (5) and (8). The constants β_j appearing in these functionals play the role of scaling factors, which may allow the introduction of *a priori* information about the solution. In particular, the factor β_3 in (8) relates to a very relevant *a priori* information, namely the fact that the unknown parameter is piecewise constant (Remark 5). In Section 4 we describe in detail the factors β_j which are effectively used in the numerical experiments with exact and noisy data.

This article is outlined as follows. In Section 2 we introduce the concept of generalized minimizers for the functional $\mathcal{G}_{\alpha,s}$ in (5). Basic properties of the generalized minimizers are verified, as well as regularization properties of the penalization term of $\mathcal{G}_{\alpha,s}$. Moreover, we derive a convergence analysis for the Tikhonov method related to the sLS approach. We prove a well-posedness result, and also convergence results for exact and noisy data. In Section 3 we derive the convergence analysis for the pcLS approach. Section 4 is devoted to numerical experiments. Level-set-type methods based on the sLS and pcLS approaches are implemented for solving a two-dimensional inverse potential problem.

2. The sLS approach

We shall consider the model problem described as in Section 1 under the following general assumptions:

(A1) $\Omega \subseteq \mathbb{R}^d$, $d = 2$, is bounded with piecewise C^1 boundary $\partial\Omega$.
(A2) The operator F: $\mathbb{D} \subset L_p(\Omega) \to Y$ is continuous and Fréchet-differentiable on $\mathbb{D}$ with respect to the L_p-topology, where $1 \le p < d/(d-1) = 2$.
(A3) ε, α and β_j, $j = 1, 2, 3$ denote positive parameters.
(A4) Equation (1) has a solution, i.e. there exists a $u \in L_\infty(\Omega)$ satisfying $F(u) = y$; there exists a function $\phi \in L_2(\Omega)$ satisfying $|\nabla\phi| \neq 0$ in a neighbourhood of $\{\phi = 0\}$ such that $H(\phi) = z \in L_\infty(\Omega)$ and there exist constants values $c^j \in \mathbb{R}$ such that $P_s(z, c^j) = u$.

For each $\varepsilon > 0$, we define the operator

$$P_{s,\varepsilon}(\phi, c^j) := c^1 H_\varepsilon(\phi) + c^2(1 - H_\varepsilon(\phi)) , \tag{9}$$

where H_ε is the smooth approximation to H given by

$$H_\varepsilon(t) := \begin{cases} 1 + t/\varepsilon & \text{for } t \in [-\varepsilon, 0] \\ H(t) & \text{for } t \in \mathbb{R}/[-\varepsilon, 0] \end{cases}.$$

2.1. *The concept of generalized minimizers*

In order to guarantee the existence of a minimizer of $\mathcal{G}_{\alpha,s}$ in (5), we adapt to the level-set framework described above, the concept of generalized minimizers formulated in [3].

Definition 1 Let the operators H, P_s, H_ε and $P_{s,\varepsilon}$ be defined as above.

(a) A **vector** $(z, \phi, c^j) \in L_\infty(\Omega) \times L_2(\Omega) \times \mathbb{R}^2$ is called **admissible** when there exists a sequence $\{\phi_k\}$ of $L_2(\Omega)$-functions satisfying $\lim_{k\to\infty} \|\phi_k - \phi\|_{H^{-1}(\Omega)} = 0$, and there exists a sequence $\{\varepsilon_k\} \in \mathbb{R}^+$ converging to zero such that $H_{\varepsilon_k}(\phi_k) \in L_p(\Omega)$ and $\lim_{k\to\infty} \|H_{\varepsilon_k}(\phi_k) - z\|_{L_p(\Omega)} = 0$.

(b) A **generalized minimizer** of $\mathcal{G}_{\alpha,s}$ is considered to be any admissible vector (z, ϕ, c^j) minimizing

$$\mathcal{G}_\alpha(z, \phi, c^j) := \|F(q(z, c^j)) - y^\delta\|_Y^2 + \alpha R(z, \phi, c^j) \tag{10}$$

over the set of admissible vectors, where q: $L_\infty(\Omega) \times \mathbb{R}^2 \ni (z, c^j) \mapsto c^1 z + c^2(1 - z) \in L_\infty(\Omega)$, and the functional R is defined by

$$R(z, \phi, c^j) := \rho(z, \phi) + \beta_3 \|c^j\|_{\mathbb{R}^2}^2, \tag{11}$$

with $\rho(z, \phi) := \inf\{\liminf_{k\to\infty}(\beta_1 |H_{\varepsilon_k}(\phi_k)|_{\mathrm{BV}} + \beta_2 \|\phi_k\|_{L_2}^2)\}$. Here the infimum is taken over all sequences $\{\varepsilon_k\}$ and $\{\phi_k\}$ characterizing (z, ϕ, c^j) as an admissible vector.

2.2. *Preliminary results*

In the sequel we investigate relevant properties of the admissible vectors as well as properties of the penalization functional R in (11). We start by verifying some basic properties of the operators $P_{s,\varepsilon}$, H_ε and q that will be necessary in the subsequent analysis.

Lemma 1 *Let Ω and p be given as in (A1), (A2). The following assertions hold true.*

(i) *Let $\{z_k\}$ be a sequence in $L_\infty(\Omega)$ converging to some element $z \in L_\infty(\Omega)$ in the L_p-topology and $\{c_k^j\}$ be sequences of real numbers converging to c^j, $j = 1, 2$. Then $q(z_k, c_k^j)$ converges to $q(z, c^j)$ in the L_p-topology.*

(ii) *Let $(z, \phi) \in L_\infty(\Omega) \times L_2(\Omega)$ be such that $H_\varepsilon(\phi) \to z$ in $L_p(\Omega)$ as $\varepsilon \to 0$ and let $c^j \in \mathbb{R}$. Then $P_{s,\varepsilon}(\phi, c^j) \to q(z, c^j)$ in $L_p(\Omega)$ as $\varepsilon \to 0$.*

Proof It is enough to prove assertion (i). Since Ω is bounded, the constant functions are in $L_p(\Omega)$. Therefore,

$$\begin{aligned}
&\|q(z_k, c_k^j) - q(z, c^j)\|_{L_p(\Omega)} \\
&= \|c_k^1 z_k + c_k^2(1-z_k) - c^1 z - c^2(1-z)\|_{L_p(\Omega)} \\
&= \|c_k^1(z_k - z) + (c_k^1 - c^1)z + c_k^2\big[(1-z_k) - (1-z)\big] + (c_k^2 - c^2)(1-z)\|_{L_p(\Omega)} \\
&\le |c_k^1|\|z_k - z\|_{L_p(\Omega)} + |c_k^1 - c^1|\|z\|_{L_p(\Omega)} + |c_k^2|\|z_k - z\|_{L_p(\Omega)} + |c_k^2 - c^2|\|1-z\|_{L_p(\Omega)}
\end{aligned}$$

and the assertion follows. ■

Lemma 2 *Let (z_k, ϕ_k, c_k^j) be a sequence of admissible vectors converging in $L_p(\Omega) \times H^{-1}(\Omega) \times \mathbb{R}^2$ to some (z, ϕ, c^j) in $L_\infty(\Omega) \times L_2(\Omega) \times \mathbb{R}^2$. Then (z, ϕ, c^j) is also an admissible vector.*

