

On inertial iterated Tikhonov methods for solving ill-posed problems

J.C. Rabelo[†]

A. Leitão^{‡§¶}

A. L. Madureira[¶]

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Abstract

In this manuscript we propose and analyze an implicit two-point type method (or inertial method) for obtaining stable approximate solutions to linear ill-posed operator equations. The method is based on the iterated Tikhonov (iT) scheme. We establish convergence for exact data, and stability and semi-convergence for noisy data. Regarding numerical experiments we consider: i) a 2D Inverse Potential Problem, ii) an Image Deblurring Problem; the computational efficiency of the method is compared with standard implementations of the iT method.

Keywords. Ill-posed problems; Two-point methods; Inertial methods; Iterated Tikhonov method.

AMS Classification: 65J20, 47J06.

This manuscript is dedicated to Professor Johann Baumeister (Oberzeitldorn, Bavaria, 7. March 1944) on the occasion of his 80th birthday and in recognition of his valuable contributions to Numerical Analysis, Inverse Problems, Control Theory and Variational Calculus.

He is the author of one of the first books on inverse problems [5], published by Vieweg in '87, which influenced a whole generation of mathematicians. A zealous adviser, his door has always been open to his students, welcoming them with attention and patience.

Johann Baumeister is currently Emeritus Professor at the Department of Mathematics of the Johann-Wolfgang von Goethe Universität, Frankfurt am Main, Germany.



1 Introduction

Problems of interest

In a typical *inverse problem* setting [5, 32, 42], let X and Y be Hilbert spaces, and consider the problem of determining an unknown quantity $x \in X$ from given data $y \in Y$, i.e. an unknown quantity of interest x (which cannot be directly measured) has to be identified, based on information obtained from some set of measured data y .

[†]Department of Mathematics, Federal Univ. of Piauí, 64049-550 Teresina, Brazil

[‡]EMAp, Getúlio Vargas Foundation, Praia de Botafogo 190, 22250-900 Rio de Janeiro, Brazil

[§]On leave from Department of Mathematics, Federal Univ. of St. Catarina, 88040-900 Florianópolis, Brazil

[¶]LNCC, Av. Getúlio Vargas 333, P.O. Box 95113, 25651-070 Petrópolis, Brazil

Emails: joelrabelo@ufpi.edu.br, acgleitao@gmail.com, alm@lncc.br.

A relevant point is that, in practice, the exact data $y \in Y$ is unavailable. Instead, only approximate measured data $y^\delta \in Y$ satisfying

$$\|y^\delta - y\| \leq \delta, \quad (1)$$

is accessible. Here, the known $\delta > 0$ represents the level of noise (or uncertainty in the measurements). The available noisy data $y^\delta \in Y$ are obtained by indirect measurements of $x \in X$; a process represented by the model

$$Ax = y^\delta, \quad (2)$$

where $A : X \rightarrow Y$, is a bounded linear ill-posed operator, whose inverse $A^{-1} : Y \rightarrow X$ either does not exist, or is not continuous.

Standard iterations

We recall two families of iterative methods for obtaining stable approximate solutions to the linear ill-posed operator equation (2), namely (I) the family of explicit iterative methods defined by

$$x_{k+1}^\delta = x_k^\delta - \gamma_k A^*(Ax_k^\delta - y^\delta), \quad k = 0, 1, \dots;$$

(II) the family of implicit iterative methods

$$x_{k+1}^\delta = x_k^\delta - \lambda_k A^*(Ax_{k+1}^\delta - y^\delta), \quad k = 0, 1, \dots$$

Here $A^* : Y \rightarrow X$ is the adjoint operator to A . The iterations start at a given initial guess $x_0 \in X$ (which may incorporate *a priori* information about the exact solution of $Ax = y$).

Both families of methods, defined by choices of γ_k and λ_k , can be interpreted as iterative schemes for solving the normal equation $A^*Ax = A^*y^\delta$, which is the optimally condition for the least square problem $\min_x \|Ax - y^\delta\|^2$ [5].

In the explicit schemes, step computations are not expensive and the parameters γ_k play the role of step size control. Appropriate choices of γ_k lead to different methods, e.g., the Landweber iteration [33], the steepest descent method [45], the minimal error method [17] or generalizations of these methods [34]. Explicit iterative type methods are extensively discussed in the inverse problems literature. Their regularization properties are well established for both linear and nonlinear operator equations [5, 17, 32, 29]. It is well known in particular that the above mentioned Landweber, steepest descent and minimal error methods present slow convergence rates [17, 29].

On the other hand, each step of the implicit methods (also known as iterated Tikhonov (iT) methods [24, 11] or proximal point (PP) methods [36, 43]) corresponds to computing $x_{k+1}^\delta := \arg \min_x \{ \lambda_k \|Ax - y^\delta\|^2 + \|x - x_k^\delta\|^2 \}$. The parameters λ_k are appropriately chosen Lagrange multipliers [11]. Here, the computation of the iterative step is more demanding than in the explicit type methods, since x_{k+1}^δ is obtained by solving the linear system $(\lambda_k A^*A + I)x = x_k^\delta + \lambda_k A^*y^\delta$ at each step. However, the iT type methods require less iterations than the explicit methods, to compute an approximate solution of similar quality [17, 29].

The literature on iT type methods is extensive. Various aspects are investigated, including regularization properties [16, 22, 31, 29], convergence rates [24, 44], *a posteriori* strategies for choosing the Lagrange multipliers [11], and a cyclic version of the iT method [14].

Two-point iterations

Two-point iteration schemes can be interpreted as a generalization of the above explicit/implicit methods. In 1983 Nesterov discussed in [38] a strategy to accelerate the convergence of

explicit methods (I), where the computation of x_{k+1}^δ depends on the last two iterates x_k^δ and x_{k-1}^δ . In the notation of (2) the Nesterov accelerated forward-backward scheme reads

$$x_{k+1}^\delta = w_k^\delta - \gamma A^*(Aw_k^\delta - y^\delta), \quad \text{where} \quad w_k^\delta = x_k^\delta + \alpha_k(x_k^\delta - x_{k-1}^\delta), \quad k \geq 0. \quad (3)$$

Here $x_0^\delta \in X$ is an initial guess, $x_{-1}^\delta = x_0^\delta$, $\alpha_k = (k-1)/(k-1+\alpha)$ with $\alpha \geq 3$, and γ is a scaling parameter. The calculation of x_k and w_k is explicit, and as expensive to compute as the step of the (explicit) Landweber method. This explicit two-point iteration scheme improves the theoretical rate of convergence for the functional values $\|Ax_k - y^\delta\|^2$ from the standard $\mathcal{O}(1/k)$ to $\mathcal{O}(1/k^2)$ [38]; in 2016, Attouch and Peypouquet [4] were able to improve this rate to $o(1/k^2)$ for $\alpha > 3$.

Explicit two-point schemes were successfully considered in the context of *Iterative Shrinkage-Thresholding Algorithms* (ISTAs); among the resulting explicit two-point methods we mention the *Fast ISTA* (or FISTA), considered in 2009 by Beck and Teboulle [7], and the *Two-Step ISTA* (or TwIST) proposed in 2007 by Bioucas-Dias and Figueiredo [10]. These methods are designed for solving linear inverse problems arising in image processing and image restoration (a large amount of inertial schemes for optimization and inverse problems is available in the literature; we refer to the monograph of Chambolle and Pock [12], the book of Beck [6] and the references therein).

More recently, the Nesterov accelerated scheme was considered as an alternative for obtaining stable solutions to ill-posed operator equations. In 2017, Neubauer considered the Nesterov scheme with a stopping rule (the discrepancy principle) and proved convergence rates under standard source conditions [39]. In 2021 Kindermann revisited the Nesterov scheme for linear ill-posed problems [30] and proved that this explicit two-point method is an optimal-order iterative regularization method. In 2017 Hubmer and Ramlau [26] considered the Nesterov scheme (with discrepancy principle) for nonlinear ill-posed problems (under the Scherzer condition [17, Eq. (11.6)]). Convergence for exact data is proven as well as semi-convergence in the noisy data case. The same authors considered in [27] the Nesterov scheme for nonlinear ill-posed problems with a locally convex residual functional.

The implicit two-point iterative schemes are known in the optimization literature under the name of *inertial proximal methods* and have been well analyzed by many authors over the last two decades (see, e.g., [1, 4, 40, 37]). To the best of our knowledge, implicit two-point type methods have not yet been considered in the inverse problems literature, and this manuscript represents an attempt to do so.

The inertial iterated Tikhonov method

In this article we propose and analyze an implicit two-point iteration scheme that can be interpreted as a generalization of the iT method; our iteration relates to the inertial method proposed in 2001 by Alvarez and Attouch [2]. We propose this implicit two-point method as a viable alternative for computing stable approximate solutions to the ill-posed operator equation (2), and investigate its numerical efficiency.

The method under consideration consists in choosing appropriate non-negative sequences (α_k) , (λ_k) , and defining at each iterative step the extrapolation $w_k^\delta := x_k^\delta + \alpha_k(x_k^\delta - x_{k-1}^\delta)$; the next iterate x_{k+1} is defined by

$$x_{k+1}^\delta := \arg \min_x \{ \lambda_k \|Ax - y^\delta\|^2 + \|x - w_k^\delta\|^2 \}, \quad k = 0, 1, \dots \quad (4)$$

where $x_{-1} = x_0 \in X$ are given. For obvious reasons we refer to this implicit two-point method as *inertial iterated Tikhonov* (iniT) method.

The outline of the manuscript is as follows: In Section 2 we introduce and analyze the inertial iterated Tikhonov (iniT) method, proving convergence for exact data in Section 2.2, and presenting stability and semi-convergence results in Section 2.3. In Section 3 the iniT method is tested in a two-dimensional Inverse Potential Problem and an Image Deblurring Problem. Section 4 is devoted to final remarks and conclusions.

2 The iteration

In this section we introduce and analyze the inertial iterated Tikhonov (iniT) method considered in this manuscript. In Section 2.1 the iniT method is presented and some basic inequalities are established. Convergence for exact data is proven in Section 2.2. Stability and semi-convergence results are proven in Section 2.3.

2.1 Description of method

We begin by addressing the exact data case $y^\delta = y$. Define the quadratic (square residual) functional $f : X \ni x \rightarrow f(x) := \frac{1}{2} \|Ax - y\|^2 \in \mathbb{R}^+$.

Denoting the current iterate by $x_k \in X$, for $k \geq 0$, the step of the proposed iniT method consists in two parts: first compute the extrapolation point $w_k \in X$,

$$w_k := x_k + \alpha_k (x_k - x_{k-1}); \quad (5a)$$

in the sequel, the next iterate $x_{k+1} \in X$ is defined as the solution of

$$\lambda_k \nabla f(x_{k+1}) + x_{k+1} - w_k = 0. \quad (5b)$$

Notice that (5b) is equivalent to $\lambda_k \nabla f(x_{k+1}) + (x_{k+1} - x_k) - \alpha_k (x_k - x_{k-1}) = 0$, i.e. (5b) corresponds to an inertial proximal point update (compare with the iterative step of the inertial proximal method in [2, Equation (A₁)]).

Here $x_0 \in X$ plays the role of an initial guess and $x_{-1} := x_0$. Moreover, $(\alpha_k) \in [0, \alpha]$ for some $\alpha \in (0, 1)$, and $(\lambda_k) \in \mathbb{R}^+$ are given sequences. Notice that, if $\alpha_k \equiv 0$ then (5b) corresponds to the standard iT iteration for exact data, i.e. x_{k+1} is defined as the solution of $\lambda_k \nabla f(x_{k+1}) + x_{k+1} - x_k = 0$.

