Dynamic Inverse Problems: Efficient Algorithms and Approximate Inverse

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Content

Definitions

Static Inverse Problems Continuous and semi-discrete versions

$$Af(t,x) = \int k(t,x,y)f(y)dy = g(t,x)$$

$$A_\ell f = g_\ell$$

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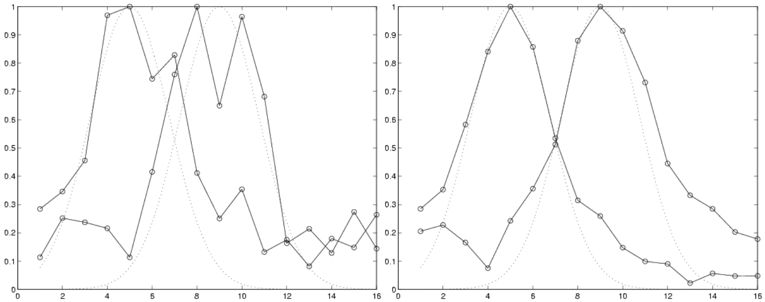
Dynamic Inverse Problems

$$Af(t,x) = \int\int k(t,\tau,x,y)f(\tau,y)dyd\tau$$

$$A_\ell f_\ell = g_\ell$$

NEED OF REGULARIZATION BOTH IN TIME AND SPACE

Temporal Regularization



Activation curves without and with temporal regularization

Undertermined Problems

Consider

- A is matrix which has more colums than rows
- A maps from an infintite-dimensional Hilbert space to finitely many data
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$\Rightarrow$ Prefer to solve $Af = g$ as

$$AA^*u = g$$

and put

$$f = A^*u$$

or its Tikhonov - Phillips variants.

Content

Two Constraints

Minimize

$$\|Af - g\|^2 + \gamma^2 \|f\|^2 + \mu^2 \|Bf\|^2$$

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Only when A and B commute then this can be written in the above mentioned form.

Formulation with slack variable

Minimize with respect to f and d

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Change of variables $d := \frac{\gamma}{\mu} y$ Then this is

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With the new variables

$$\xi = \begin{pmatrix} f \\ y \end{pmatrix} \text{and h} = \begin{pmatrix} g \\ 0 \end{pmatrix}$$

this is equivalent to

Compact Formulation

Minimze

$$\|T_\alpha \xi - h\|^2 + \gamma^2 \|\xi\|^2$$

with

$$T_\alpha = \begin{pmatrix} A & 0 \\ \alpha B & -\alpha \frac{\gamma}{\mu} I \end{pmatrix}$$

The solution of

$$T_\alpha T_\alpha^* \begin{pmatrix} u \\ v \end{pmatrix} = h$$

with

$$\xi = T_\alpha^* \begin{pmatrix} u \\ v \end{pmatrix}$$

leads to the desired result.

Efficient Minimization with two constraints

Theorem (L., Schmitt, 2002, 2007)

The minimzation of

$$\|Af - g\|^2 + \gamma^2\|f\|^2 + \mu^2\|Bf\|^2$$

is computed in two steps. First we solve for u with $\omega = \frac{\gamma^2}{\mu^2}$

$$A\big(I - B^*(BB^* + \omega I)^{-1}B\big)A^*u + \gamma^2 u = g$$

and put

$$f_{\gamma,\mu} = \big(I - B^*(BB^* + \omega I)^{-1}B\big)A^*u$$

Semi - discrete Case

$$A_\ell : H \to G_\ell, \ \ell = 1, \dots, L$$

$$A_\ell F_\ell = G_\ell$$

Minimize

$$J(f) = \sum_{\ell=1}^{L} \|A_\ell f_\ell - g_\ell\|^2 + \gamma^2 \sum_{\ell=1}^{L} \|f_\ell\|^2 + \mu^2 \sum_{\ell=1}^{L-1} \frac{\|f_{\ell+1} - f_\ell\|^2}{(t_{\ell+1} - t_\ell)^2}$$

Define

$$A = \mathrm{diag}(A_\ell) \in \mathcal{L}(H^L, G_1 \oplus \cdots \oplus G_L)$$

$$f = (f_1, \cdots f_L)$$

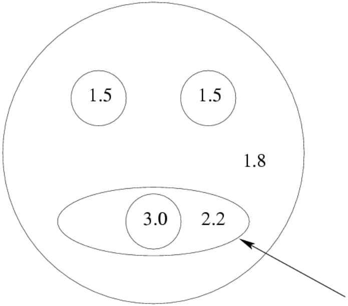
$$B = D \oplus I_H \in \mathcal{L}(H^L, H^{L-1}$$

$$D = \begin{pmatrix} \tau_1 & -\tau_1 & & & \\ & \tau_2 & -\tau_2 & & \\ & & \cdot & \cdot & \\ & & & \tau_{L-1} & -\tau_{L-1} \end{pmatrix} \in \mathbb{R}^{L \times (L-1)}$$