Sketch of the proof For each $k \in \mathbb{N}$, it follows from Definition 1 that there exists a sequence $\{\phi_k^l\}$ in $L_2(\Omega)$ and a sequence $\{\varepsilon_k^l\}$ in $\mathbb{R}^+$such that as $l \to \infty$ we have $\phi_k^l \to \phi_k$ in $H^{-1}(\Omega)$ and $H_{\varepsilon_k^l}(\phi_k^l) \to z_k$ in $L_p(\Omega)$. Thus, we can select a monotone increasing index function γ: $\mathbb{N} \to \mathbb{N}$ such that

$$\varepsilon_k^{\gamma(k)} \le \frac{1}{2}\varepsilon_{k-1}^{\gamma(k-1)}, \quad \big\|\phi_k^{\gamma(k)} - \phi_k\big\|_{H^{-1}(\Omega)} \le k^{-1}, \quad \big\|H_{\varepsilon_k^{\gamma(k)}}\big(\phi_k^{\gamma(k)}\big) - z_k\big\|_{L_p(\Omega)} \le k^{-1},$$

for every $k \in \mathbb{N}$. Now, the lemma follows arguing with the triangular inequality. ■

In the sequel, we prove coercivity and weak lower semi-continuity of the penalization functional R. These properties are fundamental for the convergence analysis in Section 2.3. First, however, we briefly recall some facts about the space $\mathtt{BV}(\Omega)$. For a proof, we refer the reader to [15, Chap. 5].

Lemma 3 *The following assertions hold true*:

(i) *The semi-norm $|\cdot|_{BV}$ is weakly lower semi-continuous with respect to L_p-convergence, i.e., if $\{x_k\} \in \mathtt{BV}(\Omega)$ converges to x in the L_p-norm, then $x \in \mathtt{BV}(\Omega)$ and $|x|_{BV} \le \liminf_{k\to\infty}|x_k|_{BV}$.*

(ii) *$\mathtt{BV}(\Omega)$ is compactly embedded in $L_p(\Omega)$ for $1 \le p < d/(d-1)$. Consequently, any bounded sequence $\{x_k\} \in \mathtt{BV}(\Omega)$ has a subsequence converging in $L_p(\Omega)$ to some $x \in \mathtt{BV}(\Omega)$.*

Lemma 4 *The functional R in* (11) *is coercive on the set of admissible vectors.*

Sketch of the proof Let (z, ϕ, c^j) be an admissible vector. From the definition of $\rho(z, \phi)$ and the definition of admissible vectors, we can guarantee the existence of sequences $\{\phi_k\} \in L_2(\Omega)$ and $\{\varepsilon_k\} \in \mathbb{R}^+$such that $\varepsilon_k \to 0$, $\phi_k \to \phi$ in $H^{-1}(\Omega)$, $H_{\varepsilon_k}(\phi_k) \to z$ in $L_p(\Omega)$, and [11, Lemma 3]

$$\rho(z, \phi) = \liminf_{k\to\infty}\big\{\beta_1 |H_{\varepsilon_k}(\phi_k)|_{\mathtt{BV}} + \beta_2\|\phi_k\|_{L_2(\Omega)}^2\big\}. \tag{12}$$

From (12), the weak lower semi-continuity of the L_2-norm, and part (i) of Lemma 3, it follows that

$$\rho(z, \phi) \ge \ \beta_1 \liminf_{k\to\infty} |H_{\varepsilon_k}(\phi_k)|_{\mathtt{BV}} + \beta_2 \liminf_{k\to\infty}\|\phi_k\|_{L_2(\Omega)}^2 \ge \beta_1|z|_{\mathtt{BV}} + \beta_2\|\phi\|_{L_2(\Omega)}^2. \tag{13}$$

Thus, it follows from (11), (13) that $\beta_1|z|_{\mathrm{BV}} + \beta_2\|\phi\|^2_{L_2(\Omega)} + \beta_3\|c^j\|^2_{\mathbb{R}^2} \le R(z,\phi,c^j)$, concluding the proof. ■

Lemma 5 *The functional R in* (11) *is weak lower semi-continuous on the set of admissible vectors, i.e. given a sequence $\{(z_k,\phi_k,c^j_k)\}$ of admissible vectors such that $z_k \to z$ in $L_p(\Omega)$, $\phi_k \rightharpoonup \phi$ in $L_2(\Omega)$, $c^j_k \to c^j$ in $\mathbb{R}$, for some admissible vector (z,ϕ,c^j), then*

$$R(z,\phi,c^j) \le \liminf_{k\to\infty} R(z_k,\phi_k,c^j_k).$$

Sketch of the proof Since the norm in $\mathbb{R}^2$ is lower semi-continuous, it is enough to prove the weak lower semi-continuity of ρ. We argue by contradiction. Let $\{(z_k,\phi_k,c^j_k)\}$ and (z,ϕ,c^j) be given as above and assume that $\rho(z,\phi) > \liminf_{k\to\infty}\rho(z_k,\phi_k)$. Consequently, there exists a constant $\underline{c} > 0$ such that $\rho(z,\phi) \ge \underline{c} > \liminf_{k\in\mathbb{N}}\rho(z_k,\phi_k)$. Arguing as in [11, Lemma 5] we prove the following Claim.

Claim For every sequence $\{(z_l,\phi_l,c^j_l)\}$ of admissible vectors satisfying $z_l \to z$ in $L_p(\Omega)$ and $\phi_l \to \phi$ in $H^{-1}(\Omega)$ such that $\rho(z_l,\phi_l) \le \overline{c}$, we have $\rho(z,\phi) \le \overline{c}$.

Notice that this claim is a sufficient condition for the weak lower semi-continuity of ρ. Indeed, if the claim holds true, the constant $\underline{c}$ above cannot exist. ■

2.3. *Convergence analysis*

Our first goal is to prove that for any positive parameters α, β_1, β_2, β_3, the functional $\mathcal{G}_{\alpha,s}$ in (5) is well posed.