It is straightforward to see that (5b) is equivalent to $G_k x_{k+1} = \lambda_k A^* y + w_k$, where $G_k := (\lambda_k A^* A + I) : X \rightarrow X$ is a positive definite operator with spectrum contained in the interval $[1, 1 + \lambda_k \|A\|^2]$. Consequently, the iterate x_{k+1} is uniquely defined by (5b).

In what follows we present the iniT method in algorithmic form

Remark 2.1. *In Algorithm 1, if $Ax_{k_0} = y$ for some $k_0 \in \mathbb{N}$, the iteration stops after computing $x_{-1}, x_0, \dots, x_{k_0}$ and $w_0, \dots, w_{k_0-1}$. Notice that Algorithm 1 generates (infinite) sequences $(x_k)_{k \in \mathbb{N}}$ and $(w_k)_{k \in \mathbb{N}}$ if and only if $Ax_k \neq y$, for all $k \in \mathbb{N}$.*

The remaining of this subsection is devoted to investigating preliminary properties of the sequences (x_k) , (w_k) generated by Algorithm 1 (see Lemma 2.2 and Proposition 2.5).

Lemma 2.2. *Let (x_k) , (w_k) be sequences generated by Algorithm 1. Given $x \in X$, it holds*

$$\|w_k - x\|^2 = (1 + \alpha_k) \|x_k - x\|^2 - \alpha_k \|x_{k-1} - x\|^2 + \alpha_k (1 + \alpha_k) \|x_k - x_{k-1}\|^2, \quad k = 0, 1, \dots \quad (6)$$

Proof. From (5a) follows

$$x_k = (1 + \alpha_k)^{-1} w_k + \alpha_k (1 + \alpha_k)^{-1} x_{k-1}, \quad k = 0, 1, \dots \quad (7)$$

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[0] choose an initial guess  $x_0 \in X$ ; set  $x_{-1} := x_0$ ;  $k := 0$ ;
[1] choose  $\alpha \in [0, 1)$  and  $(\lambda_k)_{k \geq 0} \in \mathbb{R}^+$ ;
[2] while  $\|Ax_k - y\| > 0$  do
    [2.1] choose  $\alpha_k \in [0, \alpha]$ ;
    [2.2]  $w_k := x_k + \alpha_k(x_k - x_{k-1})$ ;
    [2.3] compute  $x_{k+1} \in X$  as the solution of
            
$$\lambda_k \nabla f(x_{k+1}) + x_{k+1} - w_k = 0;$$

    [2.4]  $k := k + 1$ ;
end while

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Algorithm 1: Inertial iterated Tikhonov method (iniT) in the exact data case.

Consequently,

$$w_k - x_{k-1} = (1 + \alpha_k)(x_k - x_{k-1}), \quad k = 0, 1, \dots \quad (8)$$

Arguing with (7), together with the identity $(1 + \alpha_k)^{-1} + \alpha_k(1 + \alpha_k)^{-1} = 1$ and the strong convexity of the functional $\|\cdot\|^2$,¹ we obtain

$$\begin{aligned} \|x_k - x\|^2 &= \|(1 + \alpha_k)^{-1}(w_k - x) + \alpha_k(1 + \alpha_k)^{-1}(x_{k-1} - x)\|^2 \\ &= (1 + \alpha_k)^{-1}\|w_k - x\|^2 + \alpha_k(1 + \alpha_k)^{-1}\|x_{k-1} - x\|^2 \\ &\quad - \alpha_k(1 + \alpha_k)^{-2}\|w_k - x_{k-1}\|^2, \end{aligned}$$

for $k \geq 0$. To complete the proof, it is enough to substitute (8) in the last identity. $\square$

Assumption 2.3. Given $\alpha \in [0, 1)$ and a convergent series $\sum_k \theta_k$ of nonnegative terms, let

$$\alpha_0 = \alpha \quad \text{and} \quad \alpha_k := \begin{cases} \min \left\{ \frac{\theta_k}{\|x_k - x_{k-1}\|^2}, \theta_k, \alpha \right\} & , \text{ if } \|x_k - x_{k-1}\| > 0 \\ 0 & , \text{ otherwise} \end{cases}, \quad k \geq 1.$$

Remark 2.4. Assumption 2.3 implies the summability of the series $\sum_{k \geq 0} \alpha_k \|x_k - x_{k-1}\|^2$.

To prove the next proposition, an auxiliary result is needed (see Appendix A). In order to apply this result, the summability of the above mentioned series is required.

Proposition 2.5. Let $(x_k), (w_k)$ be sequences generated by Algorithm 1, with $(\lambda_k), (\alpha_k)$ defined as in Steps [1] and [2.1] respectively. The following assertions hold true

(a) If $x^* \in X$ is a solution of $Ax = y$ then

$$\|w_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 = \|\lambda_k A^*(Ax_{k+1} - y)\|^2 + 2\lambda_k \|Ax_{k+1} - y\|^2, \quad k = 0, 1, \dots$$

Additionally, if (α_k) satisfies Assumption 2.3, it holds

(b) The sequences (x_k) and (w_k) are bounded.

(c) The series

$$\sum_{k=0}^{\infty} \lambda_k \|Ax_{k+1} - y\|^2, \quad \sum_{k=0}^{\infty} \|\lambda_k A^*(Ax_{k+1} - y)\|^2, \quad \sum_{k=0}^{\infty} \|x_{k+1} - w_k\|^2, \quad \sum_{k=0}^{\infty} \|x_{k+1} - x_k\|^2 \quad (9)$$

are summable.

¹For $0 < p < 1$ and $z, w \in X$ it holds $\|pz + (1-p)w\|^2 = p\|z\|^2 + (1-p)\|w\|^2 - p(1-p)\|z - w\|^2$.

Proof. From (5b) follows

$$\begin{aligned}\|w_k - x\|^2 - \|x_{k+1} - x\|^2 &= \|w_k - x_{k+1}\|^2 + 2\langle w_k - x_{k+1}, x_{k+1} - x \rangle \\ &= \|\lambda_k A^*(Ax_{k+1} - y)\|^2 + 2\lambda_k \langle Ax_{k+1} - y, Ax_{k+1} - Ax \rangle,\end{aligned}$$

for $x \in X$ and $k \geq 0$. Assertion (a) is a direct consequence of this equation with $x = x^*$.

Assertion (b): Define $\varphi_k := \|x_k - x^*\|^2$, for $k \geq -1$, and $\eta_k := \alpha_k \|x_k - x_{k-1}\|^2$, for $k \geq 0$. It follows from Assertion (a) and (6) with $x = x^*$ that

$$\begin{aligned}\varphi_{k+1} - \varphi_k + \|\lambda_k A^*(Ax_{k+1} - y)\|^2 + 2\lambda_k \|Ax_{k+1} - y\|^2 &= \\ \alpha_k(\varphi_k - \varphi_{k-1}) + \alpha_k(1 + \alpha_k)\|x_k - x_{k-1}\|^2, &k \geq 0.\end{aligned}\quad (10)$$

Arguing with (10) and the fact that $\alpha_k \leq \alpha < 1$, we obtain

$$\varphi_{k+1} - \varphi_k < \alpha_k(\varphi_k - \varphi_{k-1}) + 2\eta_k, \quad k \geq 0. \quad (11)$$

Now, the summability of $\sum_{k \geq 0} \eta_k$ (see Remark 2.4) together with (11), allow us to apply Lemma A.1 to the sequences (α_k) , (φ_k) , (η_k) . This lemma guarantees the existence of $\bar{\varphi} \in \mathbb{R}$ s.t. $\varphi_k \rightarrow \bar{\varphi}$ as $k \rightarrow \infty$. Consequently, the boundedness of (x_k) follows. The boundedness of (w_k) follows from the one of (x_k) , together with (5a) and the fact that $\alpha_k \in [0, \alpha]$.

Assertion (c): We verify the summability of the four series in (9). From (10) follows

$$\|\lambda_k A^*(Ax_{k+1} - y)\|^2 + 2\lambda_k \|Ax_{k+1} - y\|^2 < \alpha_k \varphi_k + (\varphi_k - \varphi_{k+1}) + 2\eta_k, \quad k \geq 0.$$

Adding up this inequality for $k = 0, \dots, l$ we obtain

$$\begin{aligned}\sum_{k=0}^l \|\lambda_k A^*(Ax_{k+1} - y)\|^2 + 2 \sum_{k=0}^l \lambda_k \|Ax_{k+1} - y\|^2 &< \sum_{k=0}^l \alpha_k \varphi_k + (\varphi_0 - \varphi_{l+1}) + 2 \sum_{k=0}^l \eta_k \\ &\leq M \sum_{k=0}^l \theta_k + \varphi_0 + 2 \sum_{k=0}^l \eta_k,\end{aligned}\quad (12)$$

where (θ_k) is the sequence in Assumption 2.3 and $M = \sup_k \varphi_k$ (since $\varphi_k \rightarrow \bar{\varphi}$ as $k \rightarrow \infty$, it holds $M < \infty$). Taking the limit $l \rightarrow \infty$ in (12) we obtain the summability of the first two series in (9).

The summability of $\sum_k \|x_{k+1} - w_k\|^2$, the third series in (9), is a consequence of (5b) and the fact that $\sum_k \|\lambda_k A^*(Ax_{k+1} - y)\|^2 < \infty$ (the second series in (9)).

Since $\alpha_k < 1$, we argue with (5a) to estimate

$$\|x_{k+1} - x_k\|^2 = \|x_{k+1} - w_k + \alpha_k(x_k - x_{k-1})\|^2 \leq 2\|x_{k+1} - w_k\|^2 + 2\|x_k - x_{k-1}\|^2, \quad k \geq 0.$$

Summing up this inequality we obtain

$$\sum_{k \geq 0} \|x_{k+1} - x_k\|^2 \leq 2 \sum_{k \geq 0} \|x_{k+1} - w_k\|^2 + 2 \sum_{k \geq 0} \eta_k < \infty,$$

establishing the summability of the last series in (9). $\square$

2.2 The exact data case

In what follows we prove a (strong) convergence result for the iniT method (Algorithm 1) in the case of exact data.

Remark 2.6. It is well known that problem $Ax = y$ admits an x_0 -minimal norm solution [17], i.e. there exists an element $x^\dagger \in X$ satisfying $\|x^\dagger - x_0\| = \inf\{\|x - x_0\|; Ax = y\}$; notice that $x^\dagger$ is the only solution of (2) with this property.

Remark 2.7. It holds $x_k - x_{k-1} \in \text{Rg}(A^*)$,² for all $k \in \mathbb{N}$. Indeed, for $k = 0$ it holds $x_0 - x_{-1} = 0 = A^*(0)$; assume that $x_j - x_{j-1} \in \text{Rg}(A^*)$ holds true for $j = 0, \dots, k$; it follows from (5a) and (5b) that $x_{k+1} - x_k = \alpha_k(x_k - x_{k-1}) - \lambda_k A^*(Ax_{k+1} - y) \in \text{Rg}(A^*)$.

Theorem 2.8 (Convergence for exact data). Let (x_k) , (w_k) be sequences generated by Algorithm 1, with (λ_k) , (α_k) defined as in Steps [1] and [2.1] respectively. Moreover, assume that:

- (A1) The sequence (α_k) satisfies Assumption 2.3;
- (A2) (α_k) is monotone non-increasing and $\alpha < \frac{1}{2}$ (see Step [1] of Algorithm 1);
- (A3) $\lambda_k \geq \lambda > 0$, for $k \geq 0$.

Then either the sequence (x_k) stops after a finite number $k_0 \in \mathbb{N}$ of steps (in this case it holds $Ax_{k_0} = y$), or it converges strongly to $x^\dagger$ the x_0 -minimal norm solution of $Ax = y$.

Proof. We have to consider two cases.

Case I: $Ax_{k_0} = y$ for some $k_0 \in \mathbb{N}$.

