

Optical Tomography: the Forward Model Based on the Radiative Transfer Equation

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29Coloquio, Rio de Janeiro, Brasil

Introduction

- Optical tomography (OT) : “is a medical imaging modality that calculates (3D) maps of absorption and scattering coefficients in biological tissue by using a radiative transfer model (RTE) for visible or near-infrared light” (Klose, 2010);
- The reconstruction of optical properties (images) may help the clinical diagnosis ;
- Mathematical Framework: combination of
 - ① Forward model for light propagation;
 - ② Inverse model for reconstructing the optical properties (from boundary currents);
- Forward model: Diffusion equation (low order approximation);
- Recent developments and application of different solution methods based on the RTE;

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This talk

Optical Tomography:
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- On the use of **analytical approaches** to develop closed form solutions for **two-dimensional** transport problems;
- Our choice: **Nodal Schemes**: Badruzamann, Azmy, Barros & Larsen;
- Natural extension : concise and accurate results for a wide range of problems (radiative transfer; rarefied gas dynamics) by the application of the **ADO method** – Analytical Discrete Ordinates Method (Barichello & Siewert, 1999);
 - ① easy to implement;
 - ② analytical solution in terms of the spatial variables;
 - ③ reduction in the order of the associated eigenvalue systems;
 - ④ no use of iterative schemes;
 - ⑤ no division of the domain into cells.
 - ⑥ alternative proposals for the auxiliary equations

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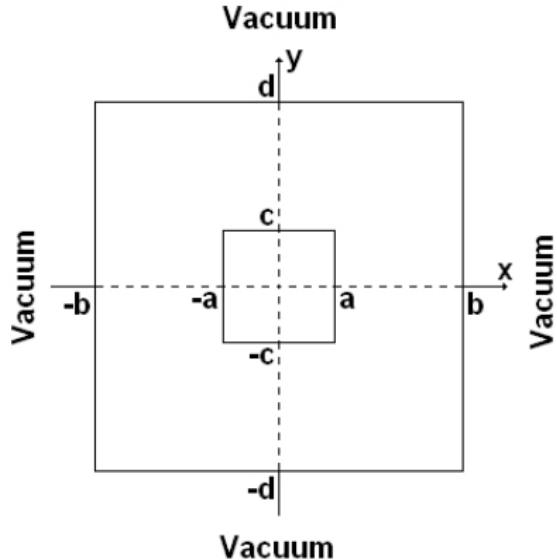
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Test Problem



- simple problem;
- proposed in the literature;
- different approaches.

Figure: Two-dimensional geometry.

Discrete ordinates equation: Level Symmetric QS

$$\mu_i \frac{\partial}{\partial x} \Psi(\mathbf{r}, \Omega_i) + \eta_i \frac{\partial}{\partial y} \Psi(\mathbf{r}, \Omega_i) + \sigma_t \Psi(\mathbf{r}, \Omega_i) = Q(\mathbf{r}) + \frac{\sigma_s}{4} \sum_{k=1}^M w_k \Psi(\mathbf{r}, \Omega_k) \quad (1)$$

for $i = 1, \dots, M$, $M = N(N+2)/2$ and N refers to $N - th$ order of the S_N approximation. $\mathbf{r} = (x, y)$, $\Omega_i = (\mu_i, \eta_i)$

One-dimensional nodal equation: x direction

$$\mu_i \frac{d}{dx} \Psi_y(x, \Omega_i) + \sigma_t \Psi_y(x, \Omega_i) = Q_y(x, \Omega_i) + \frac{\sigma_s}{4} \sum_{k=1}^M w_k \Psi_y(x, \Omega_k) \quad (2)$$

$$\Psi_y(x, \Omega_i) = \frac{1}{2d} \int_{-d}^d \Psi(\mathbf{r}, \Omega_i) dy \quad (3)$$

$$Q_y(x) = \frac{1}{2d} \int_{-d}^d Q(\mathbf{r}) dy \quad (4)$$

$$Q_y(x, \Omega_i) = Q_y(x) - \frac{\eta_i}{2d} [\Psi(x, d, \Omega_i) - \Psi(x, -d, \Omega_i)] \quad (5)$$

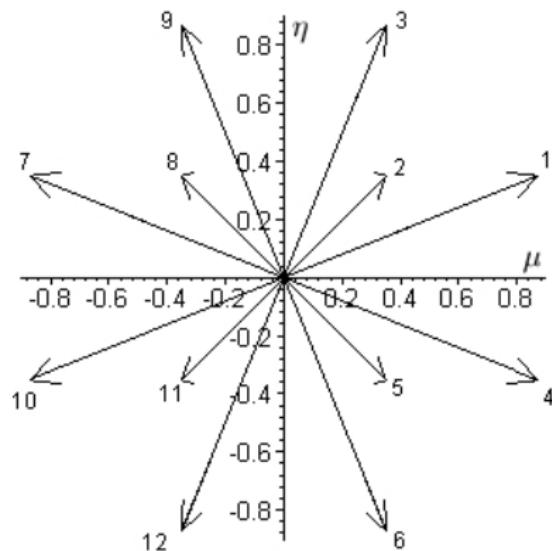
for $i = 1, \dots, M$.

