On level set regularization approaches and some applications

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Inverse Problems: Banach spaces and Hybrid Tomography
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- Recover $u : \Omega \to R$ from the "nonlinear"ill-posed equation

$$F(u) = y^\delta \tag{1}$$

$$F : D(F) \subset X \to Y$$

s.t.

$$\|y - y^\delta\|_Y \leq \delta. \tag{2}$$

- Assumption (A1): $F : D(F) \subset X \to Y$ is continuous w.r.t. the $L^1(\Omega)$ - topology.

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Ex.: Tomography problems, IPP, etc.

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- Under this framework the Inverse Problem consist in recover $\chi_{\mathbb{D}_1}$ and the values $\{c^1, c^2\}$.

The level set idea

- parameterize u using a (smooth) **level set** function $\phi : \Omega \to R$ s.t.

$$\mathbb{D}_1 : \{x \in \Omega : \phi(x) \geq 0\}$$

$$\mathbb{D}_2 : \{x \in \Omega : \phi(x) < 0\}$$

$$u = P_{ls}(\phi, c^j)\,. \tag{3}$$

where $P_{ls}(\phi, c^j) = c^1 H(\phi) + c^2(1 - H(\phi))$.

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- standard level set approach!!!
- in this presentation:
  piecewise constant level set approach (PCLS)

(PCLS)

- $\phi \in L^2(\Omega)$ – (non-smooth) such that

$$\phi(x) = i - 1 \quad x \in \mathbb{D}_i$$

- rewritten u as

$$u = c^1\psi_1(\phi) + c^2\psi_2(\phi) := P(\phi, c^j)\,. \tag{4}$$

where $\psi_1(t) = 1 - t$ and $\psi_2(t) = t$.

(PCLS)

- The inverse problem: can be rewritten as:
  find $\phi \in L^2(\Omega)$ ("and c^j ") s.t.

$$F(P(\phi, c^j)) = y^\delta . \qquad (5)$$

- the piecewise constant assumption of ϕ correspond to the constraint

$$\mathcal{K}(\phi) = \phi(\phi - 1) = 0, \quad \text{smooth}$$

or

$$K(\phi) := \sqrt{|\phi||\phi - 1|} = 0, \quad \text{non-smooth}$$

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- Assumption (A2): $\exists \phi^* \in L^2(\Omega)$ and $c^j_* \in R$ s.t. $P(\phi^*, c^j_*) = u^*$ $F(u^*) = y$ and $\mathcal{K}(\phi^*) = 0$.

penalty method

- Tikhonov regularization + penalty method

$$\textit{minimize } \mathcal{G}_\alpha(\phi, c^j) := \|F(P(\phi, c^j)) - y^\delta\|_Y^2 + \mu\|\mathcal{K}(\phi)\|_{L^1} \qquad (6)$$
$$+\alpha\left(|P(\phi, c^j)|_{BV} + \|c^j\|_{R^2}^2\right).$$

where $\mu > 0$ plays the role of a scaling factor.

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- the choice of μ is crucial in practical applications!!
Notice that the first part of the misfit depend on the data, while $\|\mathcal{K}(\phi)\|_{L^1}$ does not.

Regularization properties of penalty method

Definition (Admissible solution)

A pair $(\phi, c^j) \in L^2(\Omega) \times R^2$ is admissible if $\phi \in BV_0(\Omega)$ and $|c^1 - c^2| \geq \tau > 0$.

Here $BV_0(\Omega) := \{\phi \in BV(\Omega) : \phi(x) = 0 \text{ a.e. } x \in \tilde{\mathbb{D}}, |\tilde{\mathbb{D}}| > \gamma > 0\}$.

Theorem (Existence, Stability and Convergence)

Let Assumptions (A1)-(A2), and $\mu > 0$.

- $\exists(\phi, c^j)$ *admissible that minimizes the functional* $\mathcal{G}_\alpha$.
- *If* $\alpha(\delta) \to 0$ *and* $\delta^2/\alpha(\delta) \to 0$ *as* $\delta \to 0$ *then the corresponding minimizers of* $\mathcal{G}_\alpha$ *has a subsequence that converges in* $L^1(\Omega) \times R^2$ *to a solution of* $F(P(\phi, c^j)) = y$.