In this case, as observed in Remark 2.1, the sequence (x_k) generated by Algorithm 1 reads $x_{-1}, x_0, \dots, x_{k_0}$, and it holds $Ax_{k_0} = y$.

Case II: $Ax_k \neq y$, for every $k \geq 0$.

Notice that, in this case, the sequence $(\|Ax_k - y\|) \in \mathbb{R}$ is strictly positive. Moreover, it follows from Assumption (A3) together with Proposition 2.5 (c) that $\lim_k \|Ax_k - y\| = 0$. Therefore, there exists a strictly monotone increasing sequence $(l_j) \in \mathbb{N}$ satisfying

$$\|Ax_{l_j} - y\| \leq \|Ax_k - y\| \quad \text{for } 0 \leq k \leq l_j. \quad (13)$$

Next, given $k > 0$ and $z \in X$, we estimate

$$\begin{aligned} \|w_k - z\|^2 - \|x_{k+1} - z\|^2 &= -\|x_{k+1} - w_k\|^2 - 2\langle x_{k+1} - w_k, w_k - z \rangle \\ &\leq -2\langle x_{k+1} - w_k, w_k - z \rangle \\ &= 2\langle \lambda_k A^*(Ax_{k+1} - y), w_k - z \rangle \\ &= 2\lambda_k \langle Ax_{k+1} - y, A(w_k - x_{k+1}) + (Ax_{k+1} - y) + (y - Az) \rangle \\ &\leq 2\langle \lambda_k A^*(Ax_{k+1} - y), w_k - x_{k+1} \rangle + 2\lambda_k \|Ax_{k+1} - y\|^2 \\ &\quad + 2\lambda_k \|Ax_{k+1} - y\| \|Az - y\| \\ &\leq \|\lambda_k A^*(Ax_{k+1} - y)\|^2 + \|w_k - x_{k+1}\|^2 + 2\lambda_k \|Ax_{k+1} - y\|^2 \\ &\quad + 2\lambda_k \|Ax_{k+1} - y\| \|Az - y\|. \end{aligned}$$

Taking $z = x_{l_j}$ in the last inequality and arguing with (13), we obtain

$$\begin{aligned} \|w_k - x_{l_j}\|^2 - \|x_{k+1} - x_{l_j}\|^2 &\leq \\ &\|\lambda_k A^*(Ax_{k+1} - y)\|^2 + \|w_k - x_{k+1}\|^2 + 4\lambda_k \|Ax_{k+1} - y\|^2 =: \mu_k, \end{aligned} \quad (14)$$

for $0 \leq k \leq l_j - 1$. On the other hand, we conclude from (6) (with $x = x_{l_j}$) that

$$\|w_k - x_{l_j}\|^2 \geq \|x_k - x_{l_j}\|^2 + \alpha_k (\|x_k - x_{l_j}\|^2 - \|x_{k-1} - x_{l_j}\|^2), \quad k \geq 0. \quad (15)$$

²Here $\text{Rg}(A^*)$ denotes the range of the operator A^* .

Now, combining (15) with (14), and arguing with Assumption (A2), we conclude that

$$\begin{aligned}\|x_k - x_{l_j}\|^2 - \|x_{k+1} - x_{l_j}\|^2 &\leq \alpha_k (\|x_{k-1} - x_{l_j}\|^2 - \|x_k - x_{l_j}\|^2) + \mu_k \\ &\leq \alpha_{k-1} \|x_{k-1} - x_{l_j}\|^2 - \alpha_k \|x_k - x_{l_j}\|^2 + \mu_k,\end{aligned}\quad (16)$$

for $0 \leq k \leq l_j - 1$.

Let $0 \leq m \leq l_j - 1$. Adding up (16) for $k = m, \dots, l_j - 1$ gives us

$$\|x_m - x_{l_j}\|^2 - \|x_{l_j} - x_{l_j}\|^2 \leq \alpha_{m-1} \|x_{m-1} - x_{l_j}\|^2 - \alpha_{l_j-1} \|x_{l_j-1} - x_{l_j}\|^2 + \sum_{k=m}^{l_j-1} \mu_k,$$

from where we derive

$$\begin{aligned}\|x_m - x_{l_j}\|^2 &\leq \alpha_{m-1} \|x_{m-1} - x_{l_j}\|^2 + \sum_{k=m}^{l_j-1} \mu_k \\ &\leq 2\alpha_{m-1} (\|x_{m-1} - x_m\|^2 + \|x_m - x_{l_j}\|^2) + \sum_{k=m}^{l_j} \mu_k \\ &\leq 2 \sum_{k=m}^{l_j} \alpha_{k-1} \|x_{k-1} - x_k\|^2 + 2\alpha_{m-1} \|x_m - x_{l_j}\|^2 + \sum_{k=m}^{l_j} \mu_k.\end{aligned}$$

Consequently, whenever $m < l_j$, it holds

$$(1 - 2\alpha_{m-1}) \|x_m - x_{l_j}\|^2 \leq 2 \sum_{k=m}^{l_j} \alpha_{k-1} \|x_k - x_{k-1}\|^2 + \sum_{k=m}^{l_j} \mu_k.$$

Now, defining $\beta := (1 - 2\alpha_0)^{-1}$, we argue with Assumption (A2) to conclude that

$$\begin{aligned}\|x_m - x_{l_j}\|^2 &\leq 2\beta \sum_{k=m}^{\infty} \alpha_{k-1} \|x_k - x_{k-1}\|^2 + \beta \sum_{k=m}^{\infty} \mu_k \\ &\leq \beta \sum_{k=m}^{\infty} \|x_k - x_{k-1}\|^2 + \beta \sum_{k=m}^{\infty} \mu_k, \quad m < l_j\end{aligned}\quad (17)$$

(notice that $\beta > 0$ due to Assumption (A2)).

Notice that Assumption (A1) together with Proposition 2.5 (c) guarantee the summability of both series $\sum_k \mu_k$ and $\sum_k \|x_k - x_{k-1}\|^2$. Therefore, defining $s_m := \beta \sum_{k \geq m} \|x_k - x_{k-1}\|^2 + \beta \sum_{k \geq m} \mu_k$, for $m \in \mathbb{N}$, we have $s_m \rightarrow 0$ as $m \rightarrow \infty$.

Now let $n > m$ be given. Choosing $l_j > n$ and arguing with (17) we estimate

$$\|x_n - x_m\| \leq \|x_n - x_{l_j}\| + \|x_{l_j} - x_m\| \leq \sqrt{s_n} + \sqrt{s_m} \leq 2\sqrt{s_m}.$$

Since $\lim_m s_m = 0$, this inequality allow us to conclude that (x_k) is a Cauchy sequence.

Consequently, (x_k) converges to some $\bar{x} \in X$. In order to prove that $\bar{x}$ is a solution of $Ax = y$, it suffices to verify that $\|Ax_k - y\| \rightarrow 0$ as $k \rightarrow \infty$. This fact, however, is a consequence of Proposition 2.5 (c) (see first series in (9)) together with Assumption (A3).

It follows from Remark 2.7 that $x_{k+1} - x_k \in Rg(A^*) \subset \mathcal{N}(A)^\perp$. An inductive argument allow us to conclude that $\bar{x} \in x_0 + \mathcal{N}(A)^\perp$. Thus, from Remark 2.6 follows $\bar{x} = x^\dagger$. $\square$

2.3 The noise data case

In what follows we address the noisy data case $\delta > 0$. We begin by defining the quadratic (square residual) functional $f^\delta : X \ni x \rightarrow f^\delta(x) := \frac{1}{2} \|Ax - y^\delta\|^2 \in \mathbb{R}^+$. The iniT method in the case noisy data case reads:

[0] choose an initial guess $x_0^\delta \in X$; set $x_{-1}^\delta := x_0^\delta$; $k := 0$;

[1] choose $\tau > 1$, $\alpha \in [0, 1)$ and $(\lambda_k)_{k \geq 0} \in \mathbb{R}^+$;

[2] **while** $\|Ax_k^\delta - y^\delta\| > \tau\delta$ **do**

[2.1] choose $\alpha_k^\delta \in [0, \alpha]$;

[2.2] $w_k^\delta := x_k^\delta + \alpha_k^\delta(x_k^\delta - x_{k-1}^\delta)$;

[2.3] compute $x_{k+1}^\delta \in X$ as the solution of

$\lambda_k \nabla f^\delta(x_{k+1}^\delta) + x_{k+1}^\delta - w_k^\delta = 0$;

[2.4] $k := k + 1$;

end while

[3] $k^* = k$;

Algorithm 2: Inertial iterated Tikhonov method (iniT) in the noisy data case.

The stop criterion used in Algorithm 2 is based on the discrepancy principle, i.e. the iteration is stopped at step $k^* = k^*(\delta, y^\delta)$ satisfying

$$k^* := \min\{k \in \mathbb{N} ; \|Ax_k^\delta - y^\delta\| \leq \tau\delta\},$$

where $\tau > 1$. Notice that Algorithm 2 generate sequences $(x_k^\delta)_{k=-1}^{k^*}$ and $(w_k^\delta)_{k=0}^{k^*-1}$. In Proposition 2.14 we prove that, under appropriate assumptions, the stopping index k^* in Step [3] is finite.

In Lemma 2.9 the residuals $\|Aw_k^\delta - y^\delta\|$ and $\|Ax_{k+1}^\delta - y^\delta\|$ are compared; and in Lemma 2.10 the distances $\|w_k^\delta - x^*\|$ and $\|x_{k+1}^\delta - x^*\|$ are compared (here $x^* \in X$ is a solution of $Ax = y$).

Lemma 2.9. *Let (x_k^δ) , (w_k^δ) be sequences generated by Algorithm 2. The following assertions hold true*

(a) $\|Aw_k^\delta - y^\delta\|^2 - \|Ax_{k+1}^\delta - y^\delta\|^2 = \|Aw_k^\delta - Ax_{k+1}^\delta\|^2 + 2\langle A(w_k^\delta - x_{k+1}^\delta), Ax_{k+1}^\delta - y^\delta \rangle$, for $k = 0, \dots, k^* - 1$;

(b) $\|Ax_{k+1}^\delta - y^\delta\| \leq \|Aw_k^\delta - y^\delta\|$, for $k = 0, \dots, k^* - 1$.

Proof. For $k \leq k^*$ it follows from Step [2.3] of Algorithm 2 that

$$\begin{aligned} \|Aw_k^\delta - y^\delta\|^2 - \|Ax_{k+1}^\delta - y^\delta\|^2 &= \|Aw_k^\delta - Ax_{k+1}^\delta\|^2 + 2\langle A(w_k^\delta - x_{k+1}^\delta), Ax_{k+1}^\delta - y^\delta \rangle \\ &\geq \langle w_k^\delta - x_{k+1}^\delta, A^*(Ax_{k+1}^\delta - y^\delta) \rangle = \lambda_k \|A^*(Ax_{k+1}^\delta - y^\delta)\|^2 \geq 0, \end{aligned}$$

proving assertions (a) and (b). $\square$

Lemma 2.10. *Let (x_k^δ) , (w_k^δ) be sequences generated by Algorithm 2. If $x^* \in X$ is a solution of $Ax = y$ then*

$$\|w_k^\delta - x^*\|^2 - \|x_{k+1}^\delta - x^*\|^2 \geq \|\lambda_k A^*(Ax_{k+1}^\delta - y^\delta)\|^2 + 2\lambda_k \|Ax_{k+1}^\delta - y^\delta\| [\|Ax_{k+1}^\delta - y^\delta\| - \delta],$$

for $k = 0, \dots, k^* - 1$.

