# Online Parameter Identification in ODEs and PDEs

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# Problem Statement

- **Given a dynamical system**

$$\begin{aligned} u_t &= f(u, q_*) \\ u(0) &= u_0 \end{aligned}$$

**that describes the time evolution of the physical state variable $u$, e.g.,**

$\star$ **an ODE model from aircraft dynamics**

$\star$ **a PDE model from heat conduction**

- **and the possibility to perform measurements of**

$$z(t) = Hu(t),$$

**e.g.,**

$\star$ **observations of airspeed and altitude of the aircraft**

$\star$ **observations of boundary temperatures**

- **Task: Given an initial guess $q_0$ at $t = 0$, identify the unknown model parameters $q_*$, e.g.,**
  - $\star$ **lift and drag coefficents**
  - $\star$ **heat conductivity**

  **simultaneously to the evolution of the system**

- **Online algorithms become necessary whenever the mathematical model is needed to support decisions that have to be taken during the operation of the real system (adaptive control)**

# Ill-Posedness

- **Parameter identification is typically ill-posed:**
  - ⋆ **the solution, if it exists, need not be unique**
  - ⋆ **and will not depend continuously on the data $z$**

- **Need for regularization methods**
  - ⋆ **regularization means to replace the ill-posed problem by a family of neighboring well-posed problems**

- **Especially crucial for computations in the presence of noisy data!**

# Basic Sketch of an Online Estimator

- see extra slide

# Approaches given in Literature

- **Linear and Finite Dimensional Problems (P. Ioannou, J. Slotine,...):**

  **Assumption of a linear relation between $q_*$ and observation $z(t)$**

  $$z(t) = A(t)q_* \text{ with } A(t) \in \mathbb{R}^{m\times n}$$

- **Online Identification in PDEs (J. Baumeister, M. Demetriou,...):**

  **Estimators exploit particular structures of underlying PDE-operators such that full state observation is necessary, i.e., $z = u$**

- **Continuous extended Kalman filter for nonlinear ODEs (M. Grewal, A. Andrews, ...):**
  **Estimation of extended state $x$ in**

  $$x_t = \begin{pmatrix} u_t \\ q_{*t} \end{pmatrix} = \begin{pmatrix} f(u, q_*) \\ 0 \end{pmatrix}$$

  **from observations of $z(t) = Bx(t) = Hu(t)$**

# Linear and Finite Dimensional Case

- **J. Slotine et al. (2000): Given a linear relation between $q_*$ and observation $z(t)$**

$$z(t) = A(t)q_*$$

- **Using estimate $q(t)$ at time $t$, the output can be simulated by**

$$z_{\textbf{sim}}(\tau) = A(\tau)q(t) \textbf{ for } \tau \leq t$$

- **Time-derivative of the first order necessary condition for**

$$\int_0^t e^{-\alpha(\tau-t)} \|z(\tau) - z_{\textbf{sim}}(\tau)\|^2 d\tau \rightarrow \min_{q(t)}$$

- **yields the parameter update law**

$$\begin{aligned} q_t &= G(t)A(t)^T(z(t) - z_{\textbf{sim}}(t)), \quad q(0) = q_0 \\ G_t &= \alpha G(t) - G(t)A(t)^T A(t)G(t), \quad G(0) = G_0 \end{aligned}$$

# Persistence of Excitation

- **Parameter convergence**

$$q(t) \to q_* \textbf{ as } t \to \infty$$

**is proven under a so-called persistence of excitation condition**

$$\exists \gamma, \beta > 0 \; \forall t \in \mathbb{R}^+ \; \int_t^{t+\beta} A(\tau)^T A(\tau) d\tau \geq \gamma I$$

- **Signals have to be strong enough in order to generate a rich data set**

- **Motivation: Consider normal equation**

$$A^T(t) A(t) q_* = A^T(t) z(t);$$

**if** $\int A^T(\tau) A(\tau)\, d\tau$ **is positive definite, then** $q_*$ **is uniquely determined**

# Nonlinear and Infinite-Dimensional Case

- Define the nonlinear prediction operator

$$F(\cdot,\tilde{t}) : q \rightarrow z_{\textbf{sim}}(\tilde{t}) = Hu(\tilde{t}),$$

evaluated via integration up to time $\tilde{t}$ of

$$u_t = f(u,q), \qquad u(0) = u_0$$

- Update law motivated by approach in linear case

$$\begin{aligned} q_t(t) &= G(t)F'(q,t)^*(z(t) - z_{\textbf{sim}}(t)), \qquad q(0) = q_0, \\ G_t(t) &= \alpha(G(t) - G(t)\bar{G}^{-1}G(t)) - G(t)F'(q,t)^*F'(q,t)G(t), \qquad G(0) = G_0. \end{aligned}$$

- $F'$ is defined via the linearized dynamical system

- $\bar{G}$ acts as an upper bound for $G(t)$

# Convergence Theory

- **In case of exact data, convergence**

$$q(t) \to q_* \text{ as } t \to \infty$$

**if persistence of excitation condition**

$$\exists \gamma,\, \beta > 0\; \forall t \in \mathbb{R}^+ \;\; \int_t^{t+\beta} F'(q,\tau)^* F'(q,\tau)\, d\tau \geq \gamma I$$

**as well as a condition restricting the nonlinearity of $F$ is satisfied** ▮

- **Idea of proof: show that**

$$V(e,t) = (G^{-1}(t)e, e)$$

**with**

$$e(t) = q(t) - q_*$$

**is a Lyapunov function**

# Case of Noisy Data

- **Given an error bound for noisy data $z^\delta$**

$$\|z^\delta(t) - z(t)\| \leq \delta \text{ for all t}$$

- **Dead zone approach**

$$q_t = \begin{cases} \text{as for exact data} & \text{if } \|z^\delta(t) - z_{\text{sim}}(t)\| \geq \tau\delta \\ 0 & \text{if } \|z^\delta(t) - z_{\text{sim}}(t)\| < \tau\delta \end{cases}$$