Algorithm

- $\mathcal{G}_\alpha$ is splitted in the sum $\mathcal{G}_\alpha(\phi, c^j) = \mathcal{G}_\alpha^1(\phi, c^j) + \mathcal{G}_\alpha^2(\phi)$

$$\mathcal{G}_\alpha^1(\phi, c^j) := \|F(P(\phi, c^j)) - y^\delta\|_Y^2 + \alpha\left(|P(\phi, c^j)|_{BV} + \|c^j\|_{R^2}^2\right)$$
$$\mathcal{G}_\alpha^2(\phi) := \mu\|\mathcal{K}(\phi)\|_{L^1}\,.$$

(i) (ϕ_k, c_k^j) is updated using an explicit gradient step w.r.t. $\mathcal{G}_\alpha^1$

(ii) $(\phi_{k+1/2}, c_{k+1}^j)$ is improved by the given gradient step w.r.t. $\mathcal{G}_\alpha^2$

Algorithm: some discussion

- If a large μ is chosen, the iterates ϕ_k satisfy the constraint $\mathcal{K}(\phi_k) = 0$ (becomes piecewise constant) after a few steps and the iteration stagnates. The corresponding solution $P(\phi_k, c_k^j)$ is far from the true parameter.
- The same applies is the gradient step w.r.t. $\mathcal{G}_\alpha^2$ is performance to often.
- If a small μ is chosen, the approximated solution $P(\phi_k, c_k^j)$ is much more precise. However, it leads to a very slow convergence of the algorithm. Many iterations are necessary for enforce the constraint $\mathcal{K}(\phi_k) = 0$.
- Alternatively, we chosen $\mu = \mu_0$ and then μ is gradually increased during the iteration, according to a pre-defined strategy.

Augmented Lagrangian

- Tiknonov regularization + penalty + Lagrangian

$$
\begin{aligned}
\mathcal{F}_\alpha(\phi, c^j; \lambda, \mu) := & \|F(P(\phi, c^j)) - y^\delta\|_Y^2 + \mu\|K(\phi)\|_{L^2} \\
& + \langle \lambda, K(\phi)\rangle + \alpha\left(|P(\phi, c^j)|_{BV} + \|c^j\|_{R^2}^2\right) \\
= & \mathcal{G}_\alpha^1(\phi, c^j) + \mu\|K(\phi)\|_{L^2} + \langle \lambda, K(\phi)\rangle
\end{aligned}
\tag{7}
$$

where (λ, μ) plays the role of "generalized multipliers".

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where (λ, μ) plays the role of "generalized multipliers".

- Note that (7) is non-convex. May exist a duality gap. Hence the classical Lagrange theory cannot be applied.

Augmented Lagrangian

- Idea: find a vector $\overline{\lambda}$ **supporting and exact penalty representation** for the dual problems, as well as a corresponding penalty factor $\overline{\mu}$.
- uses abstract convexity tools

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Augmented Lagrangian

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- uses abstract convexity tools
- if $(\overline{\lambda}, \overline{\mu})$ is known, an approximated solution to the constraint problem can be found solving an unconstraint optimization problem (as in the classical Lagrangian multiplier theory)
- Advantages of Augmented Lagrangian in comparison with the penalty method:
  1. usually AL does not require that the penalty parameter tends to infinity.
  2. This reduces (moderates) the ill-conditioning.
  3. AL has a considerable better convergence rate.

Augmented Lagrangian and Abstract Convexity

- We need introduce some notation:
  - $\Gamma(z) := \{\phi \in L^2(\Omega) : K(\phi) = z\}, \quad z \in L^2(\Omega).$
  - $\tilde{\mathcal{F}}_\alpha(\phi, c^j) := \begin{cases} \mathcal{F}_\alpha(\phi, c^j) & \phi \in \Gamma(0), \\ +\infty, & \textit{otherwise}. \end{cases}$
  - dualizing parametrization function $f(\phi, c^j, z) := \mathcal{G}^1_\alpha(\phi, c^j) + \delta_{\Gamma(z)}(\phi)$
  - perturbation function $\theta(z) := \inf_{(\phi, c^j)} f(\theta, c^j, z)$
  - coupling function $\rho(z, \lambda, \mu) := -\langle \lambda, z\rangle - \mu \|z\|_{L^2}$

Augmented Lagrangian and Abstract Convexity

- The augmented Lagrangian introduced by ρ

$$\mathcal{G}_{L,\alpha}(\phi, c^j; \lambda, \mu) := \inf_z \{ f(\phi, c^j, z) - \rho(z, \lambda, \mu) \}$$