- Additional unknowns were introduced.

Ordering the directions (for $i = 1, \dots, M/2$)

$$\Omega_i = (\mu_i, \eta_i) \quad (6)$$

$$\Omega_{i+M/2} = (-\mu_i, \eta_i) \quad (7)$$



N=4;M=12

Considering the known boundary conditions (for $i = 1, \dots, M/4$)

$$\Psi(x, -d, \Omega_i) = 0 \quad (8)$$

$$\Psi(x, d, \Omega_{i+M/4}) = 0 \quad (9)$$

$$\Psi(x, -d, \Omega_{i+M/2}) = 0 \quad (10)$$

$$\Psi(x, d, \Omega_{i+3M/4}) = 0 \quad (11)$$

and the symmetry properties (for $i = 1, \dots, M/2$)

$$\eta_i = \eta_{i+M/2} \quad (12)$$

$$\mu_i \frac{d}{dx} \Psi_y(x, \Omega_i) + \sigma_t \Psi_y(x, \Omega_i) = Q_y(x, \Omega_i) + \frac{\sigma_s}{4} \sum_{k=1}^{M/2} w_k [\Psi_y(x, \Omega_k) + \Psi_y(x, \Omega_{k+M/2})] \quad (13)$$

$$-\mu_i \frac{d}{dx} \Psi_y(x, \Omega_{i+M/2}) + \sigma_t \Psi_y(x, \Omega_{i+M/2}) = Q_y(x, \Omega_{i+M/2}) + \frac{\sigma_s}{4} \sum_{k=1}^{M/2} w_k [\Psi_y(x, \Omega_k) + \Psi_y(x, \Omega_{k+M/2})] \quad (14)$$

for $i = 1, \dots, M/2$.

Final form of the source term

$$Q_y(x, \Omega_i) = Q_y(x) - \frac{\eta_i}{2d} \Psi(x, d, \Omega_i) \quad (15)$$

$$Q_y(x, \Omega_{i+M/4}) = Q_y(x) + \frac{\eta_{i+M/4}}{2d} \Psi(x, -d, \Omega_{i+M/4}) \quad (16)$$

$$Q_y(x, \Omega_{i+M/2}) = Q_y(x) - \frac{\eta_i}{2d} \Psi(x, d, \Omega_{i+M/2}) \quad (17)$$

$$Q_y(x, \Omega_{i+3M/4}) = Q_y(x) + \frac{\eta_{i+M/4}}{2d} \Psi(x, -d, \Omega_{i+3M/4}) \quad (18)$$

for $i = 1, \dots, M/4$.

One-dimensional nodal equation: y direction

$$\eta_i \frac{d}{dy} \Psi_x(y, \Omega_i) + \sigma_t \Psi_x(y, \Omega_i) = Q_x(y, \Omega_i) + \frac{\sigma_s}{4} \sum_{k=1}^M w_k \Psi_x(y, \Omega_k) \quad (19)$$

$$\Psi_x(y, \Omega_i) = \frac{1}{2b} \int_{-b}^b \Psi(\mathbf{r}, \Omega_i) dx \quad (20)$$

$$Q_x(y) = \frac{1}{2b} \int_{-b}^b Q(\mathbf{r}) dx \quad (21)$$

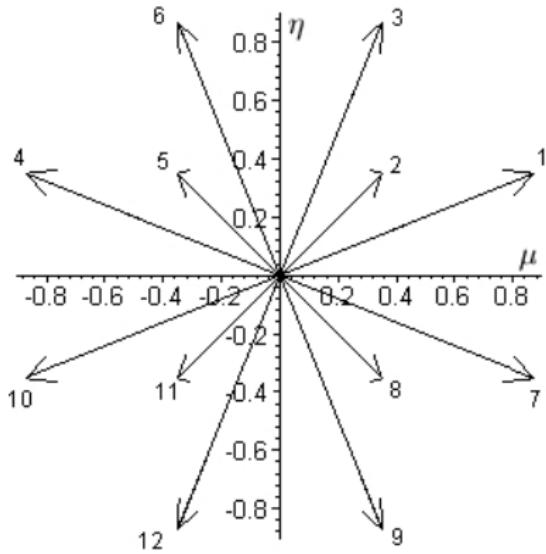
$$Q_x(y, \Omega_i) = Q_x(y) - \frac{\mu_i}{2b} [\Psi(b, y, \Omega_i) - \Psi(-b, y, \Omega_i)] \quad (22)$$

for $i = 1, \dots, M$.