Theorem 6 *The functional $\mathcal{G}_{\alpha,s}$ in* (5) *attains minimizers on the set of admissible vectors.*

Proof Notice that the set of admissible vectors is not empty, since $(0,0,0,0)$ is admissible. Let $\{(z_k,\phi_k,c^j_k)\}$ be a minimizing sequence for $\mathcal{G}_\alpha$, i.e. a sequence of admissible vectors satisfying $\mathcal{G}_\alpha(z_k,\phi_k,c^j_k) \to \inf\mathcal{G}_\alpha \le \mathcal{G}_\alpha(0,0,0,0) < \infty$. Then, $\{\mathcal{G}_\alpha(z_k,\phi_k,c^j_k)\}$ is a bounded sequence of real numbers. Therefore, $\{(z_k,\phi_k,c^j_k)\}$ is uniformly bounded in $\mathrm{BV}\times L_2\times\mathbb{R}^2$. Thus, Lemma 3, the Sobolev compact embedding theorem [16] and the Bolzano–Weierstraß theorem guarantee the existence of a subsequence (denoted again by $\{(z_k,\phi_k,c^j_k)\}$) and the existence of $(z,\phi,c^j)\in L_p(\Omega)\times L_2(\Omega)\times\mathbb{R}^2$ such that $\phi_k \rightharpoonup \phi$ in $L_2(\Omega)$, $\phi_k \to \phi$ in $H^{-1}(\Omega)$, $z_k \to z$ in $L_p(\Omega)$ and $c^j_k \to c^j$ in $\mathbb{R}$.

From Lemma 2 we conclude that (z,ϕ,c^j) is an admissible vector. Moreover, from Lemma 5 together with the continuity of F and q we obtain

$$\begin{aligned}\inf\mathcal{G}_\alpha &= \lim_{k\to\infty}\mathcal{G}_\alpha(z_k,\phi_k,c^j_k) = \liminf_{k\to\infty}\left\{\|F(q(z_k,c^j_k)) - y^\delta\|^2_Y + \alpha R(z_k,\phi_k,c^j_k)\right\}\\ &\ge \|F(q(z,c^j)) - y^\delta\|^2_Y + \alpha R(z,\phi,c^j) = \mathcal{G}_\alpha(z,\phi,c^j),\end{aligned}$$

proving that (z,ϕ,c^j) minimizes $\mathcal{G}_\alpha$. ■

In the next theorem we present the main convergence and stability results. The proofs use classical techniques from the analysis of Tikhonov-type regularization methods (see, e.g. [17,18]) and will be omitted.

Theorem 7 *The following assertions hold true.*

(i) **[Convergence for exact data]** *Assume that we have exact data, i.e. $y^\delta = y$ and $\beta_j > 0$, $j = 1,2,3$. For every $\alpha > 0$ denote by $(z_\alpha,\phi_\alpha,c^j_\alpha)$ a minimizer of $\mathcal{G}_\alpha$ on the set of* *admissible vectors. Then, for every sequence of positive numbers $\{\alpha_k\}$ converging to zero there exists a subsequence, denoted again by $\{\alpha_k\}$, such that $(z_{\alpha_k}, \phi_{\alpha_k}, c^j_{\alpha_k})$ is strongly convergent in $L_p(\Omega) \times H^{-1}(\Omega) \times \mathbb{R}^2$. Moreover, the limit is a solution of* (4).

(ii) **[Convergence for noisy data]** *Let $\alpha = \alpha(\delta)$ be a function satisfying $\lim_{\delta\to 0} \alpha(\delta) = 0$ and $\lim_{\delta\to 0} \delta^2\alpha(\delta)^{-1} = 0$. Moreover, let $\{\delta_k\}$ be a sequence of positive numbers converging to zero and $\{y^{\delta_k}\} \in Y$ be corresponding noisy data satisfying* (2). *Then, there exist a subsequence, denoted again by $\{\delta_k\}$, and a sequence $\{\alpha_k := \alpha(\delta_k)\}$ such that $(z_{\alpha_k}, \phi_{\alpha_k}, c^j_{\alpha_k})$ converges in $L_p(\Omega) \times H^{-1}(\Omega) \times \mathbb{R}^2$ to solution of* (4).

3. The pcLS approach

In the sequel, we consider the model problem described in the introduction under assumptions (A1)–(A3). Moreover, we also require

(A4′) There exists $u \in L_\infty(\Omega)$ satisfying $F(u) = y$. Moreover, there exists a function $\phi \in \mathtt{BV}(\Omega) \subset L_2(\Omega)$ and constants $c^1 \neq c^2 \in \mathbb{R}$ such that $P_{pc}(\phi, c^j) = u$ and $\mathcal{K}(\phi) = 0$.

Differently from the operator P_s, the operator $P_{pc}(\cdot, c^j)$ (for fixed constants c^j) is 1–1, continuous and continuously differentiable from $L_2(\Omega)$ onto $L_2(\Omega)$. Consequently, the set of admissible vectors for the Tikhonov functional in (8) is defined in the following way.

Definition 2 Let the operator P_{pc} be defined as in (6) and $\tau > 0$. A vector $(\phi, c^j) \in L_2(\Omega) \times \mathbb{R}^2$ is called admissible when $\phi \in \mathtt{BV}(\Omega)$ and $|c^2 - c^1| \geq \tau$.

From (6), it follows that P_{pc} maps admissible vectors to $\mathtt{BV}(\Omega)$. The next two lemmas are devoted to the investigation of relevant properties of operators $\mathcal{K}$ and P_{pc}, respectively.

Lemma 8 *Let $\mathcal{K}$ be the operator defined in* Section 1. *The following assertions hold true*:

(i) *$\mathcal{K}$ is a continuous map from $L_2(\Omega)$ to $L_1(\Omega)$.*

(ii) *If $\|\mathcal{K}(\phi)\|_{L_1(\Omega)} = 0$ for some $\phi \in L_2(\Omega)$, then $\phi(x) \in \{1, 2\}$; a.e. in Ω.*

Proof Assertion (i) follows from

$$\int_\Omega |\mathcal{K}(\phi) - \mathcal{K}(\psi)| \leq \int_\Omega |(\phi - 1)(\phi - \psi)| + \int_\Omega |(\psi - 2)(\psi - \phi)|,$$

together with the Cauchy–Schwarz inequality. Assertion (ii) follows directly from the definitions of $\mathcal{K}$ and the L_1-norm. ■

Lemma 9 *Let P_{pc} be the operator defined in* (6). *The following assertions hold true*:

(i) *For every admissible vector (ϕ, c^j) it holds $|P_{pc}(\phi, c^j)|_{BV} \geq \tau|\phi|_{BV}$.*
Moreover, if (ϕ_k, c^j_k) is a sequence of admissible vectors converging in $L_p(\Omega) \times \mathbb{R}^2$ to some admissible vector (ϕ, c^j), then

(ii) *$P_{pc}(\phi_k, c^j_k)$ converges to $P_{pc}(\phi, c^j)$ in $L_p(\Omega)$.*

(iii) *$|P_{pc}(\phi, c^j)|_{\mathtt{BV}} \leq \liminf_{k\to\infty} |P_{pc}(\phi_k, c^j_k)|_{\mathtt{BV}}$.*

(iv) *$|P_{pc}(\phi, c^j)|_{\mathtt{BV}} \geq \tau\|\phi\|_{L_2}$.*

Proof Assertion (i) follows from the identity $|P_{pc}(\phi, c^j)|_{\mathtt{BV}} = |c^2 - c^1|\ |\phi|_{\mathtt{BV}}$.