Augmented Lagrangian and Abstract Convexity

- The augmented Lagrangian introduced by ρ

$$\mathcal{G}_{L,\alpha}(\phi, c^j; \lambda, \mu) := \inf_z \{ f(\phi, c^j, z) - \rho(z, \lambda, \mu) \}$$

- Is straightforward to verify that $\mathcal{G}_{L,\alpha}$ coincides with $\mathcal{F}_\alpha$.
- Moreover, $\mathcal{G}_{L,\alpha}$ coincides with $\mathcal{G}^1_\alpha$, wherever $K(\phi) = 0$
- the dual function $Q(\lambda, \mu) := \inf_{(\phi, c^j)} \mathcal{G}_{L,\alpha}(\phi, c^j; \lambda, \mu)$.

Augmented Lagrangian: Main results

Theorem

There is no gaps of duality, i. e.,

$$\sup_{(\lambda,\mu)} Q(\lambda,\mu) = \inf_{(\phi,c^j)} \tilde{\mathcal{F}}_\alpha(\phi,c^j)$$

Augmented Lagrangian: Main results

Definition (Generalized Lagrangian multipliers)

A vector $\overline{\lambda} \in L^2(\Omega)$ is said to **support an exact penalty representation** for the problem of minimizing $\mathcal{F}_\alpha$ under the constraint $K(\phi) = 0$ if there exist a $\mu_0 > 0$ s.t.

$$\theta(0) = Q(\overline{\lambda}, \mu) \quad \text{and} \quad argmin_{(\phi, c^j)} \tilde{\mathcal{F}}_\alpha(\phi, c^j) = argmin_{(\phi, c^j)} \mathcal{G}_{L,\alpha}(\phi, c^j; \overline{\lambda}, \mu),$$

for all $\mu > \mu_0$.

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for all $\mu > \mu_0$.

Theorem

There exists a $\overline{\lambda}$ supporting an exact penalty representation.

Augmented Lagrangian: Primal-dual algorithm

- given the initial guess $(\phi_0, c_0^j; \lambda_0)$ and $\mu > 0$ sufficient large $(\mu > \mu_0)$
- update the primal components (ϕ_k, c_k^j) by minimizing $\mathcal{G}_{L,\alpha}(\cdot, \lambda_k, \mu)$ w.r.t. (ϕ, c^j)
- update the Lagrangian multiplier λ_k as a gradient step of $\mathcal{G}_{L,\alpha}(\phi_{k+1}, c_{k+1}^j; \cdot, \mu)$

$$\lambda_{k+1} = \lambda_k + \mu K(\phi_{k+1})$$

Augmented Lagrangian: Convergence and Stability

Theorem (Existence)

For any $\alpha > 0$ the Tikhonov functional $\mathcal{F}_\alpha$ attains minimizers on the set of admissible functions.

Sketch of the proof:

- the existence of a pair $(\overline{\lambda}, \mu_0)$ supporting an exactly penalty imply that the minimizers of $\tilde{\mathcal{F}}_\alpha$ and $\mathcal{G}_{L,\alpha}(\cdot, \overline{\lambda}, \mu)$ coincides.
- Assumption (A2) imply that $\mathcal{G}_\alpha^1$ (and hence $\tilde{\mathcal{F}}_\alpha$) is proper.
- Note that, for any sequence of minimizers of $\tilde{\mathcal{F}}_\alpha$ with $K(\phi_k) = 0$ then $K(\lim_k \phi_k) = 0$.
- Now the proof follows "more or less"the standard Tikhonov approach.

Augmented Lagrangian: Convergence and Stability

Theorem (Convergence and Stability)

Let $\alpha_k := \alpha(\delta_k) \longrightarrow 0$ *and* $\delta_k^2/\alpha_k \longrightarrow 0$ *as* $\delta_k \longrightarrow 0$. *Moreover,* $\{(\phi_{\alpha_k}, c^j_{\alpha_k})\}$ *the corresponding minimizers of* $\mathcal{G}_{L,\alpha_l}(\cdot, \overline{\lambda}_{\alpha_k}, \mu_{\alpha_k})$. *Then* $\{(\phi_{\alpha_k}, c^j_{\alpha_k})\}$ *has a strong convergent subsequence in* $L^1(\Omega) \times R^2$ *and the limit satisfies* $F(P(\tilde{\phi}, \tilde{c}^j)) = y$.