**Update of $q$ only if the error between data and prediction lies above the noise level**

- **Leakage approach**

$$q_t(t) = G(t)F'(q,t)^*(z^\delta(t) - z_{\text{sim}}(t)) - \sigma G(q - q_0)$$

**enforces $q$ to stay within a neighborhood of $q_0$**

# ODE Example

- **The longitudinal motion of an aircraft (pitching and translation in the vertical plane) can be described by**

$$\begin{aligned}
m\dot{V}_T &= F_T\cos(\alpha) - D - mg_D\sin(\theta-\alpha)\\
m\dot{\alpha}V_T &= -F_T\sin(\alpha) - L + mg_D\cos(\theta-\alpha) + mV_TQ\\
\dot{\theta} &= Q\\
\dot{Q} &= M/J_Y\\
\dot{h} &= V_T\sin(\theta-\alpha)
\end{aligned}$$

  - ⋆ **for airspeed $V_T$, angle of attack $\alpha$, pitch attitude $\theta$, pitch rate $Q$ and altitude $h$**
  - ⋆ **with mass $m$, thrust $F_T$, inertia $J_Y$**
  - ⋆ **and drag $D$, lift $L$ and moment $M$**

- **Drag, lift and pitching moment are given by**

$$\begin{aligned} D &\sim V_T^2 S C_D \\ L &\sim V_T^2 S C_L \\ M &\sim V_T^2 S C_M \end{aligned}$$

**with wing area** $S$**, airspeed** $V_T$ **and aerodynamic coefficients**

$$C_{D,L,M} = C_{D,L,M}(\alpha, \mathbf{mach\ number}, \mathbf{thrust}, \mathbf{altitude\ h}, \mathbf{R_e} \ldots)$$

- **Ansatz for aerodynamic coefficients**

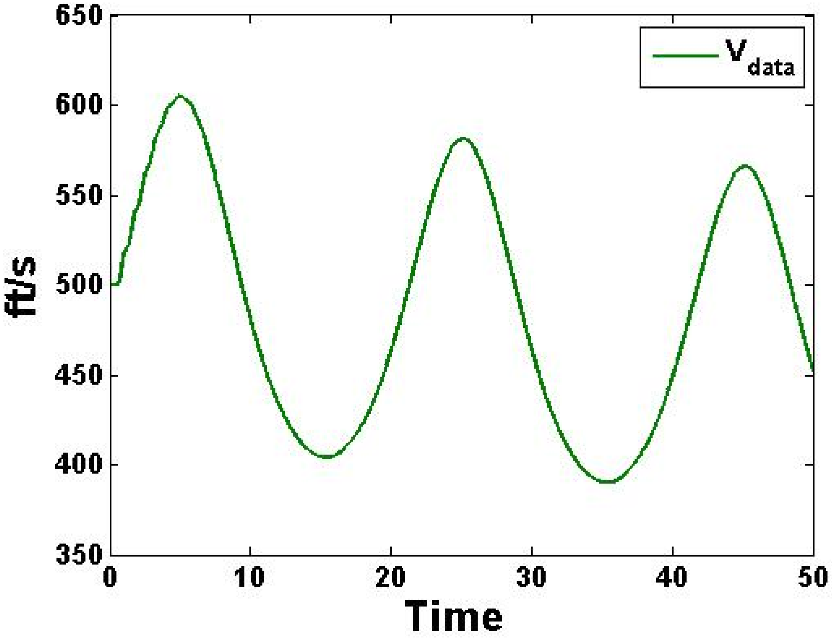
$$\begin{aligned} C_L &= q_2 + q_1 \alpha \\ C_D &= q_3 + C_L^2 \end{aligned}$$

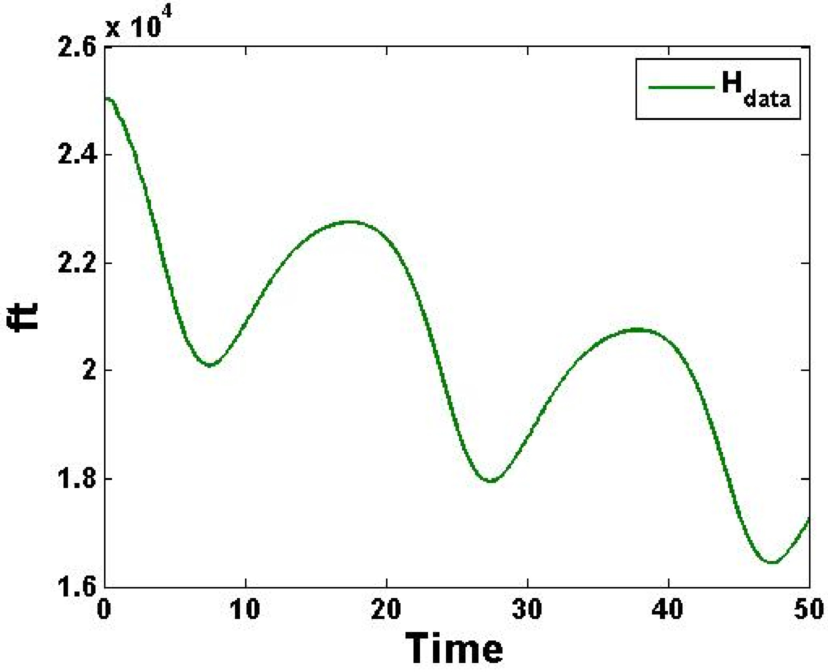
- **Short form with** $u = [V_T, \alpha, \theta, Q, h]$ **and** $q_* = [q_1, q_2, q_3]$