Assertion (ii): Since $c^j_k \to c^j$ in $\mathbb{R}^2$ and $\phi_k \to \phi$ in $L_p(\Omega)$, it follows that $c^j_k\phi_k \to c^j\phi$ in $L_p(\Omega)$ and we conclude that $P_{pc}(\phi_k, c^j_k) = c^1_k(2 - \phi_k) + c^2_k(\phi_k - 1) \to P_{pc}(\phi, c^j)$ in $L_p(\Omega)$.

Assertion (iii) follows from part (ii) together with Lemma 3 (i), while assertion (iv) is a corollary of part (i). ■

Notice that Lemma 9 (iv) guarantees the coercivity of the functional $|P_{pc}(\cdot,\cdot)|_{\mathrm{BV}}$ (w.r.t. the L_2-norm) on the set of admissible parameters.

We are now ready to state and prove the convergence analysis results for the pcLS approach. Let $R_{pc}(\phi, c^j) := \beta_1 |P_{pc}(\phi, c^j)|_{\mathrm{BV}} + \beta_2 \|c^j\|^2_{\mathbb{R}^2}$ be the penalization term of $\mathcal{G}_{\alpha,pc}$ in (8). Given α, β_1, β_2, $\beta_3 > 0$, the next result guarantees that the functional $\mathcal{G}_{\alpha,pc}$ is well posed.

THEOREM 10 *The functional $\mathcal{G}_{\alpha,pc}$ in* (8) *attains minimizers on the set of admissible vectors.*

Proof Let $\{(\phi_k, c^j_k)\}$ be a minimizing sequence for $\mathcal{G}_{\alpha,pc}$, i.e. a sequence of admissible vectors satisfying $\mathcal{G}_{\alpha,pc}(\phi_k, c^j_k) \to \inf \mathcal{G}_{\alpha,pc}$, $k \to \infty$. Then, $\{R_{pc}(\phi_k, c^j_k)\}$ is a bounded sequence of real numbers. Therefore, it follows from Lemma 9 (iv) the existence of a subsequence $\{\phi_k\}$ and $\overline{\phi} \in L_2(\Omega)$ such that $\phi_k \rightharpoonup \overline{\phi}$ in $L_2(\Omega)$. Moreover, from Lemma 9 (i) and (ii) we conclude that $\overline{\phi} \in \mathrm{BV}(\Omega)$ and that this subsequence also satisfies $\phi_k \to \overline{\phi}$ in $L_p(\Omega)$.

On the other hand, the boundedness of $\{R_{pc}(\phi_k, c^j_k)\}$ also guarantees the existence of subsequences $\{c^j_k\}$ converging to $\overline{c}^j$ in $\mathbb{R}^2$.

Clearly, $(\overline{\phi}, \overline{c}^j)$ is an admissible vector. Moreover, from (A2), Lemmas 9 (iii) and Lemma 8 (i) it follows that

$$\begin{aligned}
\inf \mathcal{G}_{\alpha,pc} &= \lim_{k\to\infty} \mathcal{G}_{\alpha,pc}(\phi_k, c^j_k) \\
&= \liminf_{k\to\infty} \left\{ \|F(P_{pc}(\phi_k, c^j_k)) - y^\delta\|^2_Y + \beta_3 \|\mathcal{K}(\phi_k)\|_{L_1} + \alpha R_{pc}(\phi_k, c^j_k) \right\} \\
&\geq \|F(P_{pc}(\overline{\phi}, \overline{c}^j)) - y^\delta\|^2_Y + \beta_3 \|\mathcal{K}(\overline{\phi})\|_{L_1} + \alpha R_{pc}(\overline{\phi}, \overline{c}^j) = \mathcal{G}_{\alpha,pc}(\overline{\phi}, \overline{c}^j),
\end{aligned}$$

proving that $(\overline{\phi}, \overline{c}^j)$ minimizes $\mathcal{G}_{\alpha,pc}$. ■

The convergence and stability results in Theorem 7 hold true for the pcLS approach, as we shall see next.

THEOREM 11 *Assume that we have exact data and $\beta_j > 0$, $j = 1, 2, 3$. For every $\alpha > 0$ denote by $(\phi_\alpha, c^j_\alpha)$ a minimizer of $\mathcal{G}_{\alpha,pc}$ on the set of admissible vectors. Then, for every sequence of positive numbers $\{\alpha_k\}$ converging to zero there exists a subsequence such that $(\phi_{\alpha_k}, c^j_{\alpha_k})$ is strongly convergent in $L_p(\Omega) \times H^{-1}(\Omega) \times \mathbb{R}^2$. Moreover, the limit is a solution of* (7).

In the case of noisy data, let $\alpha = \alpha(\delta)$ be a function chosen as in Theorem 7. Given a sequence $\{\delta_k\}$ of positive numbers converging to zero and $\{y^{\delta_k}\} \in Y$ be corresponding noisy data satisfying (2), *there exist a subsequence, denoted again by $\{\delta_k\}$, and a sequence $\{\alpha_k := \alpha(\delta_k)\}$ such that $(\phi_{\alpha_k}, c^j_{\alpha_k})$ converges in $L_p(\Omega) \times H^{-1}(\Omega) \times \mathbb{R}^2$ to solution of* (7).

Notice that the limit elements (ϕ, c^j) obtained from the convergence-stability (Theorem 11) satisfy not only $F(P_{pc}(\phi, c^j)) = y$, but also $\|\mathcal{K}(\phi)\|_{L_1} = 0$. Therefore, due to Lemma 8 (ii), we conclude that the limit level-set function ϕ is piecewise constant.

4. Numerical experiments

In this section we discuss the numerical implementations of iterative methods based on the sLS and pcLS approaches. We use an inverse potential problem as test problem, similar to the one considered in [3,11,19–21].

The forward problem consists of solving on a given Lipschitz domain $\Omega \subset \mathbb{R}^n$, for a given source function $u \in L_2(\Omega)$, the Poisson boundary value problem

$$-\Delta w = u, \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega. \tag{14}$$

This problem can be modelled by the operator F: $L_2(\Omega) \to L_2(\partial\Omega)$, $F(u) := w_\nu|_{\partial\Omega}$ [22]. The corresponding inverse problem is the so-called *inverse potential problem*, which consists of recovering an L_2-function u, from measurements of the Cauchy data of its corresponding potential w (the measurements are available only on the boundary of Ω).