Reordering the directions (for $i = 1, \dots, M/2$)

$$\Omega_i = (\mu_i, \eta_i) \quad (23)$$

$$\Omega_{i+M/2} = (\mu_i, -\eta_i) \quad (24)$$



Considering the known boundary conditions (for $i = 1, \dots, M/4$)

$$\Psi(-b, y, \Omega_i) = 0 \quad (25)$$

$$\Psi(b, y, \Omega_{i+M/4}) = 0 \quad (26)$$

$$\Psi(-b, y, \Omega_{i+M/2}) = 0 \quad (27)$$

$$\Psi(b, y, \Omega_{i+3M/4}) = 0 \quad (28)$$

and the symmetry properties (for $i = 1, \dots, M/2$)

$$\mu_i = \mu_{i+M/2} \quad (29)$$

$$\eta_i \frac{d}{dy} \Psi_x(y, \Omega_i) + \sigma_t \Psi_x(y, \Omega_i) = Q_x(y, \Omega_i) + \frac{\sigma_s}{4} \sum_{k=1}^{M/2} w_k [\Psi_x(y, \Omega_k) + \Psi_x(y, \Omega_{k+M/2})] \quad (30)$$

$$-\eta_i \frac{d}{dy} \Psi_x(y, \Omega_{i+M/2}) + \sigma_t \Psi_x(y, \Omega_{i+M/2}) = Q_x(y, \Omega_{i+M/2}) + \frac{\sigma_s}{4} \sum_{k=1}^{M/2} w_k [\Psi_x(y, \Omega_k) + \Psi_x(y, \Omega_{k+M/2})] \quad (31)$$

for $i = 1, \dots, M/2$.

Final form of the source term

$$Q_x(y, \Omega_i) = Q_x(y) - \frac{\mu_i}{2b} \Psi(b, y, \Omega_i) \quad (32)$$

$$Q_x(y, \Omega_{i+M/4}) = Q_x(y) + \frac{\mu_{i+M/4}}{2b} \Psi(-b, y, \Omega_{i+M/4}) \quad (33)$$

$$Q_x(y, \Omega_{i+M/2}) = Q_x(y) - \frac{\mu_i}{2b} \Psi(b, y, \Omega_{i+M/2}) \quad (34)$$

$$Q_x(y, \Omega_{i+3M/4}) = Q_x(y) + \frac{\mu_{i+3M/4}}{2b} \Psi(-b, y, \Omega_{i+3M/4}) \quad (35)$$

for $i = 1, \dots, M/4$.

The ADO solution: x direction

Seeking for solutions of the form

$$\Psi_y(x, \Omega_i) = \Phi_y(\nu, \Omega_i) e^{-x/\nu} \quad (36)$$

$$-\frac{1}{\nu} \mu_i \Phi_y(\nu, \Omega_i) + \sigma_t \Phi_y(\nu, \Omega_i) = \frac{\sigma_s}{4} \sum_{k=1}^{M/2} w_k [\Phi_y(\nu, \Omega_k) + \Phi_y(\nu, \Omega_{k+M/2})] \quad (37)$$

$$\frac{1}{\nu} \mu_i \Phi_y(\nu, \Omega_{i+M/2}) + \sigma_t \Phi_y(\nu, \Omega_{i+M/2}) = \frac{\sigma_s}{4} \sum_{k=1}^{M/2} w_k [\Phi_y(\nu, \Omega_k) + \Phi_y(\nu, \Omega_{k+M/2})] \quad (38)$$

for $i = 1, \dots, M/2$.

Considering

$$U_y(\nu, \Omega_i) = \Phi_y(\nu, \Omega_i) + \Phi_y(\nu, \Omega_{i+M/2}) \quad (39)$$

$$V_y(\nu, \Omega_i) = \Phi_y(\nu, \Omega_i) - \Phi_y(\nu, \Omega_{i+M/2}) \quad (40)$$

Adding and subtracting the Eqs. (37) and (38)

$$-\frac{1}{\nu} \mu_i V_y(\nu, \Omega_i) + \sigma_t U_y(\nu, \Omega_i) = \frac{\sigma_s}{2} \sum_{k=1}^{M/2} w_k U_y(\nu, \Omega_k) \quad (41)$$

$$V_y(\nu, \Omega_i) = \frac{1}{\nu} \frac{\mu_i}{\sigma_t} U_y(\nu, \Omega_i) \quad (42)$$

for $i = 1, \dots, M/2$.