$$u_t = f(u, q_*)$$

# Exact Data

- Measurements of airspeed and altitude

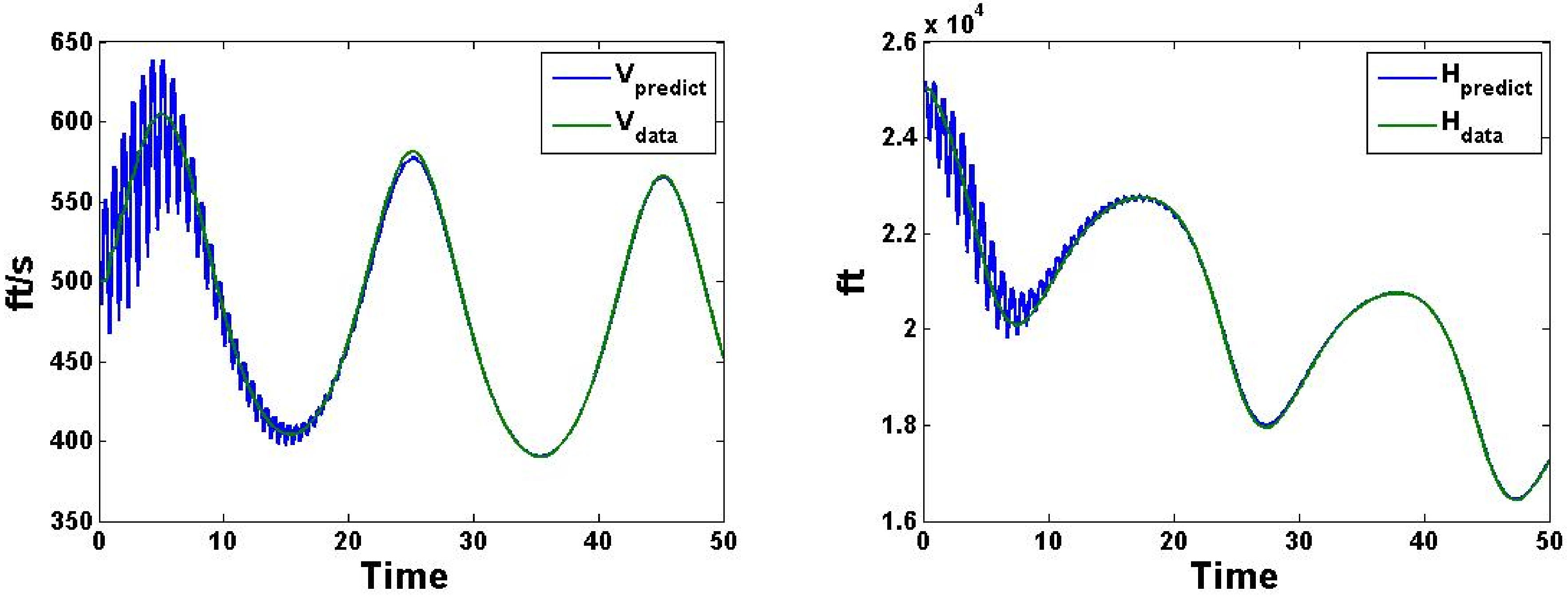




while the aircraft is climbing and falling due to changes in thrust and elevation controllers

- Simulated with true parameter $q_* = [0.85, 1., 1.6]$
- Initial guess $q_0$ with $25\%$ deviation from $q_*$ in each component

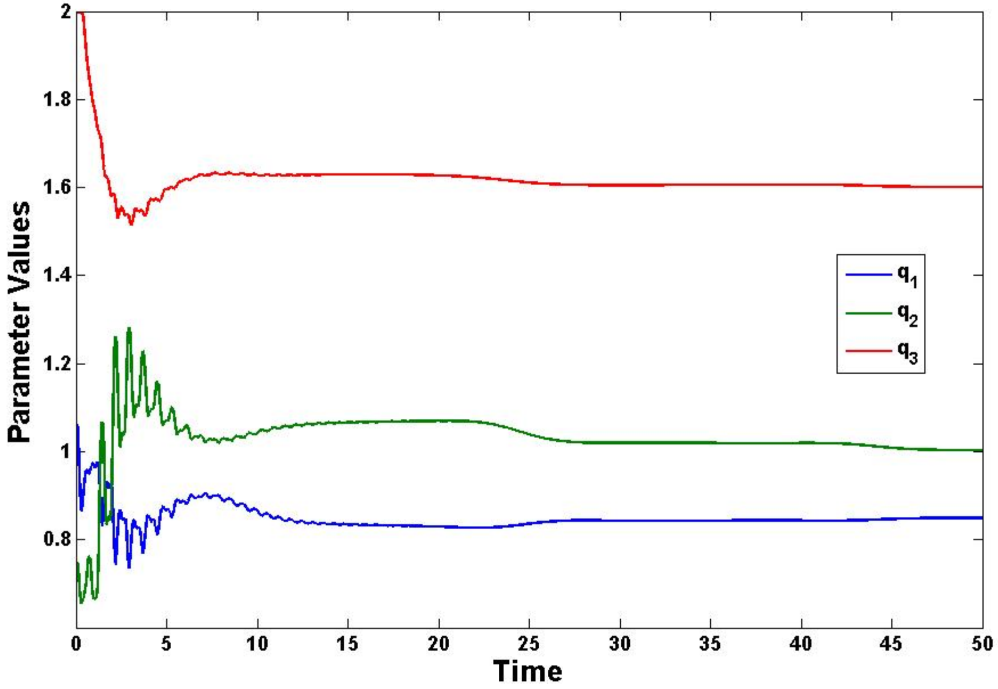
# Online Result for Exact Data



- After some time, the simulated output $z_{\text{sim}}$ follows the data $z$ $\rightarrow$ data tracking

# Online Result for Exact Data

- **Data allow for a unique identification**



- **Parameter convergence, i.e.,** $q(50) = [0.849, 1.003, 1.6008]$
- **Conjecture: persistence of excitation is given**

# Additional Parameter Component

- **Consider**

$$\begin{aligned} C_L &= q_2 + q_1\alpha \\ C_D &= q_3 + C_L^2 \\ C_M &= q_4 + c_1\alpha + c_2 C_L \end{aligned}$$