Using this notation, the inverse potential problem can be written in the abbreviated form $F(u) = y^\delta$, where the available noisy data $y^\delta \in L_2(\partial\Omega)$ have the same meaning as in (2).

It is worth noticing that this inverse problem has, in general, non-unique solution [20]. Sufficient conditions for identifiability are given in [23]. For issues related to redundancy of data as well as for an example of non-identifiability, we refer the reader to [20]. A generalization of this inverse problem, with the Laplacian replaced by a general elliptic operator, appears in many relevant applications including inverse gravimetry [22,24], EEG [25] and EMG [26].

Remark 1 Notice that the operator F above is a continuous and continuously differentiable mapping from $L_2(\Omega)$ to $L_2(\partial\Omega)$. Moreover, continuity of F w.r.t. the L_1-topology can be proved in the parameter space $\mathbb{D}$ consisting of characteristic functions (see (A2)).

In our experiments we follow [11] in the experimental setup, selecting $\Omega = (0,1) \times (0,1)$ and assuming that the unknown parameter is a piecewise constant function of the form $u = 1 + \chi_D$, where $D \subset\subset \Omega$. In particular, we allow piecewise constant functions u supported at domains consisting of several connected components. For this class of parameters no unique identifiability result is known. Nevertheless, our methods prove the ability to detect the desired (piecewise constant) solutions.

4.1. *A level set algorithm based on the sLS approach*

The iterative algorithm based on the sLS approach proposed in this article is an explicit iterative method derived from the formal conditions of optimality for a smooth Tikhonov functional approximating $\mathcal{G}_{\alpha,s}$ in (5). These optimality conditions can be written in the form of the system

$$\alpha\phi = L_{\varepsilon,\alpha,\beta}(\phi, c^1, c^2), \quad \alpha c^j = L^j_{\varepsilon,\alpha,\beta}(\phi, c^1, c^2), \ j = 1, 2, \tag{15}$$

where

$$\begin{aligned} L_{\varepsilon,\alpha,\beta}(\phi, c^1, c^2) = {} & (c^1 - c^2)\beta_2^{-1} H'_\varepsilon(\phi)^* F'(P_{s,\varepsilon}(\phi, c^1, c^2))^*(F(P_{s,\varepsilon}(\phi, c^1, c^2)) - y^\delta) \\ & - \beta_1(2\beta_2)^{-1} H'_\varepsilon(\phi)\nabla \cdot \left[\nabla H_\varepsilon(\phi)/(\varepsilon + |\nabla H_\varepsilon(\phi)|)\right], \end{aligned} \tag{16a}$$

$$L^1_{\varepsilon,\alpha,\beta}(\phi, c^1, c^2) = (2\beta_3)^{-1}\left(F'(P_{s,\varepsilon}(\phi, c^1, c^2))H_\varepsilon(\phi)\right)^*(F(P_{s,\varepsilon}(\phi, c^1, c^2)) - y^\delta), \tag{16b}$$

$$L^2_{\varepsilon,\alpha,\beta}(\phi, c^1, c^2) = (2\beta_3)^{-1}\left(F'(P_{s,\varepsilon}(\phi, c^1, c^2))(1 - H_\varepsilon(\phi))\right)^*(F(P_\varepsilon(\phi, c^1, c^2)) - y^\delta). \tag{16c}$$

Table 1. Iterative algorithm based on the sLS approach for the inverse potential problem.

(1) `Evaluate the residual` $r_k := F(P_{s,\varepsilon}(\phi_k, c_k^1, c_k^2)) - y^\delta = (w_k)_\nu|_{\partial\Omega} - y^\delta$, `where` w_k `solves`

$$\Delta w_k = P_{s,\varepsilon}(\phi_k, c_k^1, c_k^2)\ ,$$
$$\texttt{in } \Omega; \quad w_k = 0\ , \texttt{ at } \partial\Omega.$$

(2) `Evaluate` $h_k := F'(P_{s,\varepsilon}(\phi_k, c_k^1, c_k^2))^*(r_k) \in L_2(\Omega)$, `solving`

$$\Delta h_k = 0\ , \texttt{ in } \Omega\,; \quad h_k = r_k\ , \texttt{ at } \partial\Omega.$$

(3) `Calculate` $\delta\phi_k := L_{\varepsilon,\alpha,\beta}(\phi_k, c_k^1, c_k^2)$ `and` $\delta c_k^j := L^j_{\varepsilon,\alpha,\beta}(\phi_k, c_k^1, c_k^2)$, `as in` (16).

(4) `Update the level-set function` ϕ_k `and the level values` c_k^j, $j = 1, 2$:

$$\phi_{k+1} = \phi_k + \frac{1}{\alpha}\,\delta\phi_k\ , \quad c_{k+1}^j = c_k^j + \frac{1}{\alpha}\,\delta c_k^j.$$

Notice that the operators H and P in $\mathcal{G}_{\alpha,s}$ are substituted by smooth approximations H_ε and P_ε respectively.

Each step of this iterative method consists of three parts (Table 1): (1) The residual $r_k \in L_2(\partial\Omega)$ of the iterate (ϕ_k, c_k^j) is evaluated (this requires solving one elliptic BVP of Dirichlet type); (2) The L_2-solution h_k of the adjoint problem for the residual is evaluated (this corresponds to solving one elliptic BVP of Dirichlet type); (3) The update $\delta\phi_k$ for the level-set function and the updates δc_k^j for the level values are evaluated (this corresponds to multiplying two functions).

Remark 2 In order to improve the regularity of the update $\delta\phi_k$, the third step in Table 1 can be substituted by

(3′) Evaluate the update $\delta\phi_k \in H^1(\Omega)$, solving
$(I - \mu\Delta)\delta\phi_k = L_{\varepsilon,\alpha,\beta}(\phi_k, c_k^1, c_k^2)$, `in` Ω; $(\delta\phi_k)_\nu = 0$, `at` $\partial\Omega$.

where the positive constant μ satisfies $\mu \ll 1$. Notice that this corresponds to the optimality condition for the functional $\mathcal{G}_{\alpha,s}$ in (5) if we add $\beta_2\mu\|\nabla\phi\|_{L_2(\Omega)}$ to the the penalization term. In [27] such a Tikhonov functional (with $\mu = 1$) based on BV–H^1 regularization was proposed. The corresponding update $\delta\phi_k$ was very smooth and led to a slow convergence of the iteration.