Eigenvalue Problem

$$[\mathbf{D}_y - \mathbf{A}_y] \mathbf{U}_y = \lambda_y \mathbf{U}_y \quad (43)$$

$$\mathbf{D}_y = \text{diag} \left\{ \left(\frac{\sigma_t}{\mu_1} \right)^2, \left(\frac{\sigma_t}{\mu_2} \right)^2, \dots, \left(\frac{\sigma_t}{\mu_{M/2}} \right)^2 \right\} \quad (44)$$

$$\mathbf{A}_y(i, k) = \frac{\sigma_t \sigma_s w_k}{2\mu_i^2} \quad (45)$$

$$\lambda_y = \frac{1}{\nu^2} \quad (46)$$

for $i, k = 1, \dots, M/2$. Order $M/2$

Using Eqs. (39) and (40)

$$\Phi_y(\nu_j, \Omega_i) = \frac{1}{2} [U_y(\nu_j, \Omega_i) + V_y(\nu_j, \Omega_i)] \quad (47)$$

$$\Phi_y(\nu_j, \Omega_{i+M/2}) = \frac{1}{2} [U_y(\nu_j, \Omega_i) - V_y(\nu_j, \Omega_i)] \quad (48)$$

The homogeneous (explicit) solution is written

$$\begin{aligned} \Psi_y^h(x, \Omega_i) = & \sum_{j=1}^{M/2} A_j \Phi_y(\nu_j, \Omega_i) e^{-(b+x)/\nu_j} + \\ & A_{j+M/2} \Phi_y(\nu_j, \Omega_{i+M/2}) e^{-(b-x)/\nu_j} \end{aligned} \quad (49)$$

$$\begin{aligned} \Psi_y^h(x, \Omega_{i+M/2}) = & \sum_{j=1}^{M/2} A_j \Phi_y(\nu_j, \Omega_{i+M/2}) e^{-(b+x)/\nu_j} + \\ & A_{j+M/2} \Phi_y(\nu_j, \Omega_i) e^{-(b-x)/\nu_j} \end{aligned} \quad (50)$$

for $i = 1, \dots, M/2$. Arbitrary constants to be determined

The ADO solution: y direction

Proposing

$$\Psi_x(y, \Omega_i) = \Phi_x(\gamma, \Omega_i) e^{-y/\gamma} \quad (51)$$

$$[\mathbf{D}_x - \mathbf{A}_x] \mathbf{U}_x = \lambda_x \mathbf{U}_x \quad (52)$$

$$\mathbf{D}_x = \text{diag} \left\{ \left(\frac{\sigma_t}{\eta_1} \right)^2, \left(\frac{\sigma_t}{\eta_2} \right)^2, \dots, \left(\frac{\sigma_t}{\eta_{M/2}} \right)^2 \right\} \quad (53)$$

$$\mathbf{A}_x(i, k) = \frac{\sigma_t \sigma_s w_k}{2 \eta_i^2} \quad (54)$$

$$\lambda_x = \frac{1}{\gamma^2} \quad (55)$$

for $i, k = 1, \dots, M/2$.

$$\Phi_x(\gamma_j, \Omega_i) = \frac{1}{2} [U_x(\gamma_j, \Omega_i) + V_x(\gamma_j, \Omega_i)] \quad (56)$$

$$\Phi_x(\gamma_j, \Omega_{i+M/2}) = \frac{1}{2} [U_x(\gamma_j, \Omega_i) - V_x(\gamma_j, \Omega_i)] \quad (57)$$

$$\begin{aligned} \Psi_x^h(y, \Omega_i) = & \sum_{j=1}^{M/2} B_j \Phi_x(\gamma_j, \Omega_i) e^{-(d+y)/\gamma_j} + \\ & B_{j+M/2} \Phi_x(\gamma_j, \Omega_{i+M/2}) e^{-(d-y)/\gamma_j} \end{aligned} \quad (58)$$

$$\begin{aligned} \Psi_x^h(y, \Omega_{i+M/2}) = & \sum_{j=1}^{M/2} B_j \Phi_x(\gamma_j, \Omega_{i+M/2}) e^{-(d+y)/\gamma_j} + \\ & B_{j+M/2} \Phi_x(\gamma_j, \Omega_i) e^{-(d-y)/\gamma_j} \end{aligned} \quad (59)$$

for $i = 1, \dots, M/2$.

Particular solution

Source term

$$Q(\mathbf{r}) = \begin{cases} 1 & , \text{ for } x \in [-a, a] \text{ and } y \in [-c, c] \\ 0 & , \text{ otherwise} \end{cases} \quad (60)$$

Integrated source terms

$$Q_y(x) = \begin{cases} c/d & , \text{ for } x \in [-a, a] \\ 0 & , \text{ otherwise} \end{cases} \quad (61)$$

$$Q_x(y) = \begin{cases} a/b & , \text{ for } y \in [-c, c] \\ 0 & , \text{ otherwise} \end{cases} \quad (62)$$