  **with**

$$q_* = [0.85, 1, 1.6, 0.5]$$

  **which gives exactly the same data as before**

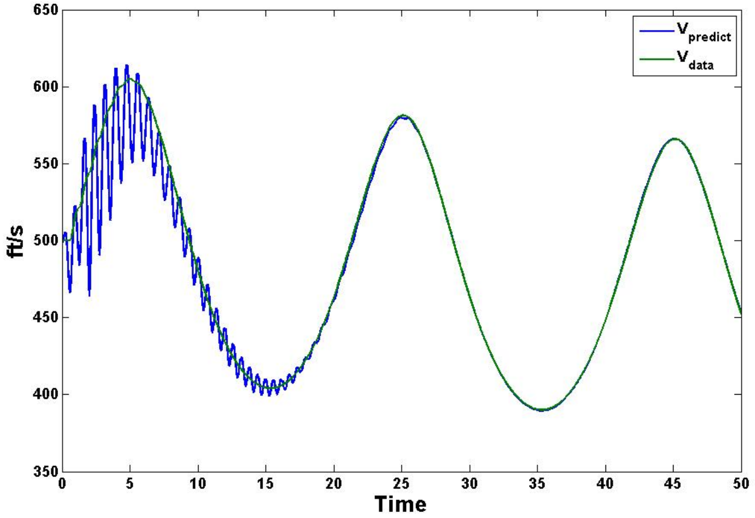
- **Offline result for exact data**
  - ⋆ **quasi Newton-technique for**