Remark 3 In [21] another level-set method was proposed in order to attack the inverse potential problem described above. The level-set method proposed in [21] is different from the one proposed in this article. The main differences are:

- Here we work with L_2 level-set functions, while [21] uses the H^1 framework.
- In [21] the regularization parameter $\alpha > 0$ is kept fixed, while here we define $\delta t = 1/\alpha$ as time increment and take the limit $\alpha \to \infty$ in order to derive a continuous evolution equation for the levelset function (a fixed point equation related to the system of optimality conditions for the Tikhonov functional).

- In [21] the iteration is based on an inexact Newton-type method, where the inner iteration is implemented using the conjugate gradient method. Here the iteration is based on a gradient type method.

4.2. *A level set algorithm based on the pcLS approach*

The iterative algorithm based on the pcLS approach proposed in this article is an explicit iterative method based on the operator splitting technique in [28] and derived from the optimality conditions for the Tikhonov functional $\mathcal{G}_{\alpha,pc}$ in (8). First, the operator $\mathcal{G}_{\alpha,pc}$ is splitted in the sum $\mathcal{G}_{\alpha,pc}(\phi, c^j) = \mathcal{G}^1_{\alpha,pc}(\phi, c^j) + \mathcal{G}^2_{\alpha,pc}(\phi)$, where

$$\mathcal{G}^1_{\alpha,pc}(\phi, c^j) := \|F(P_{pc}(\phi, c^j)) - y^\delta\|^2_Y + \alpha\beta_1 |P_{pc}(\phi, c^j)|_{\mathrm{BV}} + \alpha\beta_2 \|c^j\|^2_{\mathbb{R}^2}$$
$$\mathcal{G}^2_{\alpha,pc}(\phi) := \beta_3 \|\mathcal{K}(\phi)\|_{L_1(\Omega)}.$$

Each step of the iterative method consists of two parts: (i) The iterate (ϕ_k, c^j_k) is updated using an explicit gradient step w.r.t. the operator $\mathcal{G}^1_{\alpha,pc}$, i.e.

$$\phi_{k+1/2} := \phi_k - \frac{\partial}{\partial\phi}\mathcal{G}^1_{\alpha,pc}(\phi_k, c^j_k), \quad c^j_{k+1/2} := c^j_k - \frac{\partial}{\partial c^j}\mathcal{G}^1_{\alpha,pc}(\phi_k, c^j_k).$$

It is worth noticing that this first part is analogue to steps 1–4 in Table 1.

(ii) The obtained approximation $(\phi_{k+1/2}, c^j_{k+1/2})$ is improved by giving a gradient step w.r.t. the operator $\mathcal{G}^2_{\alpha,pc}$, i.e.

$$\phi_{k+1} := \phi_{k+1/2} - \frac{\mathrm{d}}{\mathrm{d}\phi}\mathcal{G}^2_{\alpha,pc}(\phi_{k+1/2}), \quad c^j_{k+1} := c^j_{k+1/2}.$$

In [29] a similar operator splitting strategy was use to minimize a Tikhonov functional related to an elliptic inverse problem in EIT.

Remark 4 The pcLS approach described above is characterized by a constraint enforcing either $\phi = 1$ or $\phi = 2$ in Ω. It is worth noticing that the resulting (two steps) level-set algorithm relates to the phase field method used by the dynamic interface community to analyse front propagation problems [30,31].

4.3. *First numerical example: exact data*

In this first numerical experiment we aim to identify the right-hand side u of (14) from the knowledge of the exact data $y = w_\nu|_{\partial\Omega}$. We assume that the level value $c^2 = 0$ is given, and that we have to identify only the support of u and the level value $c^1 > 0$.

The exact data $y = F(u)$ is obtained by solving numerically the elliptic boundary value problem in (14) at a very fine grid (the word 'exact' here means: up to the precision of the numerical method used for solving the direct problem). In order to avoid inverse crimes, the direct problem (14) is solved on an adaptively refined finite element grid with 8.804 nodes. However, in the numerical implementation of the level-set method, all boundary value problems are solved at an uniform grid with 545 nodes (33 nodes at each boundary side).

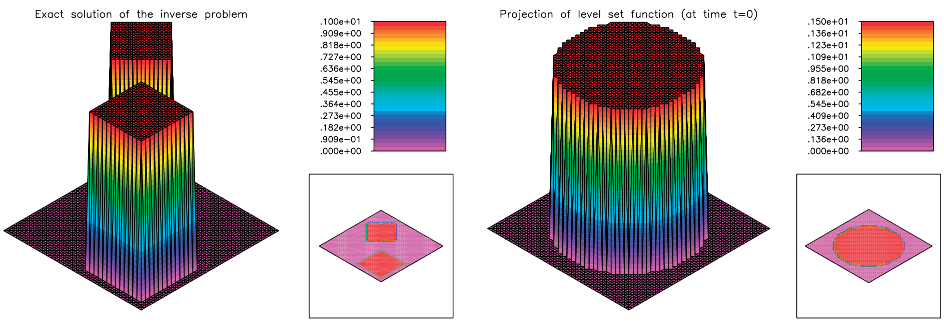

Figure 1. First experiment: the picture on the left-hand side shows the coefficient u_{exact} to be reconstructed. On the right-hand side, the initial condition for the sLS level-set method.

For this experiment with exact data, the level-set method was tested without the BV regularization term: we set $\beta_1 = 0$, $\beta_2 = 1$ (the choice of β_3 is discussed in Remark 5). Moreover, we chose $\varepsilon = 2^{-4}$ in (9).

In Figure 1 the solution u_{exact} of the inverse problem and the initial guess for the iterative method based on the sLS approach are presented (the initial guess $c_0^1 = 1.5$ is used for the unknown level value). Notice that the support of u is a non-connected proper subset of Ω. In Figure 2 the evolution of the sLS level-set method for the first 1500 iterative steps is presented. Notice, the shapes of both inclusions are reasonably reconstructed, and the level value c^1 is accurately reconstructed as well. The iteration is stopped when the residual drops below the predefined precision $\|F(P_{s,\varepsilon}(\phi_k, c_k^1)) - y\|_{L_2} < 10^{-2}$. For comparison purposes we present in the second line of this figure the evolution of the BV–H^1 level-set method [27] for the same initial guess.

The same stop criteria is used. Both methods deliver good approximations for the support of u as well as for the unknown level c^1. However, the sLS level-set method uses a less regular update and converges much faster. In Figure 2 (last line) we present the iteration error after $k = 200$ steps for sLS level-set method, and after $k = 1900$ steps for the BV–H^1 level-set method.

We performed other numerical simulations with different choice of initial guess (ϕ_0, c_0^1), and observed that the number of iterative steps required in order to obtain a reasonable approximation (up to the predefined precision of 10^{-2} in the L_2-norm) strongly depends on the choice of the initial guess c_0^1. On the other hand, the final result is not sensitive with respect to the choice of the initial guess ϕ_0.