Proof. Let $0 \leq k \leq k^* - 1$. From Step [2.3] of Algorithm 2 and (1) follow

$$\begin{aligned} \|w_k^\delta - x^*\|^2 - \|x_{k+1}^\delta - x^*\|^2 &= \|w_k^\delta - x_{k+1}^\delta\|^2 + 2\langle w_k^\delta - x_{k+1}^\delta, x_{k+1}^\delta - x^* \rangle \\ &= \|\lambda_k A^*(Ax_{k+1}^\delta - y^\delta)\|^2 + 2\lambda_k \langle Ax_{k+1}^\delta - y^\delta, Ax_{k+1}^\delta - y + y^\delta - y^\delta \rangle \\ &= \|\lambda_k A^*(Ax_{k+1}^\delta - y^\delta)\|^2 + 2\lambda_k \|Ax_{k+1}^\delta - y^\delta\|^2 + 2\lambda_k \langle Ax_{k+1}^\delta - y^\delta, y^\delta - y \rangle \\ &\geq \|\lambda_k A^*(Ax_{k+1}^\delta - y^\delta)\|^2 + 2\lambda_k \|Ax_{k+1}^\delta - y^\delta\| [\|Ax_{k+1}^\delta - y^\delta\| - \delta], \end{aligned}$$

proving the assertion. $\square$

The next assumption considers a particular choice of the inertial parameter α_k^δ in Step [2.1] of Algorithm 2. It plays a key role in the forthcoming analysis (see Proposition 2.14 and Theorems 2.15 and 2.16).

Assumption 2.11. Given $\alpha \in [0, 1)$, define α_k^δ in Step [2.1] of Algorithm 2 by:

$$\alpha_0^\delta = \alpha \quad \text{and} \quad \alpha_k^\delta := \begin{cases} \min \left\{ \frac{\theta_k}{\|x_k^\delta - x_{k-1}^\delta\|^2}, \theta_k, \alpha \right\} & , \text{ if } \|x_k^\delta - x_{k-1}^\delta\| > 0 \\ 0 & , \text{ otherwise} \end{cases} \quad (18)$$

for $k \geq 1$. Here (θ_k) is a nonnegative sequence such that $\sum_k \theta_k < \infty$.

In the sequel we state and prove the main results of this section, namely stability (Theorem 2.15) and regularization (Theorem 2.16). First, however, in the next two remarks we extend to the noisy data case some results discussed in Lemma 2.2 and Proposition 2.5.

Remark 2.12. Arguing as in Lemma 2.2 we prove that the sequences (x_k^δ) , (w_k^δ) generated by Algorithm 2 (with (α_k^δ) defined as in Step [2.1] of this algorithm) satisfy

$$\|w_k^\delta - x\|^2 = (1 + \alpha_k^\delta) \|x_k^\delta - x\|^2 - \alpha_k^\delta \|x_{k-1}^\delta - x\|^2 + \alpha_k^\delta (1 + \alpha_k^\delta) \|x_k^\delta - x_{k-1}^\delta\|^2, \quad (19)$$

for $x \in X$, and $k = 0, \dots, k^*$.

Remark 2.13. If the inertial parameters α_k^δ satisfy Assumption 2.11, then the sequences (x_k^δ) , (w_k^δ) generated by Algorithm 2 are bounded.

If k^* in Step [3] of Algorithm 2 is finite, the statement is obvious. Otherwise, define the (infinite) sequences $\varphi_k^\delta := \|x_k^\delta - x^*\|^2$, for $k \geq -1$, and $\eta_k^\delta := \alpha_k^\delta \|x_k^\delta - x_{k-1}^\delta\|^2$, for $k \geq 0$ (here $x^* \in X$ is a solution of $Ax = y$). Arguing as in the proof of Proposition 2.5 (b), we apply Lemma A.1 to the sequences (α_k^δ) , (φ_k^δ) and (η_k^δ) to conclude that φ_k^δ converges strongly.³ The boundedness of (x_k^δ) follows from this fact. The boundedness of (w_k^δ) follows from the one of (x_k^δ) and the fact that $\|w_k^\delta\| = \|x_k^\delta + \alpha_k^\delta(x_k^\delta - x_{k-1}^\delta)\|$ for $k \geq 0$.

Proposition 2.14. Let (x_k^δ) , (w_k^δ) be sequences generated by Algorithm 2, and α_k^δ satisfy Assumption 2.11. If $\sum_k \lambda_k = \infty$, then the stopping index k^* defined in Step [3] is finite. Moreover, if $\lambda_k \geq \lambda > 0$ it holds

$$k^* \leq [2\lambda\tau\delta^2(\tau - 1)]^{-1} \left(\|x_0 - x^*\|^2 + M^\delta \sum_k \alpha_k^\delta + 2 \sum_k \theta_k \right),$$

where $x^* \in X$ is a solution of $Ax = y$, $M^\delta := \sup_k \|x_k^\delta - x^*\|^2$, and (θ_k) is the sequence in Assumption 2.11.

Proof. Assume by contradiction that k^* is not finite. It follows from Lemma 2.10 that

$$2\lambda_k\tau\delta^2(\tau - 1) \leq 2\lambda_k \|Ax_{k+1}^\delta - y^\delta\| \left[\|Ax_{k+1}^\delta - y^\delta\| - \delta \right] \leq \|w_k^\delta - x^*\|^2 - \|x_{k+1}^\delta - x^*\|^2,$$

for $k \geq 0$. From this inequality and (19) with $x = x^*$ we obtain

$$\begin{aligned} 2\lambda_k\tau\delta^2(\tau - 1) &\leq (1 + \alpha_k^\delta) \|x_k^\delta - x^*\|^2 + \alpha_k^\delta (1 + \alpha_k^\delta) \|x_k^\delta - x_{k-1}^\delta\|^2 - \|x_{k+1}^\delta - x^*\|^2 \\ &\leq \|x_k^\delta - x^*\|^2 - \|x_{k+1}^\delta - x^*\|^2 + \alpha_k^\delta \|x_k^\delta - x^*\|^2 + 2\alpha_k^\delta \|x_k^\delta - x_{k-1}^\delta\|^2 \\ &\leq \|x_k^\delta - x^*\|^2 - \|x_{k+1}^\delta - x^*\|^2 + \alpha_k^\delta M^\delta + 2\theta_k, \end{aligned} \quad (20)$$

³Notice that Lemma 2.10 is used together with Remark 2.12 to derive (10) for x_k^δ , y^δ , φ_k^δ , α_k^δ and λ_k ; additionally Assumption 2.11 is used to guarantee $\sum_k \eta_k^\delta < \infty$.

for $k \geq 0$ (to obtain the second inequality we used the fact $\alpha_k^\delta \leq \alpha < 1$). Adding up (20) for $k = 0, \dots, l$ we obtain

$$2\tau(\tau - 1)\delta^2 \sum_{k=0}^l \lambda_k \leq \|x_0^\delta - x^*\|^2 + M^\delta \sum_{k=0}^l \alpha_k^\delta + 2 \sum_{k=0}^l \theta_k. \quad (21)$$

Notice that the right hand side of (21) is bounded due to Assumption 2.11. Consequently, due to the assumption $\sum_k \lambda_k = \infty$, inequality (21) leads to a contradiction when $l \rightarrow \infty$, proving that k^* is finite.

To prove the second assertion, it is enough to take $l = k^*$ in (21) and argue with the additionally assumption $\lambda_k \geq \lambda > 0$. $\square$

Theorem 2.15 (Stability). *Let $(\delta^j)_j$ be a zero sequence, and $(y^{\delta^j})_j$ be a corresponding sequence of noisy data satisfying (1) for some $y \in \text{Rg}(A)$. For each $j \in \mathbb{N}$, let $(x_l^{\delta^j})_{l=-1}^{k_j^*}$ and $(w_l^{\delta^j})_{l=0}^{k_j^*-1}$ be sequences generated by Algorithm 2, with $(\alpha_l^{\delta^j})_{l=0}^{k_j^*}$ satisfying Assumption 2.11 (here $k_j^* = k^*(\delta^j, y^{\delta^j})$ are the corresponding stopping indices defined in Step [3]). Moreover, let $(x_k), (w_k)$ be the sequences generated by Algorithm 1 (with (α_k) satisfying Assumption 2.3). Then, either the sequences $(x_k), (w_k)$ are not finite and*

$$\lim_{j \rightarrow \infty} x_k^{\delta^j} = x_k, \quad \lim_{j \rightarrow \infty} w_k^{\delta^j} = w_k, \quad \text{for } k = 0, 1, \dots \quad (22)$$

or the sequences $(x_k) = (x_k)_{k=-1}^{k_0}$, $(w_k) = (w_k)_{k=0}^{k_0-1}$ are finite and it holds

$$\lim_{j \rightarrow \infty} x_k^{\delta^j} = x_k, \quad 0 \leq k \leq k_0 \quad \text{and} \quad \lim_{j \rightarrow \infty} w_k^{\delta^j} = w_k, \quad 0 \leq k \leq k_0 - 1, \quad (23)$$

for some $k_0 \in \mathbb{N}$ (in this case, $x_{k_0} \in X$ is a solution of $Ax = y$).

Proof. We present here a proof by induction. Notice that $w_0^{\delta^j} = w_0 = x_0 = x_0^{\delta^j}$ and $\alpha_0^{\delta^j} = \alpha_0 = \alpha$, for all $j \in \mathbb{N}$. Thus, (22) holds for $k = 0$. Next, assume the existence of $(x_l)_{l \leq k}$ and $(w_l)_{l \leq k}$ generated by Algorithm 1 (with $(\alpha_l)_{l \leq k}$ satisfying Assumption 2.3) such that $\lim_j x_l^{\delta^j} = x_l$ and $\lim_j w_l^{\delta^j} = w_l$, for $l = 0, \dots, k$.

Define $x_{k+1} := (\lambda_k A^* A + I)^{-1}(w_k + \lambda_k A^* y)$ as in Step [2.2] of Algorithm 1. Thus, from Step [2.3] of Algorithm 2 follows

$$\begin{aligned} x_{k+1}^{\delta^j} - x_{k+1} &= w_k^{\delta^j} - \lambda_k A^* (Ax_{k+1}^{\delta^j} - y^{\delta^j}) - [w_k - \lambda_k A^* (Ax_{k+1} - y)] \\ &= w_k^{\delta^j} - w_k - \lambda_k A^* A (x_{k+1}^{\delta^j} - x_{k+1}) + \lambda_k A^* (y^{\delta^j} - y). \end{aligned}$$

Consequently, $\|(I + \lambda_k A^* A)(x_{k+1}^{\delta^j} - x_{k+1})\| \leq \|w_k^{\delta^j} - w_k\| + \|A\| \lambda_k \delta^j$. This inequality together with the inductive hypothesis and the fact that $\lim_j \delta^j = 0$, allow us to estimate

$$\lim_{j \rightarrow \infty} \|(I + \lambda_k A^* A)(x_{k+1}^{\delta^j} - x_{k+1})\| \leq \lim_{j \rightarrow \infty} \|w_k^{\delta^j} - w_k\| + \|A\| \lambda_k \lim_{j \rightarrow \infty} \delta^j = 0. \quad (24)$$

Since $\|(I + \lambda_k A^* A)\| \geq 1$, we conclude from (24) that $\lim_j \|x_{k+1}^{\delta^j} - x_{k+1}\| = 0$.

At this point, we have to consider two complementary cases:

Case 1. $\|Ax_{k+1} - y\| = 0$. In this case, Algorithm 1 stops (see Remark 2.1). Consequently, (23) holds true with $k_0 = k + 1$ (it is immediate to see that $Ax_{k_0} = y$).