$$\tau_i = (t_{i+1} - t_i)^{-1}$$

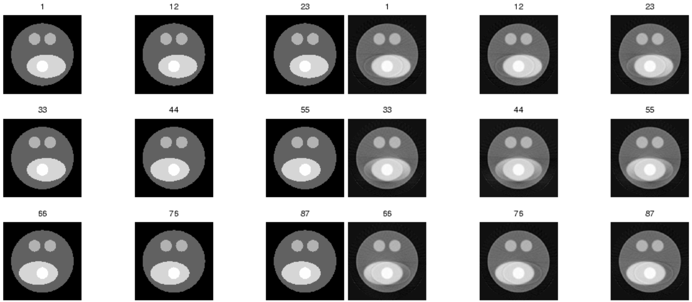
Generalized Sylvester Equations

Dynamic Computerized Tomography



$$R_\ell f(s) = Rf(\omega_\ell, s) = \int f(s\omega + t\omega^\perp)dt$$

Dynamic Computerized Tomography



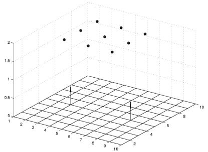
Current Density Reconstruction

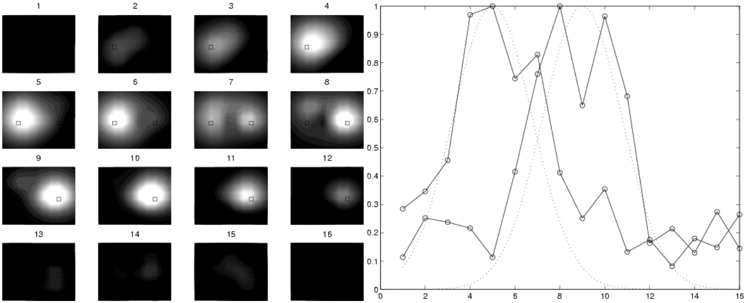
Application: study of neurological activity in the brain
Forward Model

$$\mathrm{div}(\sigma\nabla\Phi) = \mathrm{div} j \text{ in } \Omega$$

$$\langle \sigma\nabla\Phi, n\rangle = 0 \text{ at } \delta\Omega$$

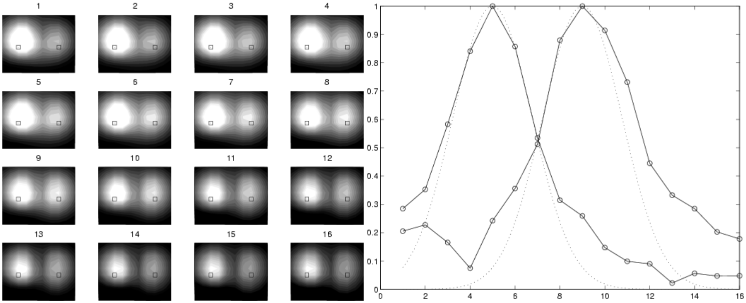
σ conductivity tensor
Φ electrical potential
Data Φ at the boundary





Reconstruction without temporal smoothness constrains

Current Density Reconstruction



Reconstruction with temporal smoothness constrains

Approximate Inverse

L., Inverse Problems,1996, 1999
Compare: Backus-Gilbert, 76, Grünbaum, 80, Smith 80
Given:

- $A : L^2(\Omega_1, \mu_1) \to L^2(\Omega_2, \mu_2)$ linear, continuous
  Mollifier $\delta_x \approx e_\gamma(x, \cdot)$ or $\delta'_x \approx e_\gamma(x, \cdot)$

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- Compute

$$f_\gamma(x) = \langle f, e_\gamma(x, \cdot)\rangle_{L^2(\Omega_1,\mu_1)} = \langle Af, \psi_\gamma(x)\rangle_{L^2(\Omega_2,\mu_2)}$$

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Idea:

- Solve

$$A^*\psi_\gamma(x) = e_\gamma(x, \cdot)$$

Approximate Inverse in L^2-Spaces

Data g given

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Approximate Inverse

$$S_\gamma g(x) := \langle g, \psi_\gamma(x) \rangle_{L^2(\Omega_2,\mu_2)}$$

Theorem (L, 1997)

Let the operators T_1, T_2 intertwine with A^*; i.e.,

$$T_1 A^* = A^* T_2$$

and solve for a reference mollifier E_γ the equation

$$A^* \Psi_\gamma = E_\gamma$$

Then the general reconstruction kernel for the general mollifier $e_\gamma = T_1 E_\gamma$ is

Linear Regularization

$$
\begin{array}{cccccc}
 & & & & Y_1 & \\
 & & & \overset{A^+}{\swarrow} & \uparrow & \tilde{M}_\gamma \\
A: & & X & \longrightarrow & Y & \\
 & M_\gamma & \uparrow & \overset{\overline{A^+}}{\swarrow} & & \\
 & & X_{-1} & & &
\end{array}
$$

$$S_\gamma = M_\gamma \overline{A^+} = A^+ \tilde{M}_\gamma$$

Tikhonov - Phillips as Approximate Inverse

Solve

$$E_\gamma = \Psi_\gamma A$$

via Tikhonov - Phillips, then

$$S_\gamma = A^*\left(AA^* + \lambda I\right)^{-1}$$

Use Invariances to further speed up the method