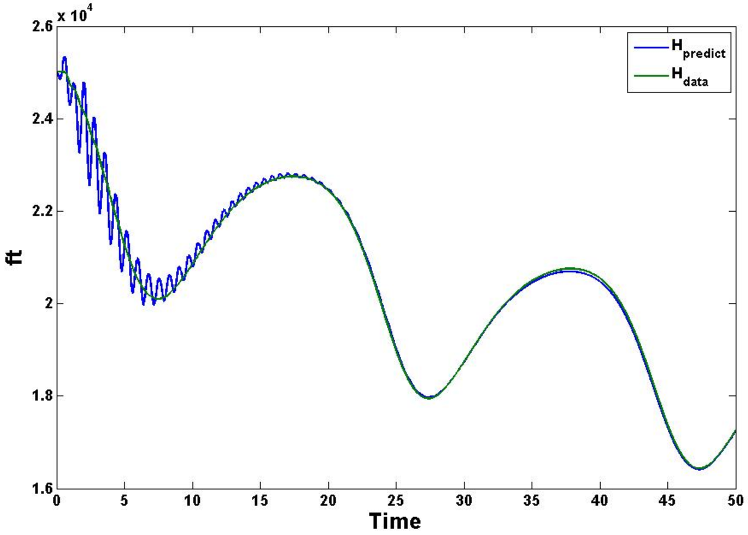
$$\|z - F(q)\|^2 \to \min_q$$

    **yields** $q = [0.8496, 0.8312, 1.5994, 0.5872]$

- **Non-uniqueness**

# Online Result for Exact Data - Output Space

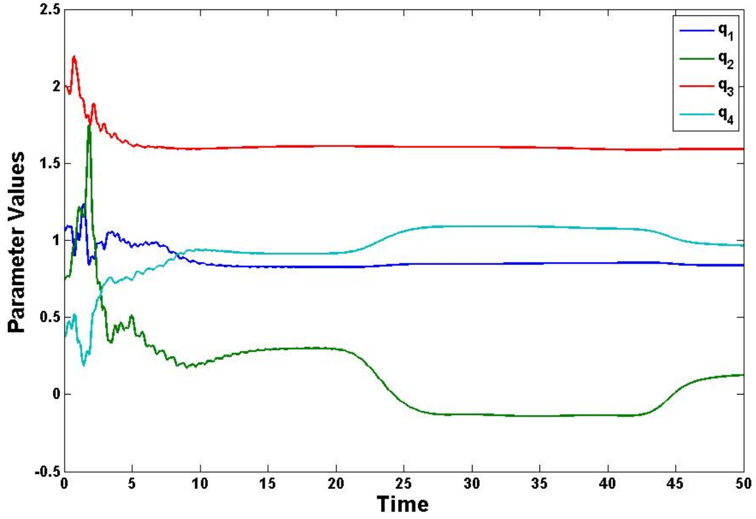


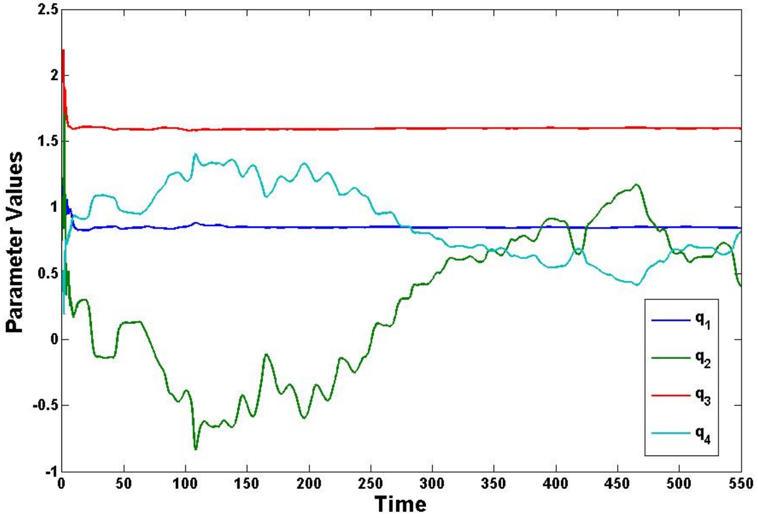


- **Again, tracking property is given**

# Online Result for Exact Data - Parameter Space

- **However, convergence only in two components**

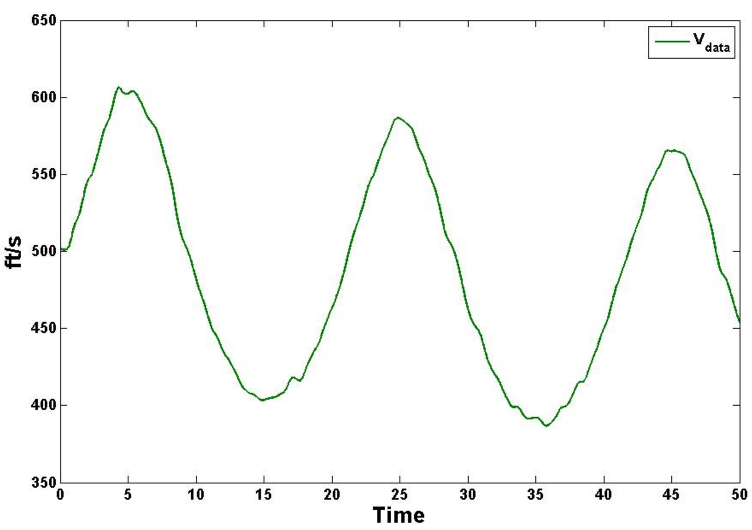


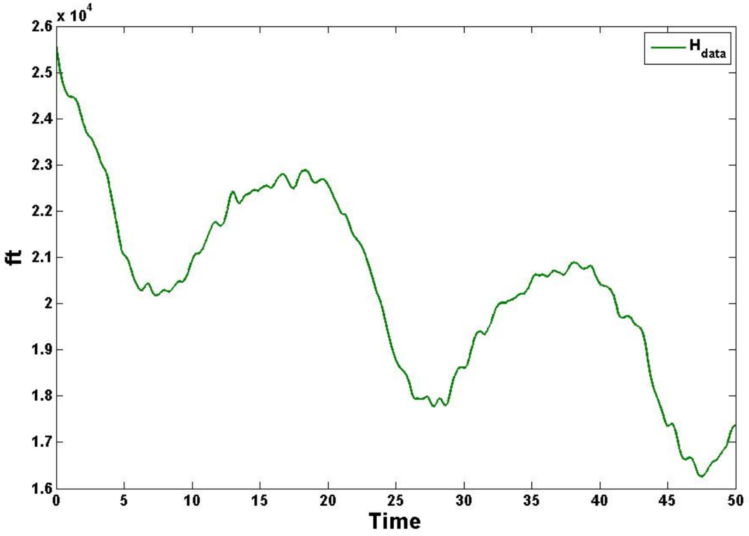


- **Conjecture: no persistence of excitation**

# Noisy Data

- **Perturbed measurements of airspeed and altitude**



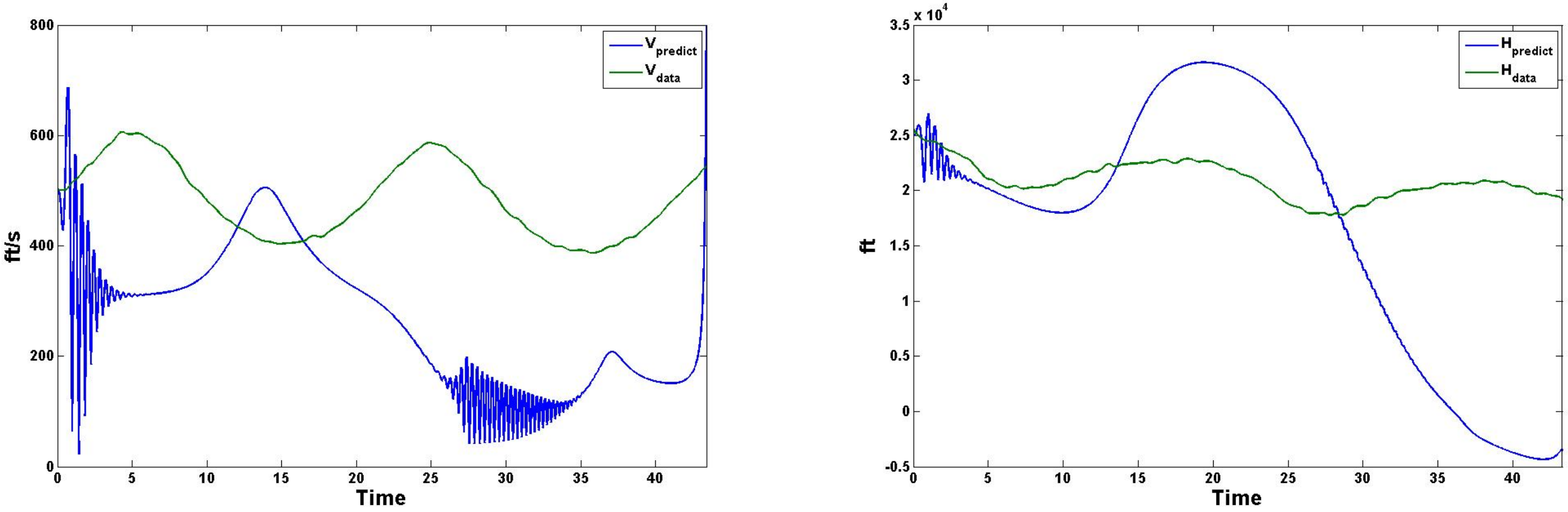


- **Error bounds**

$$\frac{|z_V^\delta(t) - z_V(t)|}{|z_V(t)|} \le 0.0142 \quad \textbf{and} \quad \frac{|z_H^\delta(t) - z_H(t)|}{|z_H(t)|} \le 0.025$$