In Figure 3 we present the results obtained for the exact data case which concerns the level-set method based on the pcLS approach. The initial guess is a smooth (polynomial) function attaining values in the interval $(1, 2)$. The initial guess for c_0^1 is the same as before. The evolution of the pcLS level-set method is shown for the first 1000 iterative steps of the algorithm presented in Subsection 4.2. As in the previous methods, the shape of the inclusions could be well reconstructed. The level value c^1 could be accurately reconstructed as well. For comparison purposes, we used the same stop criterion as before, i.e. $\|F(P_{pc}(\phi_k, c_k^1)) - y\|_{L_2} < 10^{-2}$.

Remark 5 What concerns the numerical implementation of the level-set method based on the pcLS approach, some facts have to be observed:

(1) Due to the operator splitting technique, we compute several times the step-part (i) before a single calculation of step-part (ii) is performed.

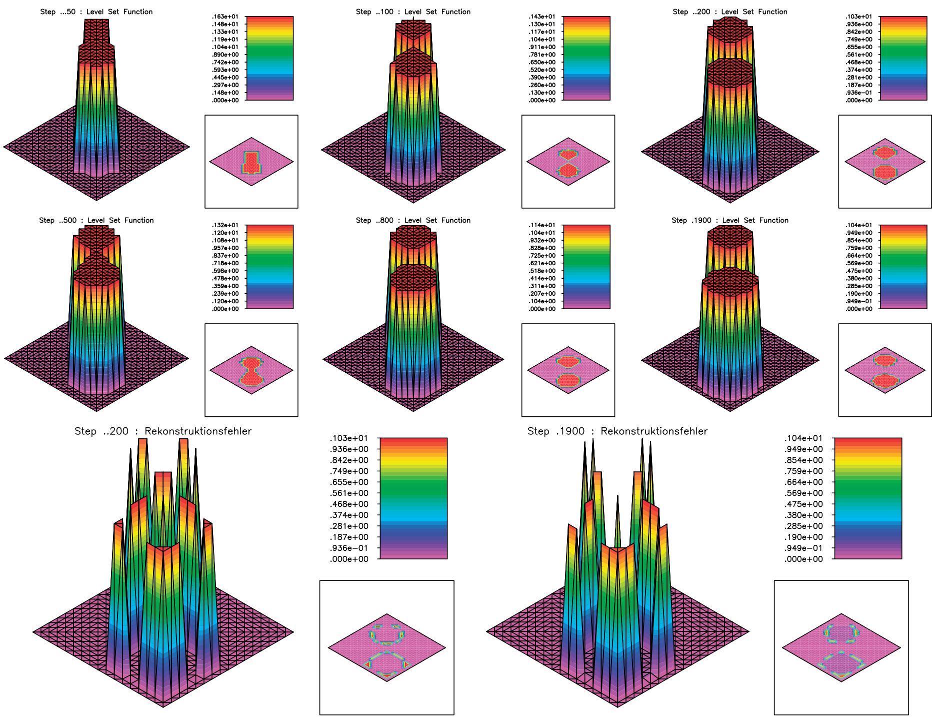

Figure 2. First experiment sLS: on the first line, plots of $P_{s,\varepsilon}(\phi_k, c_k^1)$, $k = 50, 100, 200$, for the sLS level-set method. The pictures on the second line show $P_{s,\varepsilon}(\phi_k, c_k^1)$, $k = 500, 800, 1900$, for the BV–H^1 level-set method in [27]. On the third line, the picture on the left hand side shows the iteration error for the sLS level-set method after $k = 200$ iterations, while the other picture shows the iteration error for the BV–H^1 level-set method after $k = 1900$ iterations.

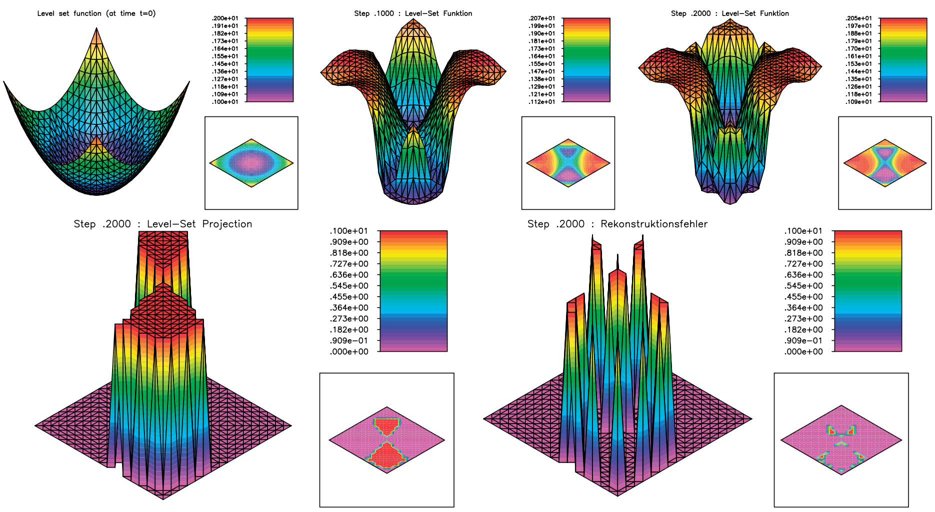

Figure 3. First experiment pcLS: the picture on the top left shows the initial condition for the pcLS level-set method. On the two subsequent pictures of the first line, plots of ϕ_k, for $k = 1000, 2000$. The bottom left picture shows $P_{pc}(\phi_k, c_k^1)$ for $k = 2000$. The bottom right picture shows the iteration error after $k = 2000$ iterations.

(2) Step-part (i) aims to minimize the misfit in the iteration and is the most relevant component of the iteration step described in Subsection 4.2.
(3) Step-part (ii) aims to drag the iterate ϕ_k to a piecewise constant (integer valued) function. If step-part (ii) is implemented too often, all the iterates ϕ_k become piecewise constant functions and the operator equation is not satisfied in an satisfactory way. On the other hand, if step-part (ii) is implemented only seldom, the iterates ϕ_k become too smooth and may be trapped in some local minimizer (due to the non-uniqueness of the inverse potential problem). Therefore, the determination of how often the step-part (ii) should be implemented is crucial for the good performance of the algorithm. In our numerical experiments the step-part (ii) was committed in computation of the initial 100 iterations; then we started computing the step-part (ii) after every 20 iterations. For all test problems considered in our experiments, this strategy brought good results.
(4) The constant β_3 should be chosen in such a way that $\beta_3 \ll 1$ in step-part (ii). This choice guarantees that the dragging effect resulting from step-part (ii) is not enforced too strongly. If $\alpha\beta_2 \approx 2$ the iterates once again become piecewise constant functions and the misfit does not decrease.