Case 2. $\|Ax_{k+1} - y\| > 0$. In this case we must consider two scenarios:

(2a) $x_{k+1} \neq x_k$. Choose α_{k+1} in agreement with Assumption 2.3, i.e.

$$\alpha_{k+1} := \min \{ \theta_{k+1} \|x_{k+1} - x_k\|^{-2}, \theta_{k+1}, \alpha \} > 0, \quad (25)$$

and define w_{k+1} as in Step [2.1] of Algorithm 1, i.e. $w_{k+1} := x_{k+1} + \alpha_{k+1}(x_{k+1} - x_k)$. Since $x_k^{\delta^j} \rightarrow x_k$ and $x_{k+1}^{\delta^j} \rightarrow x_{k+1}$ as $j \rightarrow \infty$, it follows from Assumption 2.11 and (25) that $\alpha_{k+1}^{\delta^j} \rightarrow \alpha_{k+1}$ as $j \rightarrow \infty$. Consequently,

$$\lim_{j \rightarrow \infty} w_{k+1}^{\delta^j} = \lim_{j \rightarrow \infty} (x_{k+1}^{\delta^j} + \alpha_{k+1}^{\delta^j}(x_{k+1}^{\delta^j} - x_k^{\delta^j})) = x_{k+1} + \alpha_{k+1}(x_{k+1} - x_k) = w_{k+1}. \quad (26)$$

(2b) $x_{k+1} = x_k$. Choose $\alpha_{k+1} := 0$, in agreement with Assumption 2.3, and define w_{k+1} as in Step [2.1] of Algorithm 1, i.e. $w_{k+1} := x_{k+1} + \alpha_{k+1}(x_{k+1} - x_k) = x_{k+1}$. Since $\lim_j x_k^{\delta^j} = x_k$, $\lim_j x_{k+1}^{\delta^j} = x_{k+1}$, and $(\alpha_{k+1}^{\delta^j})_j$ is bounded (see Assumption 2.11), it follows from Step [2.2] of Algorithm 2 that

$$\lim_{j \rightarrow \infty} w_{k+1}^{\delta^j} = \lim_{j \rightarrow \infty} (x_{k+1}^{\delta^j} + \alpha_{k+1}^{\delta^j}(x_{k+1}^{\delta^j} - x_k^{\delta^j})) = x_{k+1} = w_{k+1}. \quad (27)$$

Thus, it follows from (26) and (27) that, in Case 2, $w_{k+1}^{\delta^j} \rightarrow w_{k+1}$ as $j \rightarrow \infty$. This completes the inductive proof. Consequently, in Case 2, the assertions in (22) hold true. $\square$

Theorem 2.16 (Semi-convergence). *Let $(\delta^j)_j$ be a zero sequence, $(y^{\delta^j})_j$ be a corresponding sequence of noisy data satisfying (1) for some $y \in \text{Rg}(A)$, and assume that (A1), (A2) and (A3) in Theorem 2.8 hold. For each $j \in \mathbb{N}$, let $(x_l^{\delta^j})_{l=-1}^{k_j^*}$ and $(w_l^{\delta^j})_{l=0}^{k_j^*-1}$ be sequences generated Algorithm 2, with $(\alpha_l^{\delta^j})_{l=0}^{k_j^*}$ satisfying Assumption 2.11 (here $k_j^* = k^*(\delta^j, y^j)$ are the corresponding stopping indices defined in Step [3] of Algorithm 2). Then, the sequence $(x_{k_j^*}^{\delta^j})_j$ converges strongly to $x^\dagger$, the x_0 -minimal norm solution of $Ax = y$.*

Proof. It suffices to prove that every subsequence of $(x_{k_j^*}^{\delta^j})_j$ has itself a subsequence converging strongly to $x^\dagger$. In what follows, we denote a subsequence of $(x_{k_j^*}^{\delta^j})_j$ again by $(x_{k_j^*}^{\delta^j})_j$.

Let (x_k) , (w_k) be sequences generated by Algorithm 1 with exact data y and (α_k) as in the theorem assumptions. Two cases are considered:

Case 1. The corresponding subsequence $(k_j^*)_j \in \mathbb{N}$ has a finite accumulation point.

In this case, we can extract a subsequence $(k_{j_m}^*)_j$ of $(k_j^*)_j$ such that $k_{j_m}^* = n$, for some $n \in \mathbb{N}$ and all indices j_m . Applying Theorem 2.15 to (δ^{j_m}) and $(y^{\delta^{j_m}})$, we conclude that $x_{k_{j_m}^*}^{\delta^{j_m}} = x_n^{\delta^{j_m}} \rightarrow x_n$, as $j_m \rightarrow \infty$. We claim that $Ax_n = y$. Indeed, notice that $\|Ax_n - y\| = \lim_{j_m \rightarrow \infty} \|Ax_n^{\delta^{j_m}} - y\| \leq \lim_{j_m \rightarrow \infty} (\|Ax_n^{\delta^{j_m}} - y^{\delta^{j_m}}\| + \|y^{\delta^{j_m}} - y\|) \leq \lim_{j_m \rightarrow \infty} (\tau + 1)\delta^{j_m} = 0$.

Case 2. The corresponding subsequence $(k_j^*)_j$ has no finite accumulation point.

In this case we can extract a monotone strictly increasing subsequence, again denoted by $(k_j^*)_j$, such that $k_j^* \rightarrow \infty$ as $j \rightarrow \infty$.

Take $\varepsilon > 0$. From Theorem 2.8 follows the existence of $K_1 = K_1(\varepsilon) \in \mathbb{N}$ such that

$$\|x_k - x^\dagger\| < \frac{1}{3}\varepsilon, \quad k \geq K_1. \quad (28)$$

Since $\sum_k \theta_k$ is finite (see Assumption 2.11), there exists $K_2 = K_2(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{k \geq K_2} \theta_k \leq \frac{1}{3}\varepsilon. \quad (29)$$

Define $K = K(\varepsilon) := \max\{K_1, K_2\}$. Due to the monotonicity of the subsequence $(k_j^*)_j$, there exists $J_1 \in \mathbb{N}$ such that $k_j^* \geq K$ for $j \geq J_1$.

Theorem 2.15 applied to the subsequences $(\delta^j)_j, (y^{\delta^j})_j$ (corresponding to the subsequence $(k_j^*)_j$) implies the existence of $J_2 \in \mathbb{N}$ s.t.

$$\|x_K^{\delta^j} - x_K\| \leq \frac{1}{3}\varepsilon, \quad j > J_2. \quad (30)$$

Set $J := \max\{J_1, J_2\}$. From Lemma 2.10 (with $x^* = x^\dagger$) and Step [2.2] of Algorithm 2 follow

$$\|x_{k+1}^{\delta^j} - x^\dagger\| \leq \|w_k^{\delta^j} - x^\dagger\| \leq \|x_k^{\delta^j} - x^\dagger\| + \alpha_k^{\delta^j} \|x_k^{\delta^j} - x_{k-1}^{\delta^j}\|,$$

for $j \geq J$ and $k = 0, \dots, k_j^* - 1$. Consequently,

$$\|x_{k+1}^{\delta^j} - x^\dagger\| - \|x_k^{\delta^j} - x^\dagger\| \leq \sqrt{\alpha_k^{\delta^j}} \sqrt{\alpha_k^{\delta^j}} \|x_k^{\delta^j} - x_{k-1}^{\delta^j}\| \leq \frac{1}{2}\alpha_k^{\delta^j} + \frac{1}{2}\alpha_k^{\delta^j} \|x_k^{\delta^j} - x_{k-1}^{\delta^j}\|^2, \quad (31)$$

for $j \geq J$ and $k = 0, \dots, k_j^* - 1$. Now, adding (31) for $k = K, \dots, k_j^* - 1$ we obtain

$$\|x_{k_j^*}^{\delta^j} - x^\dagger\| \leq \|x_K^{\delta^j} - x^\dagger\| + \frac{1}{2} \sum_{k=K}^{k_j^*-1} \alpha_k^{\delta^j} + \frac{1}{2} \sum_{k=K}^{k_j^*-1} \alpha_k^{\delta^j} \|x_k^{\delta^j} - x_{k-1}^{\delta^j}\|^2, \quad j \geq J.$$

Thus, arguing with Assumption 2.11, together with (28), (29) and (30), we obtain

$$\begin{aligned} \|x_{k_j^*}^{\delta^j} - x^\dagger\| &\leq \|x_K^{\delta^j} - x^\dagger\| + \frac{1}{2} \sum_{k=K}^{k_j^*-1} \theta_k + \frac{1}{2} \sum_{k=K}^{k_j^*-1} \theta_k \\ &\leq \|x_K^{\delta^j} - x_K\| + \|x_K - x^\dagger\| + \sum_{k \geq K} \theta_k, \\ &\leq \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon, \quad j \geq J. \end{aligned}$$

Repeating the above argument for $\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \dots$ we are able to generate a sequence of indices $j_1 < j_2 < j_3 \dots$ such that

$$\|x_{k_{j_m}^*}^{\delta^{j_m}} - x^\dagger\| \leq \frac{1}{m}, \quad m \in \mathbb{N}.$$

This concludes Case 2, and completes the proof of the theorem. $\square$

3 Numerical experiments

In this section the *Inverse Potential Problem* [18, 13, 25, 46] and the *Image Deblurring Problem* [9, 8] are used to test the numerical efficiency of the iniT method. All computations are performed using MATLAB[®] R2017a, running on an Intel[®] Core[™] i9-10900 CPU.

3.1 The Inverse Potential Problem

The underlying forward problem is as follows. Let $\Omega \subset \mathbb{R}^2$ be bounded with Lipschitz boundary. For a given source $x \in L_2(\Omega)$, consider the elliptic boundary value problem (BVP) of finding u such that

$$-\Delta u = x, \text{ in } \Omega, \quad u = 0, \text{ on } \partial\Omega. \quad (32)$$

weakly.

The corresponding inverse problem is the Inverse Potential Problem (IPP). It consists of recovering an L_2 -function x , from measurements of the Neumann trace of its corresponding potential $u \in H^1(\Omega)$ on the boundary of Ω , i.e. we aim to recover $x \in L_2(\Omega)$ from the available data $y := u_\nu|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$.⁴

⁴Here u_ν denotes the outward normal derivative of u on $\partial\Omega$.

For issues related to 'redundant data' and 'identifiability' in IPP we refer the reader to [13] and the references therein. Generalizations of this linear inverse problem lead to distinct applications, e.g., Inverse Gravimetry [28, 46], EEG [15], and EMG [47].

The linear direct problem is modeled by the operator $A : L_2(\Omega) \rightarrow H^{-1/2}(\partial\Omega)$, which is defined by $Ax := u_\nu|_{\partial\Omega}$, where $u \in H_0^1(\Omega)$ is the unique weak solution of (32) (for the solution theory of this particular BVP we refer the reader to [19, 25]). Using this notation, the IPP can be modeled in the form (2), where the available noisy data $y^\delta \in H^{-1/2}(\partial\Omega)$ satisfies (1).

Discretization of the direct problem

The discretization of the BVP (32) relies on finite element techniques. Assume that $x \in L^2(\Omega)$ is piecewise constant, i.e., $x = \sum_{i=1}^N x_i \chi_i$, where $\chi_i(\cdot)$ is the characteristic function of the element $K_i \subset \Omega$. Those elements define a partition of Ω in the sense that $K_i \cap K_j = \emptyset$ for $i \neq j$, and $\bar{\Omega} = \cup_{i=1}^N \bar{K}_i$. As a preprocessing step, we determine $\gamma_i = (u_i)_\nu|_{\partial\Omega}$ for $i = 1, \dots, N$, where

$$-\Delta u_i = \chi_i \text{ in } \Omega, \quad u_i = 0 \text{ on } \partial\Omega. \quad (33)$$

Then, $\gamma = \sum_{i=1}^N x_i \gamma_i$.

Of course, solving (33) exactly is not feasible, and a further discretization step is necessary. Discretizing Ω into triangular finite elements of maximum length $h \ll \text{diameter } K_i$ for $i = 1, \dots, N$, and using a primal-hybrid finite element discretization [41, 3, 35], we compute γ_i^h as an approximation of γ_i .