Integrated source term for the one-dimensional nodal equation in the x direction (for $i = 1, \dots, M/4$)

$$Q_y(x, \Omega_i) = Q_y(x) - \frac{\eta_i}{2d} \Psi(x, d, \Omega_i) \quad (63)$$

$$Q_y(x, \Omega_{i+M/4}) = Q_y(x) + \frac{\eta_{i+M/4}}{2d} \Psi(x, -d, \Omega_{i+M/4}) \quad (64)$$

$$Q_y(x, \Omega_{i+M/2}) = Q_y(x) - \frac{\eta_i}{2d} \Psi(x, d, \Omega_{i+M/2}) \quad (65)$$

$$Q_y(x, \Omega_{i+3M/4}) = Q_y(x) + \frac{\eta_{i+M/4}}{2d} \Psi(x, -d, \Omega_{i+3M/4}) \quad (66)$$

Additional equations

$$\Psi(x, d, \Omega_i) = H_i \quad (67)$$

$$\Psi(x, -d, \Omega_i) = G_i \quad (68)$$

in the directions where the flux at the boundary is unknown .

$$Q_y(x, \Omega_i) = Q_y(x) - \frac{\eta_i}{2d} H_i \quad (69)$$

$$Q_y(x, \Omega_{i+M/4}) = Q_y(x) + \frac{\eta_{i+M/4}}{2d} G_{i+M/4} \quad (70)$$

$$Q_y(x, \Omega_{i+M/2}) = Q_y(x) - \frac{\eta_i}{2d} H_{i+M/2} \quad (71)$$

$$Q_y(x, \Omega_{i+3M/4}) = Q_y(x) + \frac{\eta_{i+M/4}}{2d} G_{i+3M/4} \quad (72)$$

for $i = 1, \dots, M/4$.

Integrated source term for the one-dimensional nodal equation in the y direction (for $i = 1, \dots, M/4$)

$$Q_x(y, \Omega_i) = Q_x(y) - \frac{\mu_i}{2b} \Psi(b, y, \Omega_i) \quad (73)$$

$$Q_x(y, \Omega_{i+M/4}) = Q_x(y) + \frac{\mu_{i+M/4}}{2b} \Psi(-b, y, \Omega_{i+M/4}) \quad (74)$$

$$Q_x(y, \Omega_{i+M/2}) = Q_x(y) - \frac{\mu_i}{2b} \Psi(b, y, \Omega_{i+M/2}) \quad (75)$$

$$Q_x(y, \Omega_{i+3M/4}) = Q_x(y) + \frac{\mu_{i+M/4}}{2b} \Psi(-b, y, \Omega_{i+3M/4}) \quad (76)$$

Additional equations

$$\Psi(b, y, \Omega_i) = F_i \quad (77)$$

$$\Psi(-b, y, \Omega_i) = E_i \quad (78)$$

in the directions where the flux at the boundary is unknown.

$$Q_x(y, \Omega_i) = Q_x(y) - \frac{\mu_i}{2b} F_i \quad (79)$$

$$Q_x(y, \Omega_{i+M/4}) = Q_x(y) + \frac{\mu_{i+M/4}}{2b} E_{i+M/4} \quad (80)$$

$$Q_x(y, \Omega_{i+M/2}) = Q_x(y) - \frac{\mu_i}{2b} F_{i+M/2} \quad (81)$$

$$Q_x(y, \Omega_{i+3M/4}) = Q_x(y) + \frac{\mu_{i+M/4}}{2b} E_{i+3M/4} \quad (82)$$

for $i = 1, \dots, M/4$.

Particular solution: Green's Function

- For the one-dimensional nodal equations in x direction

$$\Psi_y^P(x, \Omega_i) = \sum_{j=1}^{M/2} \{ A_j(x) \Phi_y(\nu_j, \Omega_i) + A_{j+M/2}(x) \Phi_y(\nu_j, \Omega_{i+M/2}) \} \quad (83)$$

$$\Psi_y^P(x, \Omega_{i+M/2}) = \sum_{j=1}^{M/2} \{ A_j(x) \Phi_y(\nu_j, \Omega_{i+M/2}) + A_{j+M/2}(x) \Phi_y(\nu_j, \Omega_i) \} \quad (84)$$

for $i = 1, \dots, M/2$.

$$A_j(x) = \int_{-b}^x \left\{ \sum_{\alpha=1}^M Q_y(\tau, \Omega_\alpha) A_{j,\alpha} \right\} e^{-(x-\tau)/\nu_j} d\tau \quad (85)$$

$$A_{j+M/2}(x) = - \int_x^b \left\{ \sum_{\alpha=1}^M Q_y(\tau, \Omega_\alpha) A_{j+M/2,\alpha} \right\} e^{-(\tau-x)/\nu_j} d\tau \quad (86)$$

for $i = j, \dots, M/2$. (LBB, Garcia and CES, 2000)

Linear system (jump condition)

$$\mu_i \sum_{j=1}^{M/2} \{ A_{j,\alpha} \Phi_y(\nu_j, \Omega_i) + A_{j+M/2,\alpha} \Phi_y(\nu_j, \Omega_{i+M/2}) \} = \delta_{i,\alpha} \quad (87)$$

$$-\mu_i \sum_{j=1}^{M/2} \{ A_{j,\alpha} \Phi_y(\nu_j, \Omega_{i+M/2}) + A_{j+M/2,\alpha} \Phi_y(\nu_j, \Omega_i) \} = \delta_{i+M/2,\alpha} \quad (88)$$

for $i = 1, \dots, M/2$ and $\alpha = 1, \dots, M$.