# Online Result for Noisy Data - Output Space

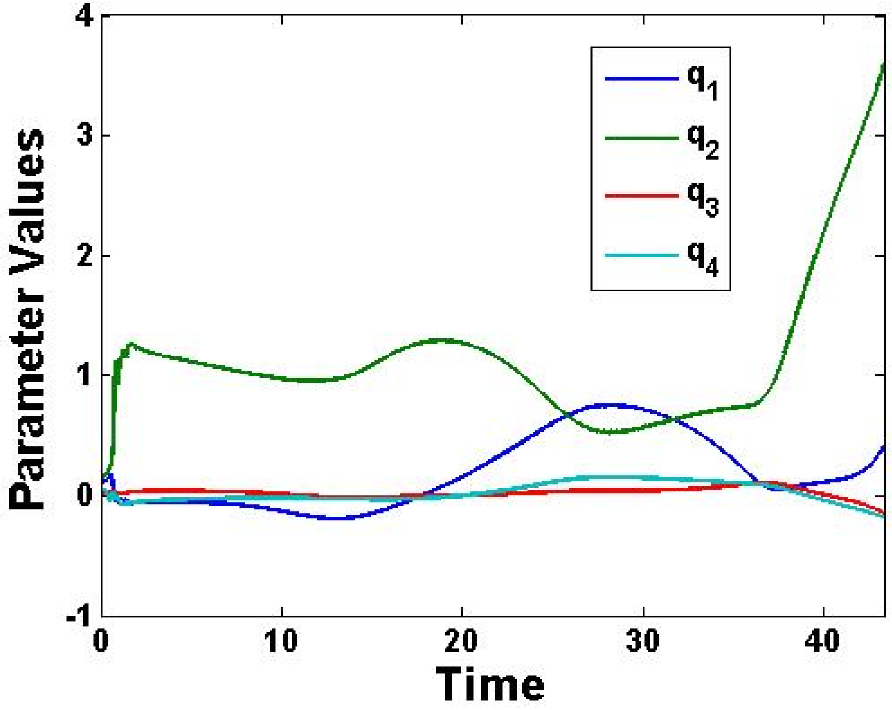
- No regularization



- Data tracking property is lost

# Online Result for Noisy Data - Parameter Space

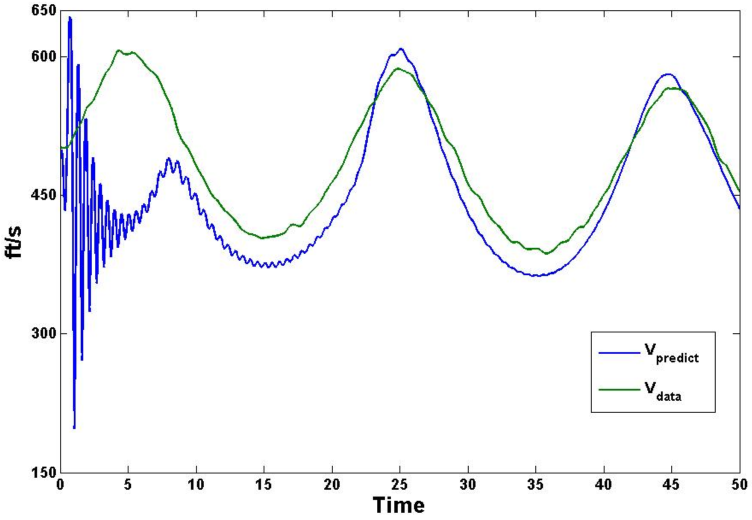
- No regularization

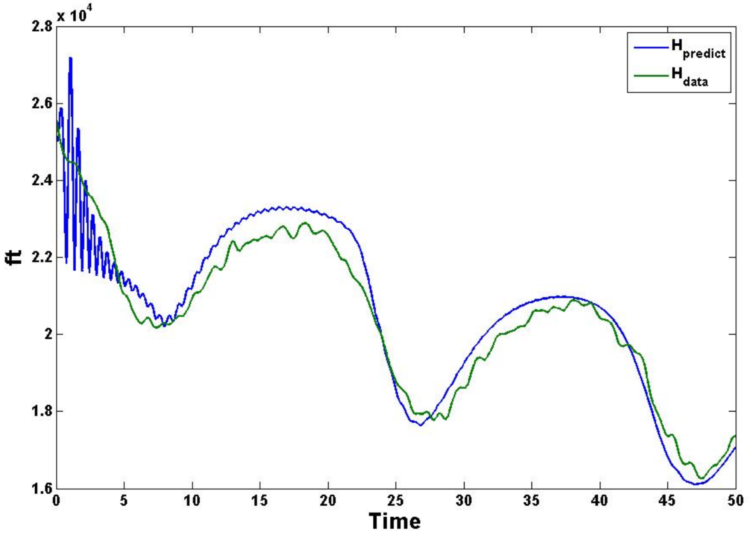


- Parameter divergence

# Online Result for Noisy Data - Output Space

- Use of leakage term

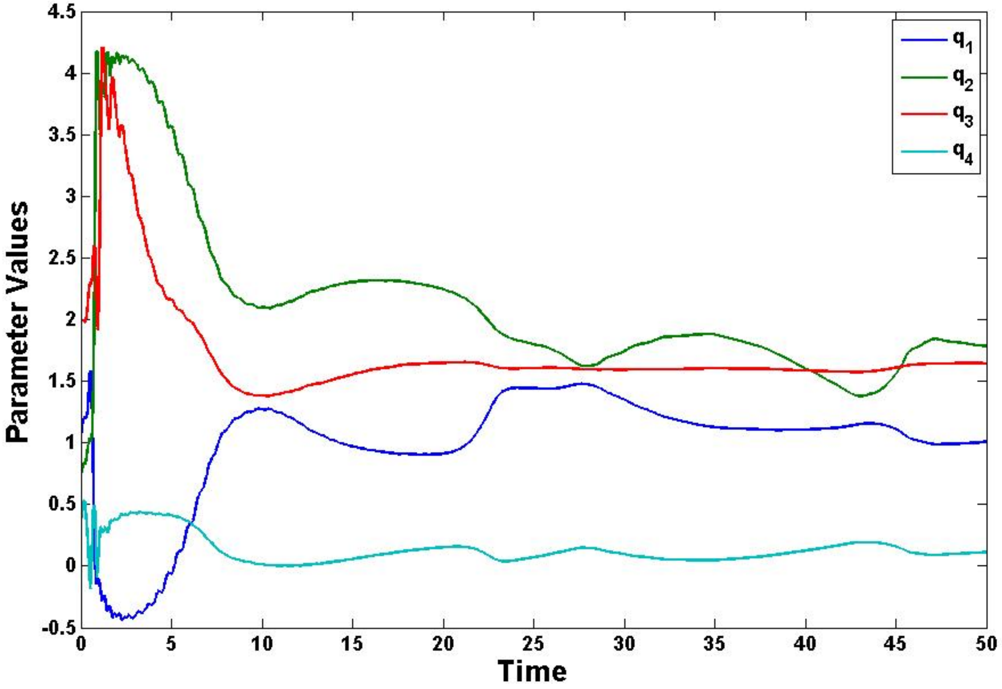




- Data tracking property within certain tolerance restored

# Online Result for Noisy Data - Parameter Space

- Use of leakage term



- Parameter divergence is suppressed
- $\hat{q}(t)$ stays in a neighborhood of $q^*$