Experiments with noisy data

The numerical experiments discussed in this section follow [11, 13]. Here, $\Omega = (0, 1) \times (0, 1)$ and the unknown ground truth x^* is assumed to be an L_2 -function (see Figure 1). The iniT method is compared with the iT method. The setup of our numerical experiments is detailed as follows:

- Solve problem (32) with $x = x^*$, and compute the exact data y .
- Add 0.1% and 5% of uniformly distributed random noise to the exact data, generating the noisy data y^δ .
- Use the constant function $x_0 = 1.5$ as initial guess for the iT and iniT methods.
- Employ the discrepancy principle with $\tau = 1.5$ as the stopping criteria for the iT and iniT methods.
- $\lambda_k = (\frac{2}{3})^k$ for both methods.
- Choose α_k^δ as in Assumption 2.11 for the iniT method with $\theta_k = (1/k)^{1.1}$.
- Compute x_{k+1}^δ in Step [2.3] of Algorithm 2 using the *conjugate gradient* (CG) method [20, 23] (in each iteration $k = 1, \dots, k^*(\delta)$, we use x_k^δ as initial guess for the CG method, to compute x_{k+1}^δ).
- In the iT method, obtain x_{k+1}^δ by solving $(\lambda_k A^* A + I)x = x_k^\delta + \lambda_k A^* y^\delta$ using the CG method (with x_k^δ as initial guess).

Numerical test 1 (0.1% noise): The iniT method and the iT method are implemented for solving the IPP under the above described setup, reaching the stop criteria after 17 and 19 steps respectively.

In Figure 1 the results obtained by the iniT method are presented. The stopping criterion is reached after $k^*(\delta) = 17$ steps. The top figure shows the exact solution, the center figure displays the approximate solution x_{17}^δ ; the relative iteration error $|x_{17}^\delta - x^*|/|x^*|$ is depicted

at the bottom figure. Note that the quality of the reconstruction is not as good close to the discontinuity curve of x^* , improving farther from it.

The progress of the corresponding *relative iteration error* $\|x_k^\delta - x^*\|/\|x^*\|$ and *relative residual* $\|Ax_k^\delta - y^\delta\|/\|y^\delta\|$ are depicted in Figure 2. In each of the subplots, we display the iT method in black, and the iniT method in red. Note the presence of a third curve, in blue. That corresponds to fixing $\alpha_k = 2/3$ constant, a choice not covered by our theory (see *Numerical tests 3* below for further discussion). In Table 1 the number of CG-steps evaluated at each iteration of the methods in Figure 2 is compared.

	Iteration number																		
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
iT	3	3	3	3	3	4	4	5	5	6	6	7	8	9	11	12	14	16	18
iniT ($\alpha_k = \frac{2}{3}$)	3	3	3	3	3	4	4	5	5	6	6	7	8	9	10	12	13		
iniT	3	3	3	3	3	4	4	5	5	6	6	7	8	9	11	13	14		

Table 1: Noise level 0.1%. Number of CG-steps required to compute x_k^δ in each step of the methods presented in Figure 2.

The accumulated number of CG-steps and the execution time of these methods read:

iT – 140 CG-steps	iniT ($\alpha_k = \frac{2}{3}$) – 104 CG-steps	iniT – 107 CG-steps
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Notice that the iT type methods require 30% more CG-steps than the iniT method.

Numerical test 2 (5.0% noise): Similarly to the previous case, we present in Figure 3 how the iniT and iT methods perform under a **5.0%** noise level. The stop criteria are reached after 9 and 12 steps respectively. The subplot display the corresponding *relative iteration error* (top) and *relative residual* (bottom). The curves in black and red correspond to the iT and iniT methods. The blue curve corresponds to a constant $\alpha_k = 2/3$ case, not covered by the theory (see *Numerical tests 3* for further discussion). In Table 2 the number of CG-steps needed in each iteration of the methods presented in Figure 3 is compared.

	Iteration number											
	1	2	3	4	5	6	7	8	9	10	11	12
iT	3	3	3	3	3	4	4	5	5	6	7	7
iniT ($\alpha_k = 0.8$)	3	3	3	3	3	4	4	5	5			
iniT	3	3	3	3	4	4	4	5	5			

Table 2: Noise level 5.0%. Number of CG-steps required to compute x_k^δ in each step of the methods presented in Figure 3.

The accumulated number of CG-steps and the execution time of these methods read:

iT – 53 CG-steps 0.80 seconds	iniT ($\alpha_k = 0.8$) – 33 CG-steps 0.53 seconds	iniT – 34 CG-steps 0.54 seconds
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Notice that the iniT type methods require 36% less CG-steps than the iT method; the computational time is reduced by 33%.

Numerical test 3 (experimenting with constant α_k^δ): Observing the choice of the scaling parameters in Nesterov’s accelerated forward-backward scheme (3), a natural question arises: “How does the iniT method performs if one chooses α_k^δ constant in Algorithm 2?”

Differently from the explicit two-point type methods (which include Nesterov's scheme), in the implicit two-point methods (or inertial methods) one must assume that $\sum_k \alpha_k < \infty$ and $\sum_k \alpha_k \|x_k - x_{k-1}\|^2 < \infty$. Indeed, these hypothesis are needed in the proof of the convergence Theorem 2.8 (they are used to obtain (9)). As a matter of fact, the summability of the series $\sum_k \alpha_k \|x_k - x_{k-1}\|^2$ is quintessential in our analysis: it is used to prove boundedness of the sequence (x_k) (see Proposition 2.5 (b)).⁵

Thus, choosing α_k^δ constant may not lead to bounded iterations. Nevertheless, we implemented the iniT method for different (constant) values of α_k^δ , ranging in the interval $(0,1)$. In Figures 4 and 5 we revisit the noise level scenarios of 0.1% and 5.0% respectively.

For the noise level of 0.1%, the best result was obtained for $\alpha_k^\delta = 2/3$, while $\alpha_k^\delta = 8/10$ was the best choice for the noise level of 5.0%. For comparison purposes, the iniT method results with these optimal choices of (constant) α_k^δ are presented in Figures 2 and 3 respectively (blue curves). Notice that:

- For (constant) α_k^δ close to zero, the iniT method performs similarly as the iT method;
- For (constant) α_k^δ close to one, the iniT method becomes unstable;
- The iniT method with constant choice $\alpha_k^\delta = 2/3$ performs similarly as the iniT method with α_k^δ satisfying Assumption 2.11.

Numerical test 4 (comparison with explicit two-point type methods): The Nesterov's scheme (3) and the FISTA method [7] are two well known methods for solving linear systems of the form (2). Another natural question that arises is: "How does the iniT method performance compare with performance of these established explicit two-point methods?"

To answer this question we revisit the 0.1% noise scenario. In Figure 6 the iniT method (implicit), the Nesterov and the FISTA methods (both explicit) are implemented for solving the IPP. In the Nesterov and FISTA methods we use $\gamma = \|A^*A\|^{-1/2}$. Moreover, in the Nesterov method we set (the frequently used value) $\alpha = 3.0$.

Due to the distinct nature (implicit/explicit) of these methods, the numerical effort cannot be compared by simply computing the number of iterations necessary to reach the stop criterion. Instead, we compute the execution time of these methods:

iniT – 1.40 seconds	Nesterov – 6.61 seconds	FISTA – 6.76 seconds
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3.2 The Image Deblurring Problem

The here considered Image Deblurring Problem (IDP) is an ill-posed inverse problem modeled by a finite-dimensional linear system of the form (2). In this problem $x^* \in X = R^n$ are the pixel values of a true image, while $y \in Y = X$ represents the pixel values of the observed (blurred) image. In real situations, blurred (noisy) data $y^\delta \in Y$ satisfying (1) are available and the noise level $\delta > 0$ is not always known.

In this setting the matrix A in (2) describes the blurring process, which is modeled by a space invariant point spread function (PSF). In the continuous model, the blurring phenomena is modeled by a convolution operator and A corresponds to an integral operator of the first kind [21, 17]. In our model, the discrete convolution is evaluated by means of the Fast Fourier Transform (FFT) algorithm. We added to the exact data, i.e. the true image convoluted with the PSF, a normally distributed noise with zero mean and suitable variance.

In our numerical implementations we follow [11] in the experimental setup, see Figure 7: (LEFT) True image $x^* \in R^n$, with $n = 2^{16}$ (Cameraman image 256×256); (CENTER) PSF

⁵An analog assumption is used in [2, Theor.2.1] to prove weak convergence of the inertial proximal method.

is the rotationally symmetric Gaussian low-pass filter of dimension $[257 \times 257]$ and standard deviation $\sigma = 4$; (RIGHT) Blurred image $y = Ax \in R^n$.

Experiments with noisy data

The iniT and the iT method are compared. The setup of our experiments is as follows:

- Exact data $y = Ax^* = PSF * x^*$ is computed.
- Noise of 0.1% and 1% is added to y , generating the noisy data y^δ .
- The constant function $x_0 = 0$ is used as initial guess for the iT and iniT methods.
- The discrepancy principle, with $\tau = 1.1$, is used as stop criterion for both methods.
- $\lambda_k = (\frac{2}{3})^k$ in both methods.
- Choose α_k^δ as in Assumption 2.11 for the iniT method with $\theta_k = (1/k)^{1.1}$.
- In both iniT and iT methods the computation of x_{k+1}^δ is performed explicitly (in the frequency domain the convolution corresponds to a multiplication).

Numerical test 5 (high/low noise): The iniT method and the iT method are implemented for solving the IDP under the above described setup. Two distinct levels of noise are considered, namely $\delta = 0.1\%$ and $\delta = 1.0\%$.

In the $\delta = 0.1\%$ noise scenario, the stop criteria are reached after 25 and 33 steps respectively. For $\delta = 1.0\%$, the stop criteria are reached after 6 and 8 steps respectively. The progress of the corresponding relative iteration error and relative residual for both methods are presented in Figure 8. It is worth noticing that the iniT method applied to this example displays similar computational savings as discussed in Section 3.1.

4 Final remarks and conclusions

In this manuscript we propose and analyze an implicit two-point type iteration, namely the *inertial iterated Tikhonov* (iniT) method, as an alternative for obtaining stable approximate solutions to linear ill-posed operator equations.

The main results discussed in this notes are: boundedness of the sequences (x_k) and (w_k) generated by the iniT method (Proposition 2.5, convergence for exact data (Theorem 2.8), stability and semi-convergence for noisy data (Theorems 2.15 and 2.16 respectively).

We faced an unexpected challenge in our analysis, namely the derivation of a monotonicity result for the iteration error (i.e. $\|x_{k+1} - x^*\| \leq \|x_k - x^*\|$, where x^* is a solution of $Ax = y$). This is a standard result in the analysis of iterative regularization methods [17, 29], and is used to establish boundedness and convergence of (x_k) . Due to the structure of the inertial iteration (5), we were only able to prove (under the current assumptions) that $\|x_{k+1} - x^*\| \leq \|w_k - x^*\|$ (see Proposition 2.5 (a)). Notice that, in order to establish the boundedness of (x_k) in Proposition 2.5 (b), we had to resort to Lemma A.1, which follows from a result by Alvarez and Attouch [2, Theorem 2.1].

The proof of the convergence result in Theorem 2.8 uses a novel strategy. The classical proof is based on a telescopic-sum argument coupled with the above mentioned monotonicity inequality. Since the monotonicity of $\|x_{k+1} - x^*\|$ is not available, we used an additional assumption on (α_k) (see Assumption (A2) in Theorem 2.8) in order to apply an alternative telescopic-sum argument (see (16)).