Matrix form (for $i, j = 1, \dots, M/2$ and $\alpha = 1, \dots, M$)

$$\begin{bmatrix} [\mu_i \Phi_y(\nu_j, \Omega_i)] & [\mu_i \Phi_y(\nu_j, \Omega_{i+M/2})] \\ [-\mu_i \Phi_y(\nu_j, \Omega_{i+M/2})] & [-\mu_i \Phi_y(\nu_j, \Omega_i)] \end{bmatrix} \times \begin{bmatrix} [A_{j,\alpha}] \\ [A_{j+M/2,\alpha}] \end{bmatrix} = Id_{M \times M} \quad (89)$$

- For the one-dimensional nodal equations in y direction

$$\Psi_x^p(y, \Omega_i) = \sum_{j=1}^{M/2} \{ B_j(y) \Phi_x(\gamma_j, \Omega_i) + B_{j+M/2}(y) \Phi_x(\gamma_j, \Omega_{i+M/2}) \} \quad (90)$$

$$\Psi_x^p(y, \Omega_{i+M/2}) = \sum_{j=1}^{M/2} \{ B_j(y) \Phi_x(\gamma_j, \Omega_{i+M/2}) + B_{j+M/2}(y) \Phi_x(\gamma_j, \Omega_i) \} \quad (91)$$

for $i = 1, \dots, M/2$.

$$B_j(y) = \int_{-d}^y \left\{ \sum_{\alpha=1}^M Q_x(\tau, \Omega_\alpha) B_{j,\alpha} \right\} e^{-(y-\tau)/\gamma_j} d\tau \quad (92)$$

$$B_{j+M/2}(y) = - \int_y^d \left\{ \sum_{\alpha=1}^M Q_x(\tau, \Omega_\alpha) B_{j+M/2,\alpha} \right\} e^{-(\tau-y)/\gamma_j} d\tau \quad (93)$$

for $j = 1, \dots, M/2$.

Linear System (jump condition)

$$\eta_i \sum_{j=1}^{M/2} \{B_{j,\alpha} \Phi_x(\gamma_j, \Omega_i) + B_{j+M/2,\alpha} \Phi_x(\gamma_j, \Omega_{i+M/2})\} = \delta_{i,\alpha} \quad (94)$$

$$-\eta_i \sum_{j=1}^{M/2} \{B_{j,\alpha} \Phi_x(\gamma_j, \Omega_{i+M/2}) + B_{j+M/2,\alpha} \Phi_x(\gamma_j, \Omega_i)\} = \delta_{i+M/2,\alpha} \quad (95)$$

for $i = 1, \dots, M/2$ and $\alpha = 1, \dots, M$.

Matrix form (for $i, j = 1, \dots, M/2$ and $\alpha = 1, \dots, M$)

$$\begin{bmatrix} [\eta_i \Phi_x(\gamma_j, \Omega_i)] & [\eta_i \Phi_x(\gamma_j, \Omega_{i+M/2})] \\ [-\eta_i \Phi_x(\gamma_j, \Omega_{i+M/2})] & [-\eta_i \Phi_x(\gamma_j, \Omega_i)] \end{bmatrix} \times \begin{bmatrix} [B_{j,\alpha}] \\ [B_{j+M/2,\alpha}] \end{bmatrix} = Id_{M \times M} \quad (96)$$

General solution

- one-dimensional nodal equations in x direction

$$\Psi_y(x, \Omega_i) = \Psi_y^h(x, \Omega_i) + \Psi_y^p(x, \Omega_i) \quad (97)$$

- one-dimensional nodal equations in y direction

$$\Psi_x(y, \Omega_i) = \Psi_x^h(y, \Omega_i) + \Psi_x^p(y, \Omega_i) \quad (98)$$

Coupling the one-dimensional problems

From de boundary conditions (Eqs. (8) - (11))

$$\Psi_x^h(-d, \Omega_i) + \Psi_x^p(-d, \Omega_i) = 0 \quad (99)$$

$$\Psi_x^h(d, \Omega_{i+M/4}) + \Psi_x^p(d, \Omega_{i+M/4}) = 0 \quad (100)$$

$$\Psi_x^h(-d, \Omega_{i+M/2}) + \Psi_x^p(-d, \Omega_{i+M/2}) = 0 \quad (101)$$

$$\Psi_x^h(d, \Omega_{i+3M/4}) + \Psi_x^p(d, \Omega_{i+3M/4}) = 0 \quad (102)$$

for $i = 1, \dots, M/4$.