# PDE-Example

- **Consider online identification of the space dependent heat conductivity**

  $$q_*(x) = 0.1 - 0.05\sin(2\pi(x-0.25)),\ x \in \Omega = (0,1),$$

  **in the linear heat equation**

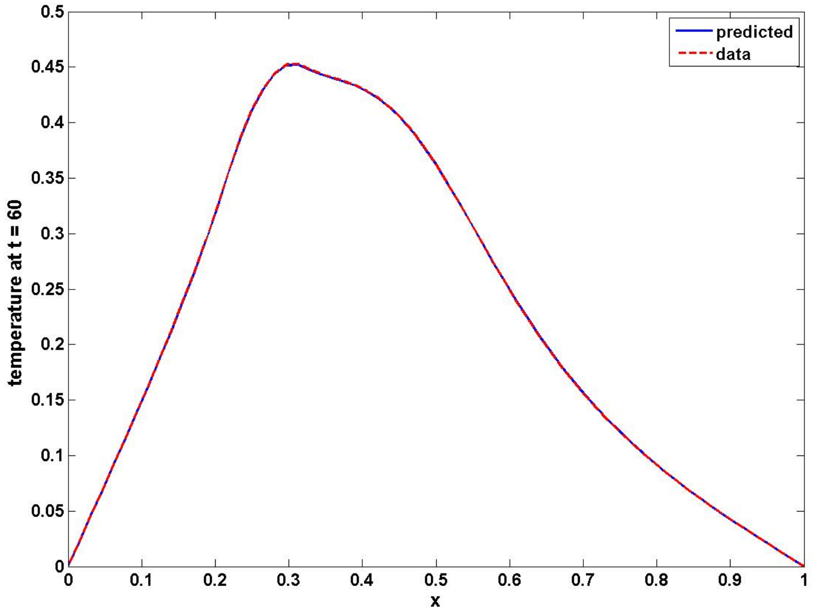
  $$\begin{aligned} u_t(x,t) - \nabla(q_*(x)\nabla u(x,t)) &= (4\sin(4\pi t) + 0.001t^2)\chi_{[0.215,0.315]} \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= 0, \end{aligned}$$

- **the data shall either result from full temperature observations, i.e,**

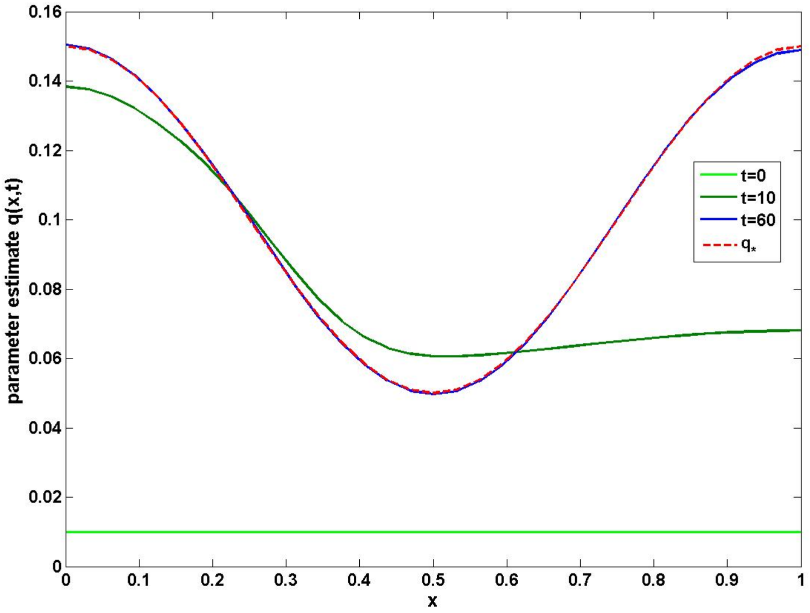
  $$z(x,t) = u_{q_*}(x,t),\ x \in [0,1],\ t > 0,$$