The lack of a monotonicity result also influences the verification of the stability and semi-convergence results (Theorems 2.15 and 2.16 respectively). The proofs presented here rely on properties of the sequences $(\alpha_k^{\delta^j})_k$ and $(\alpha_k^{\delta^j} \|x_k^{\delta^j} - x_{k-1}^{\delta^j}\|^2)_k$.

The choice of θ_k in Assumptions 2.3 and 2.11 plays a key role in the numerical performance of the iniT method. In our experiments, we tried $\theta_k = (1/k)^p$ for distinct choices of $p > 1$. The best results were obtained for values of p close to one.

Our numerical results demonstrate that the iniT method outperforms the standard iT method (for the same choice of Lagrange multipliers λ_k):

- IPP: iniT requires 10% to 20% less CG-steps than iT to reach the stop criterion (discrepancy).
- IDP: iniT requires approximately 25% less iterations than iT to reach the stop criterion.

A Appendix

In what follows we address a result, which is needed for the proof of Proposition 2.5. This result corresponds to a (small) part of the proof of [2, Theorem 2.1]. For the convenience of the reader, we present here a sketch of the proof.

Lemma A.1. *Let $(\alpha_k)_{k \geq 0} \in [0, \alpha]$, with $\alpha \in (0, 1)$. Moreover, let $(\varphi_k)_{k \geq -1}$, $(\eta_k)_{k \geq 0}$ be sequences of non-negative real numbers s.t. $\varphi_{k+1} - \varphi_k < \alpha_k(\varphi_k - \varphi_{k-1}) + 2\eta_k$, for $k \geq 0$, and $\sum_{k \geq 0} \eta_k < \infty$. There exists $\bar{\varphi} \in \mathbb{R}$ such that $\lim_{k \rightarrow \infty} \varphi_k = \bar{\varphi}$.*

Proof. Define $\gamma_k := \varphi_k - \varphi_{k-1}$, for $k \geq 0$. Thus, it follows from the assumptions

$$\gamma_{k+1} < \alpha_k \gamma_k + 2\eta_k \leq \alpha \gamma_k + 2\eta_k \leq \alpha[\gamma_k]_+ + 2\eta_k, \quad k \geq 0,$$

where $[t]_+ := \max\{t, 0\}$, $t \in \mathbb{R}$. Consequently, $[\gamma_{k+1}]_+ \leq \alpha[\gamma_k]_+ + 2\eta_k$, $k \geq 0$. A recursive use of this inequality yields $[\gamma_{k+1}]_+ \leq \alpha^{k+1}[\gamma_0]_+ + 2 \sum_{j=0}^k \alpha^j \eta_{k-j}$, for $k \geq 0$. Therefore,

$$\sum_{k=0}^{\infty} [\gamma_{k+1}]_+ \leq \frac{\alpha}{1-\alpha} [\gamma_0]_+ + \frac{2}{1-\alpha} \sum_{k=0}^{\infty} \eta_k.$$

Since $\sum_{k \geq 0} \eta_k$ is summable, the right hand side is finite.

Define $\zeta_k := \varphi_k - \sum_{j=1}^k [\gamma_j]_+$, for $k \geq 1$. Notice that (ζ_k) is bounded from below. Moreover, (ζ_k) is monotone non-increasing. Indeed,

$$\zeta_{k+1} = \varphi_{k+1} - [\gamma_{k+1}]_+ - \sum_{j=1}^k [\gamma_j]_+ \leq \varphi_{k+1} - (\varphi_{k+1} - \varphi_k) - \sum_{j=1}^k [\gamma_j]_+ = \zeta_k, \quad k \geq 1.$$

Consequently, (ζ_k) converges and we conclude $\lim_k \varphi_k = \sum_{j \geq 1} [\gamma_j]_+ + \lim_k \zeta_k =: \bar{\varphi}$. $\square$

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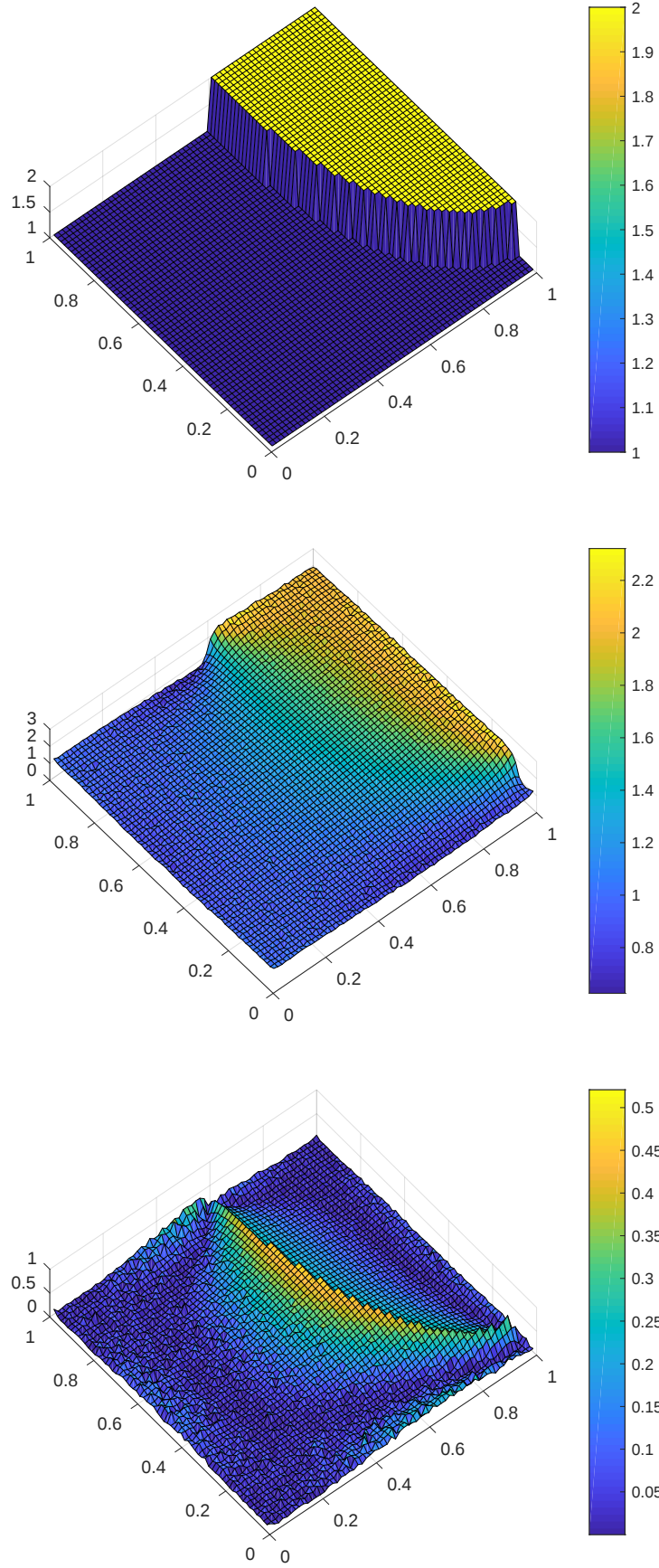


Figure 1: IPP Noise level 0.1%. Results obtained by the iniT method (the stop criterion is reached at $k^*(\delta) = 17$ steps). (TOP) Ground truth x^* ; (CENTER) Approximate solution x_{17}^δ ; (BOTTOM) Relative iteration error $|x_{17}^\delta - x^*|/|x^*|$.

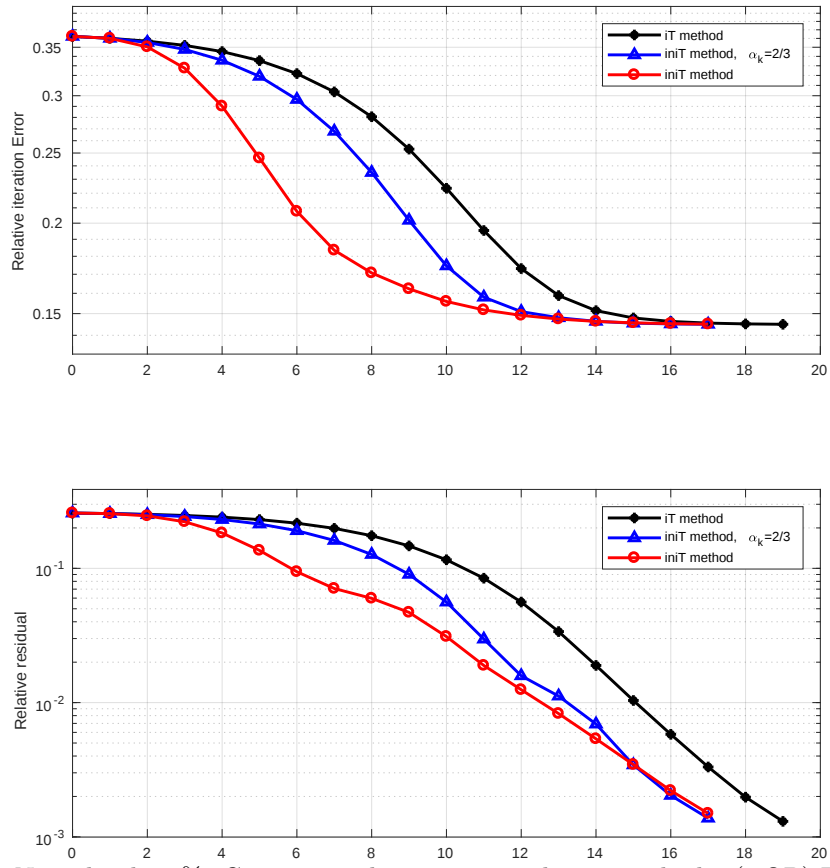


Figure 2: IPP Noise level 0.1%. Comparison between iT and iniT methods. (TOP) Relative iteration error; (BOTTOM) Relative residual.

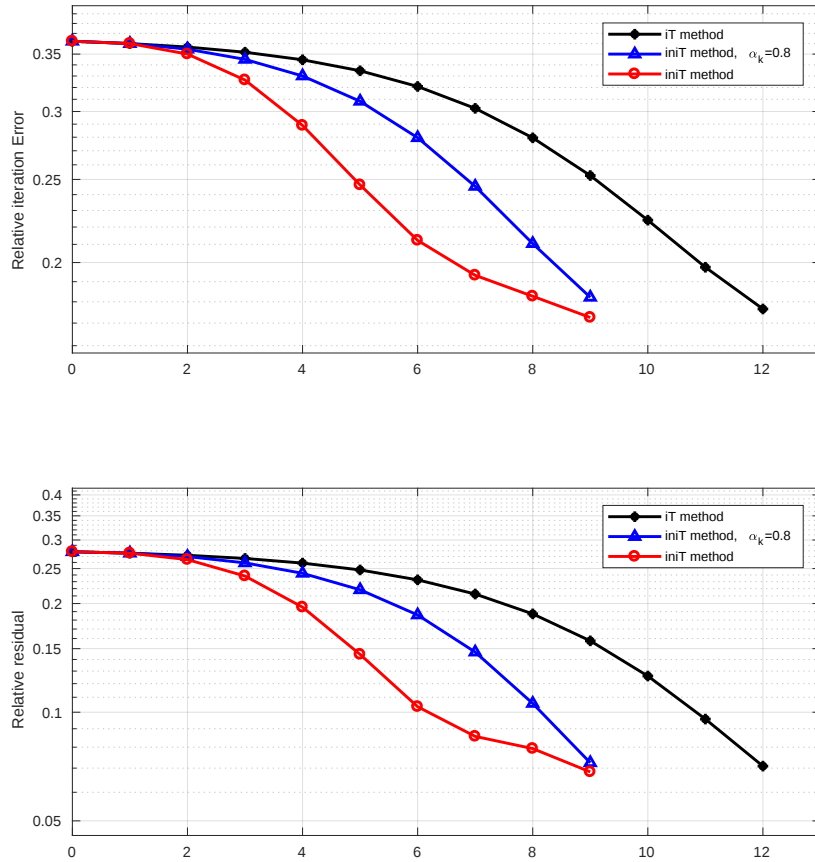


Figure 3: IPP Noise level 5.0%. Comparison between iT and iniT methods. (TOP) Relative iteration error; (BOTTOM) Relative residual.