From de boundary conditions (Eqs. (25) - (28))

$$\Psi_y^h(-b, \Omega_i) + \Psi_y^p(-b, \Omega_i) = 0 \quad (103)$$

$$\Psi_y^h(b, \Omega_{i+M/4}) + \Psi_y^p(b, \Omega_{i+M/4}) = 0 \quad (104)$$

$$\Psi_y^h(-b, \Omega_{i+M/2}) + \Psi_y^p(-b, \Omega_{i+M/2}) = 0 \quad (105)$$

$$\Psi_y^h(b, \Omega_{i+3M/4}) + \Psi_y^p(b, \Omega_{i+3M/4}) = 0 \quad (106)$$

for $i = 1, \dots, M/4$.

From the relation between the one-dimensional solutions and the auxiliary equations (Eqs. (67)-(68))

$$\Psi_x^h(d, \Omega_i) + \Psi_x^P(d, \Omega_i) = H_i \quad (107)$$

$$\Psi_x^h(d, \Omega_{i+M/4}) + \Psi_x^P(d, \Omega_{i+M/4}) = H_{i+M/2} \quad (108)$$

$$\Psi_x^h(d, \Omega_{i+M/2}) + \Psi_x^P(d, \Omega_{i+M/2}) = H_{i+M/4} \quad (109)$$

$$\Psi_x^h(d, \Omega_{i+3M/4}) + \Psi_x^P(d, \Omega_{i+3M/4}) = H_{i+3M/4} \quad (110)$$

$$\Psi_x^h(-d, \Omega_i) + \Psi_x^P(-d, \Omega_i) = G_i \quad (111)$$

$$\Psi_x^h(-d, \Omega_{i+M/4}) + \Psi_x^P(-d, \Omega_{i+M/4}) = G_{i+M/2} \quad (112)$$

$$\Psi_x^h(-d, \Omega_{i+M/2}) + \Psi_x^P(-d, \Omega_{i+M/2}) = G_{i+M/4} \quad (113)$$

$$\Psi_x^h(-d, \Omega_{i+3M/4}) + \Psi_x^P(-d, \Omega_{i+3M/4}) = G_{i+3M/4} \quad (114)$$

for $i = 1, \dots, M/4$.

From the relation between the one-dimensional solutions and the auxiliary equations (Eqs. (77)-(78))

$$\Psi_y^h(b, \Omega_i) + \Psi_y^p(b, \Omega_i) = F_i \quad (115)$$

$$\Psi_y^h(b, \Omega_{i+M/4}) + \Psi_y^p(b, \Omega_{i+M/4}) = F_{i+M/2} \quad (116)$$

$$\Psi_y^h(b, \Omega_{i+M/2}) + \Psi_y^p(b, \Omega_{i+M/2}) = F_{i+M/4} \quad (117)$$

$$\Psi_y^h(b, \Omega_{i+3M/4}) + \Psi_y^p(b, \Omega_{i+3M/4}) = F_{i+3M/4} \quad (118)$$

$$\Psi_y^h(-b, \Omega_i) + \Psi_y^p(-b, \Omega_i) = E_i \quad (119)$$

$$\Psi_y^h(-b, \Omega_{i+M/4}) + \Psi_y^p(-b, \Omega_{i+M/4}) = E_{i+M/2} \quad (120)$$

$$\Psi_y^h(-b, \Omega_{i+M/2}) + \Psi_y^p(-b, \Omega_{i+M/2}) = E_{i+M/4} \quad (121)$$

$$\Psi_y^h(-b, \Omega_{i+3M/4}) + \Psi_y^p(-b, \Omega_{i+3M/4}) = E_{i+3M/4} \quad (122)$$

for $i = 1, \dots, M/4$.

The general solution was established.

Test problems: Case 1

- Tsai, R. W., Loyalka, S. K., *A numerical method for solving the integral equation of neutron transport: Part III, Nuclear Science and Engineering*, vol. 59, pp.536-540, 1976.

Scalar flux

$$\phi(x) = \frac{1}{4} \sum_{k=1}^{M/2} w_k [\Psi_y(x, \Omega_k) + \Psi_y(x, \Omega_{k+m/2})] \quad (123)$$

$$\phi(y) = \frac{1}{4} \sum_{k=1}^{M/2} w_k [\Psi_x(y, \Omega_k) + \Psi_x(y, \Omega_{k+m/2})] \quad (124)$$

$$b = d = 1; c = d = 0.52$$

Figure: Case 1: scalar flux.