  **or from partial temperature observations on the right half of the domain, i.e.,**

  $$z(x,t) = u_{q_*}(x,t),\ x \in [0.5,1],\ t > 0,$$

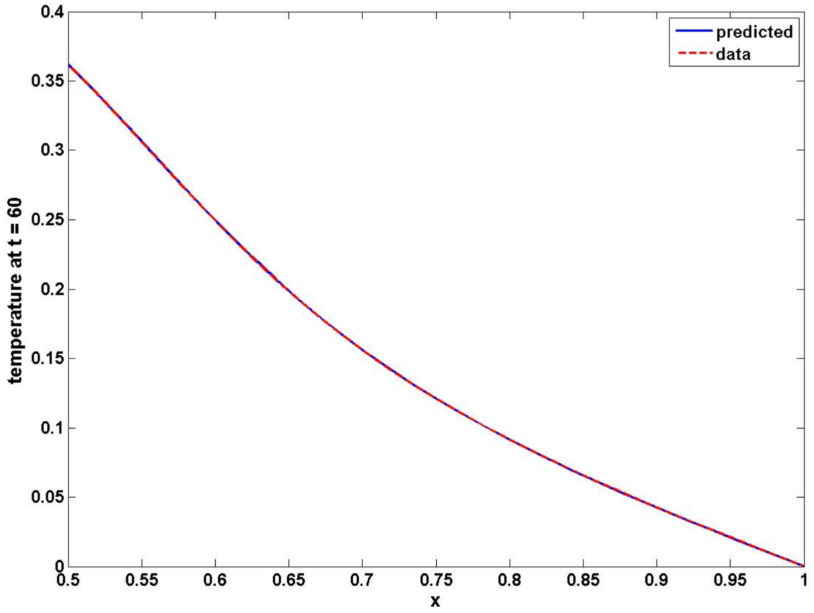


**a): data and predicted output at $t = 60$**



**b): $q(x, t_i)$ and $q_*(x)$**

**full state observation:
convergence both in output and parameter space**



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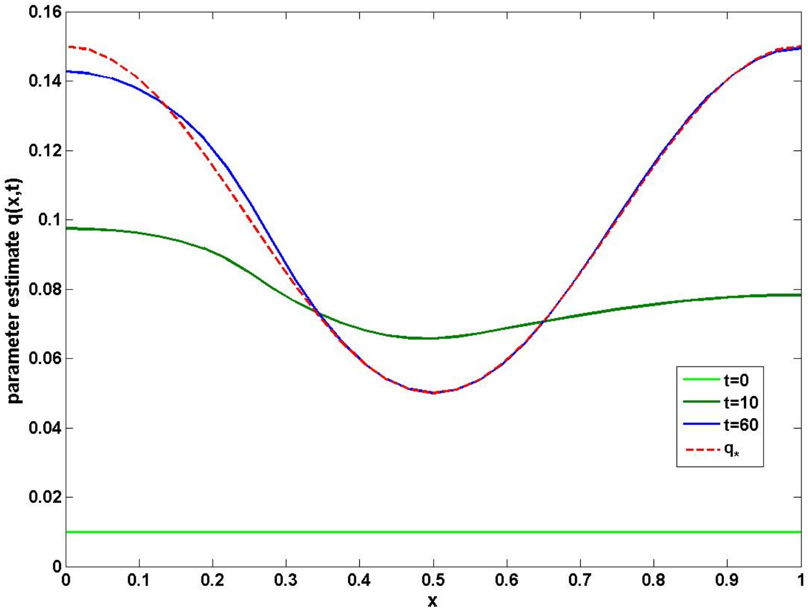
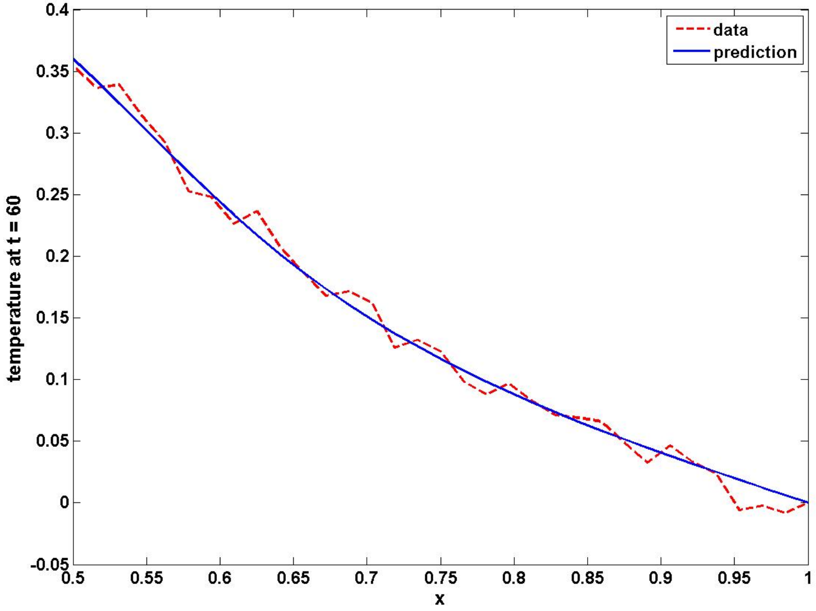


**b): $q(x, t_i)$ and $q_*(x)$**

**partial state observation:**
**convergence in output space while parameter convergence in region where data are taken**



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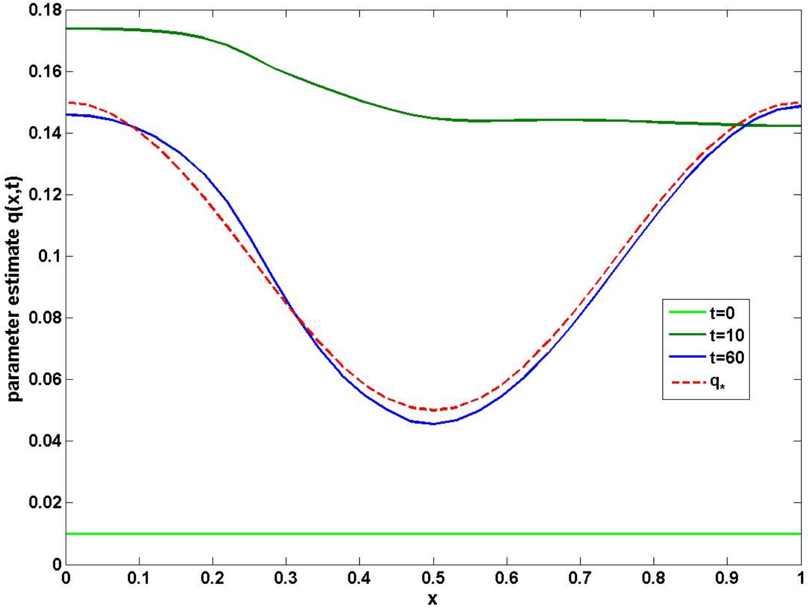

**b): $q(x, t_i)$ and $q_*(x)$**

**partial state observation:**
**stable result in case of noisy data by use of leakage approach**

# Summary and Future Topics

- We presented a nonlinear operator approach to online parameter identification in dynamical system that
  - ⋆ is an extension of the linear and finite dimensional approach
  - ⋆ applies to PDE problems without requiring full state observations

- In ODE case, method with $\alpha = 0$ can be understood as a Kalman filter for estimation of $q_*$ in
$$q_{*t} = 0$$
from nonlinear observations $z(t) = F(q_*, t)$
different to extended Kalman filter for estimation of $x = [u\, q]^T$
→ comparison

- Choice of parameters of the method in dependence on noise level
- How to ensure/verify persistence of excitation ?
- Link to Kaczmarz type methods ?