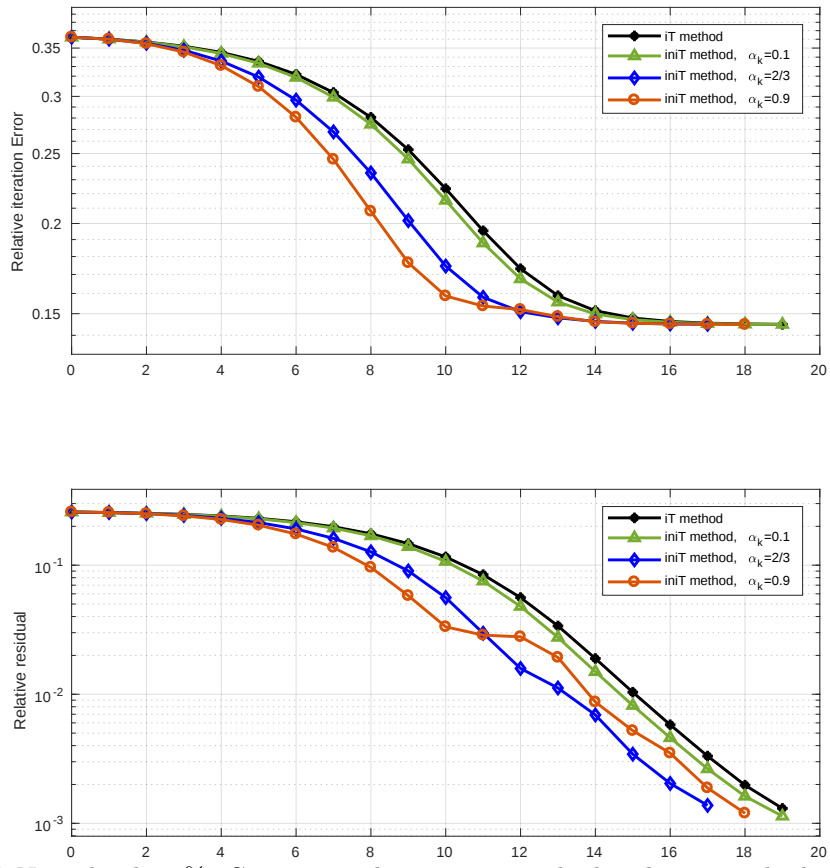


Figure 4: IPP Noise level 0.1%. Comparison between iT method and iniT method with constant α_k . (TOP) Relative iteration error; (BOTTOM) Relative residual.

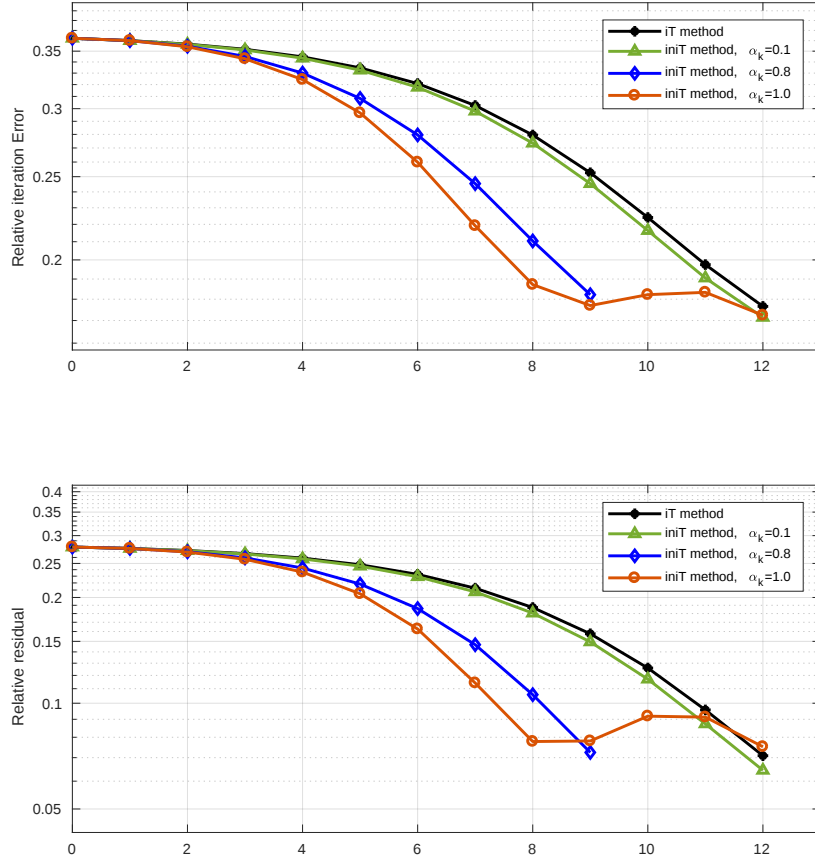


Figure 5: IPP Noise level 5.0%. Comparison between iT method and iniT method with constant α_k . (TOP) Relative iteration error; (BOTTOM) Relative residual.

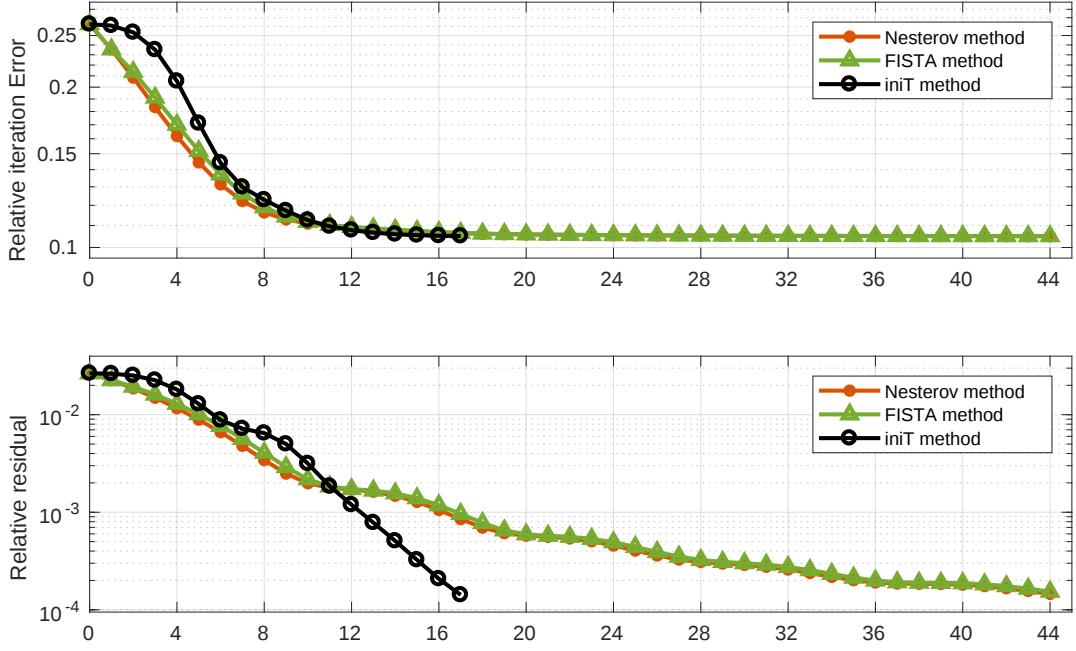


Figure 6: IPP Noise level 0.1%. Comparison between iniT, Nesterov and FISTA methods. (TOP) Relative iteration error; (BOTTOM) Relative residual.

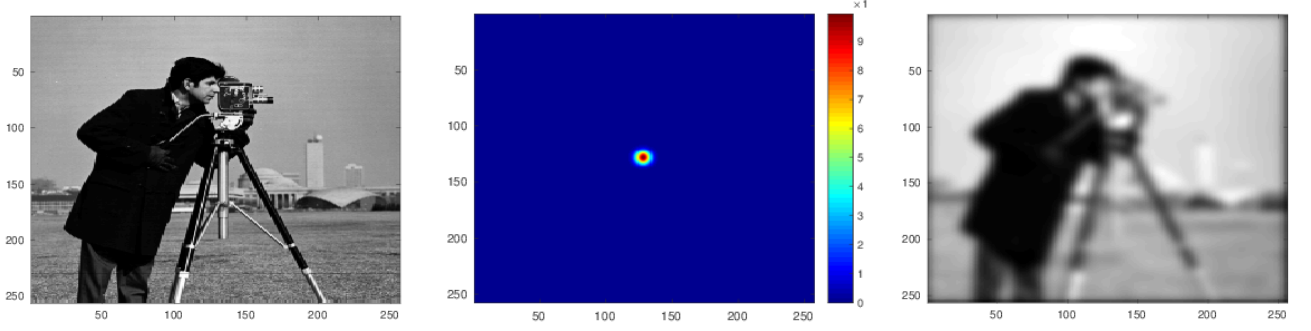


Figure 7: IDP. Setup: (LEFT) True image; (CENTER) PSF; (RIGHT) Blurred image.

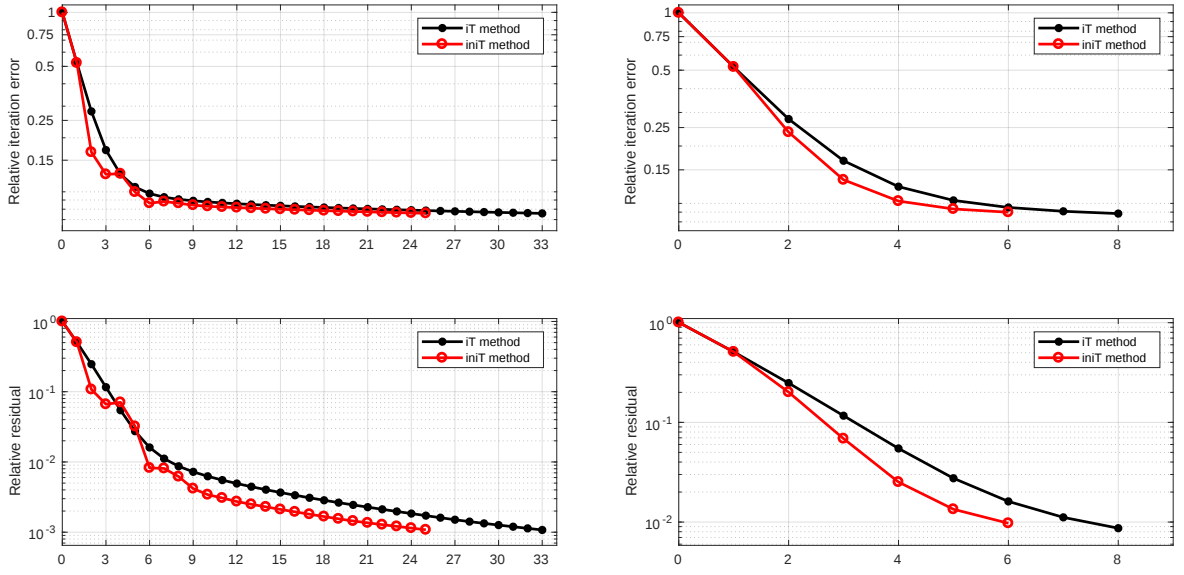


Figure 8: IDP. Relative iteration error and relative residual: (LEFT) $\delta = 0.1\%$; (RIGHT) $\delta = 1.0\%$.