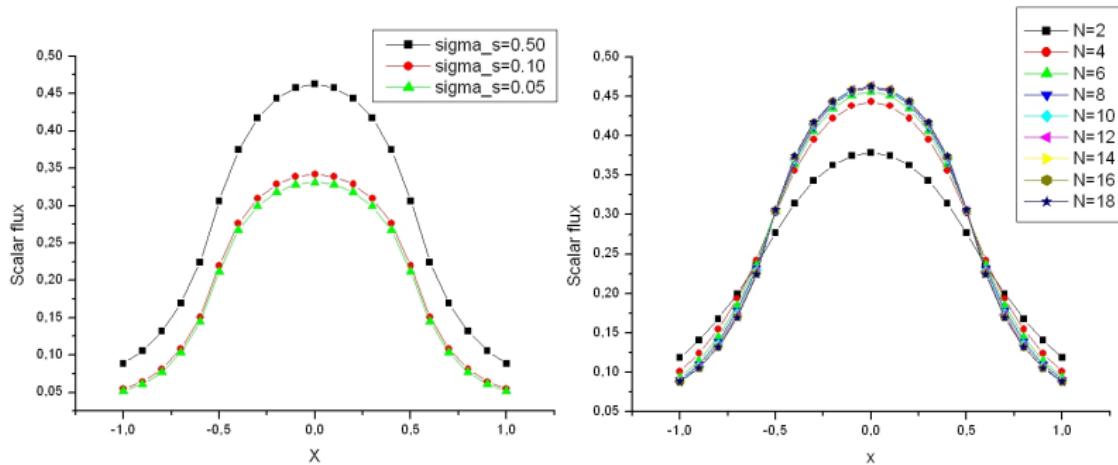
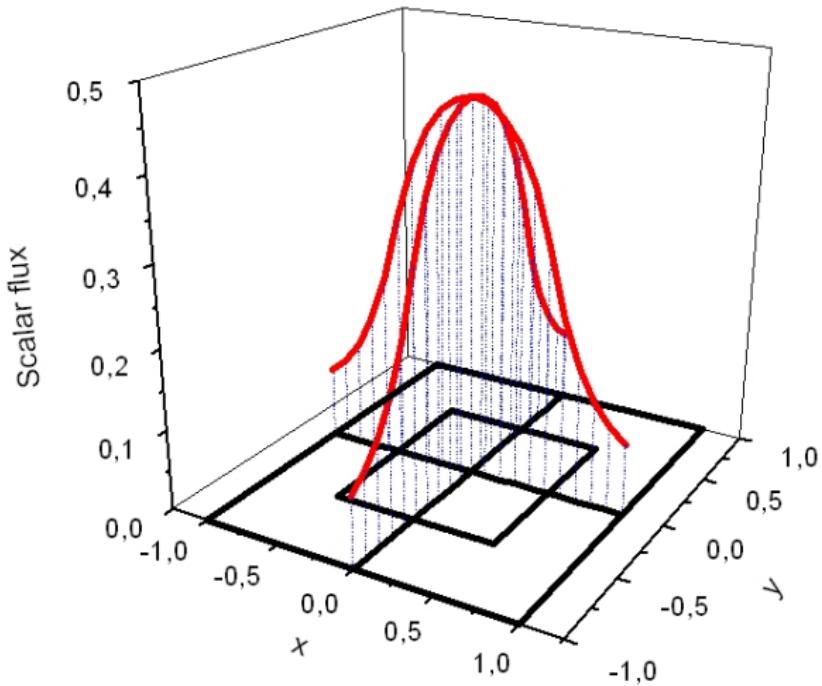


Figure: Case 1: scalar flux, $N = 18$, $\sigma_t = 1$, $\sigma_s = 0.5$.



Test problems: Case 2

- Watanabe, Y. Maynard, C. W., *The discrete cones method for two-dimensional neutron transport calculations*, **Transport Theory and Statistical Physics**, vol. 15, pp.135-156, 1986.

Net leakage

$$J_N = J_x + J_y \quad (125)$$

$$J_x = \frac{1}{4} \sum_{i=1}^{M/4} \mu_i w_i \int_0^2 [\Psi(b, y, \Omega_i) + \Psi(b, y, \Omega_{i+M/2})] dy \quad (126)$$

$$J_y = \frac{1}{4} \sum_{i=1}^{M/4} \eta_i w_i \int_0^2 [\Psi(x, d, \Omega_i) + \Psi(x, d, \Omega_{i+M/2})] dx \quad (127)$$

$$b = d = 2; a = c = 1$$

Concluding Remarks, Challenges and Future Work

- ① Nodal 2D : simple appoximation for the auxiliary equations
- ② Well conditioned systems
- ③ No iterative schemes
- ④ Easy to implement and fast
- ⑤ analytical solution (1D) in terms of the spatial variables
- ⑥ reduced order eigenvalue problems
- ⑦ Alternative check
- ⑧ Application in test cases related to OT
- ⑨ Heterogeneous media
- ⑩ Alternative Auxiliary Equations (ANE, 2011; INAC 2009)

Acknowledgements

The authors would like to thank to

- Organizers: A. Leitão, W. Muniz, A. Rieder
- SBM
- CNPq of Brazil and [FAPERGS](#) for partial financial support to this work.

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THANK YOU !