4.4. *Second numerical example: noisy data*

In the sequel we consider once again the inverse potential problem in (14) with the solution shown in Figure 1. This time, the data y^δ for the inverse problem is obtained by adding to the exact data $y = F(u)$ random generated noise of 25%.

As in the previous experiment, the direct problem is solved at a grid that is finer than the one used in the numerical implementation of the level-set method. The initial guess (ϕ_0, c_0^1) is the same as in the experiment with exact data (Figure 1), as well as the value used for ε. For this experiment with noisy data, the level-set method was tested with the BV regularization term: $\beta_1 = 10^{-3}$. Moreover, $\beta_2 = 1$. We used the generalized discrepancy with $\tau = 2$ as stop criteria, i.e. the iteration was stopped when for the first time $\|F(P_{s,\varepsilon}(\phi_k, c_k^1)) - y^\delta\|_{L_2} < \tau\delta$.

In Figure 4 we show the evolution of the level-set method based on the sLS approach, while in Figure 5 the evolution of the level-set method based on the pcLS approach is shown. The number of iterative steps required to obtain an acceptable approximation is similar for both approaches. However, the iterative method based on the sLS approach produced smoother and slightly more accurate approximate solutions.

For comparison purposes we present the evolution of the BV–H^1 level-set method [27] in this noisy data case (Figure 4). The same initial guess and the same stop criteria are used. The goal is to establish a comparison between the stability of the proposed methods and the method in [27].

The results in Figure 4 indicate that the sLS method and the method in [27] are able to produce approximate solutions with similar accuracy. However, the sLS method requires much less numerical effort. Indeed, the sLS method requires only $k = 230$ iterative steps to reach the stop criteria, while the previous method in [27] requires over $k = 2000$ iterations to reach the same accuracy. These findings corroborate the results obtained in Section 4.3.

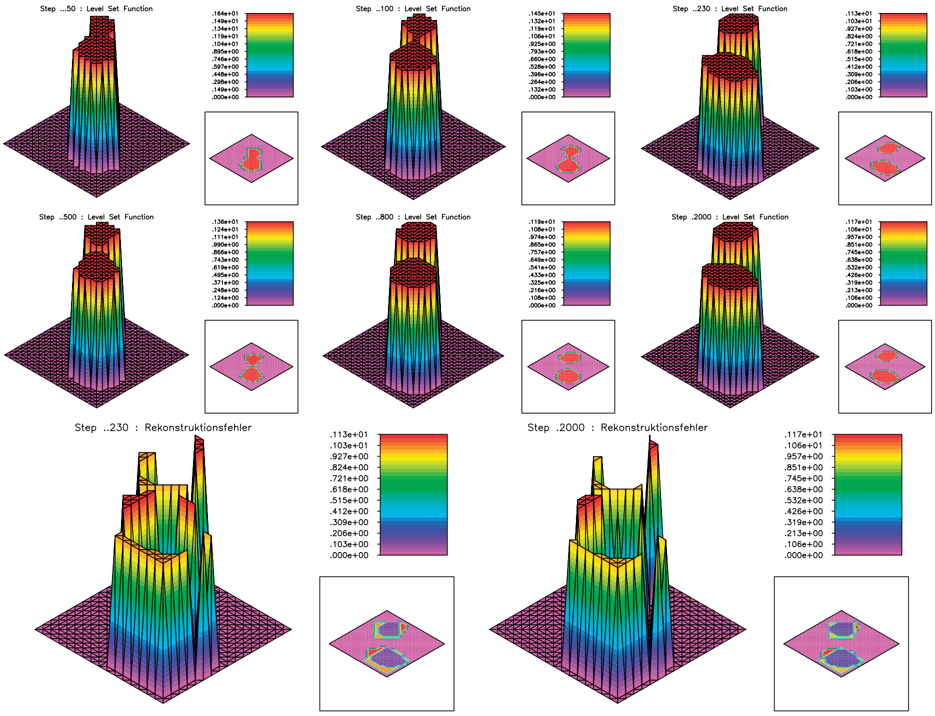

Figure 4. Second experiment sLS: on the first line, plots of $P_{s,\varepsilon}(\phi_k, c_k^1)$, $k = 50, 100, 230$, for the sLS level-set method. The pictures on the second line show $P_{s,\varepsilon}(\phi_k, c_k^1)$, $k = 500, 800, 2000$, for the BV–H^1 level-set method in [27]. On the third line, the picture on the left-hand side shows the iteration error for the sLS level-set method after $k = 230$ iterations, while the other picture shows the iteration error for the BV–H^1 level-set method after $k = 2000$ iterations.

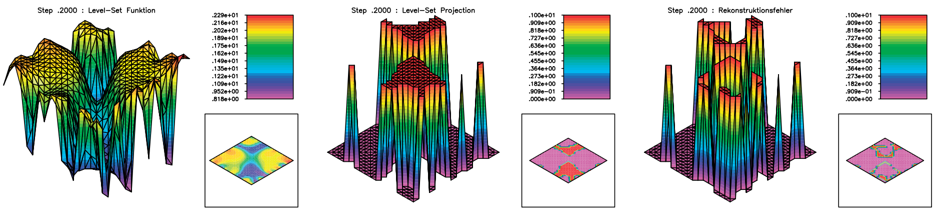

Figure 5. Second experiment pcLS: on the left, plots of ϕ_k for $k = 2000$, for the pcLS level-set method. On the centre the corresponding projection $P_{pc}(\phi_k)$. On the right-hand side, the iterative error $e_k := |P_{s,\varepsilon}(\phi_k, c_k^1) - u_{\text{exact}}|$.

5. Conclusions

Two distinct level-set-type approaches for solving ill-posed problems are proposed, where the level-set functions are chosen in L_2-spaces.

- The first approach (sLS) corresponds to an extension of the results obtained in [11,27] for H^1 level-set functions.

- In the second approach (pcLS) the parameter space consists of piecewise constant level-set functions. An extra equation is added to (4), namely $\mathcal{K}(\phi) = 0$, in order to enforce the level-set functions to become piecewise constant.

Based on each one of these two level-set approaches, corresponding Tikhonov functionals are derived. We provide convergence analysis for the resulting Tikhonov regularization methods.

Numerical experiments for an inverse potential problem are presented and the implementation of algorithms for both regularization methods is compared. Moreover, we compare our results with the BV–H^1 level-set method in [27] in the case of exact and noisy data. This comparison indicates that, although the sLS method and the BV–H^1 method are able to produce approximate solutions with similar accuracy, the numerical effort required by the sLS method is significantly smaller.

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