Sketch of the proof: Follows ”more or less”the standard Tikhonov approach.

IPP forward model

Given $u \in L^2(\Omega)$, solve the Poisson boundary problem

$$-\Delta w = u\,, \text{ in } \Omega \qquad w = 0\,, \quad \text{on } \partial\Omega\,. \tag{8}$$

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Forward operator

$$F : L^2(\Omega) \to L^2(\partial\Omega)\,, \quad F(u) = w_\nu|_{\partial\Omega} \tag{9}$$

For u piecewise constant in Ω, F is continuous w.r.t. the L^1-norm.

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The inverse potential problem: recover $u \in L^2(\Omega)$, from measurements of the Cauchy data y^δ of it corresponding potential on $\partial\Omega$.

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Assumption $u \in \{c^1, c^2\}$ in $\Omega = [0,1] \times [0,1]$
$c^1 = 0\,, c^2 = 1$ are known.

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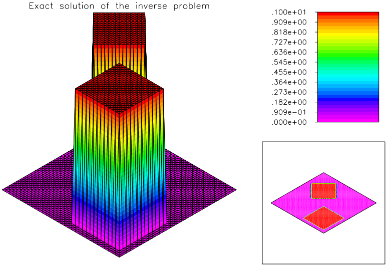
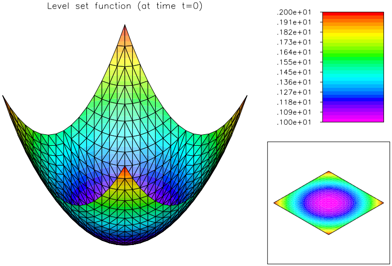
Assumption $u \in \{c^1, c^2\}$ in $\Omega = [0,1] \times [0,1]$
$c^1 = 0\,, c^2 = 1$ are known.

For this class of parameters, no uniqueness identifiability is known

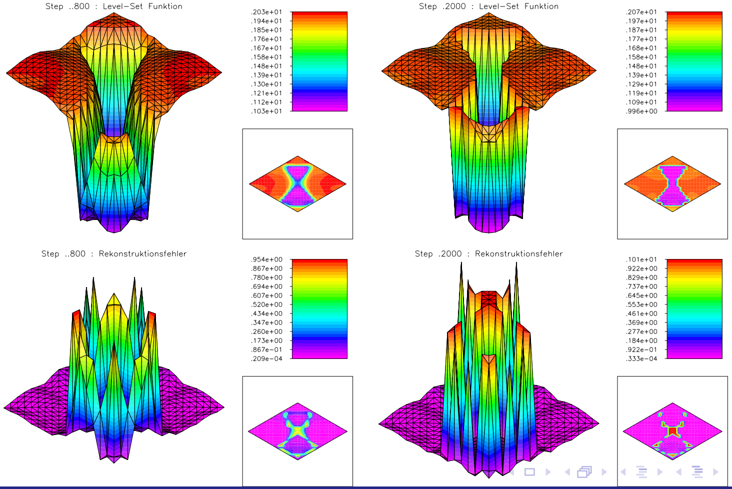
The IPP is linear, but exponential ill-posed

exact solution

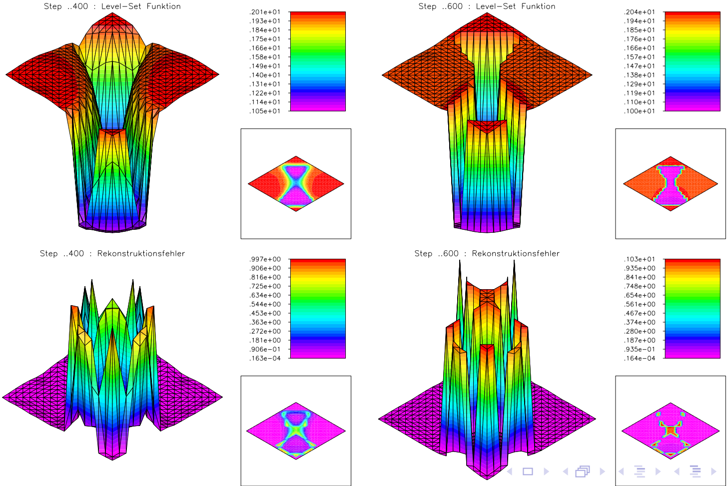




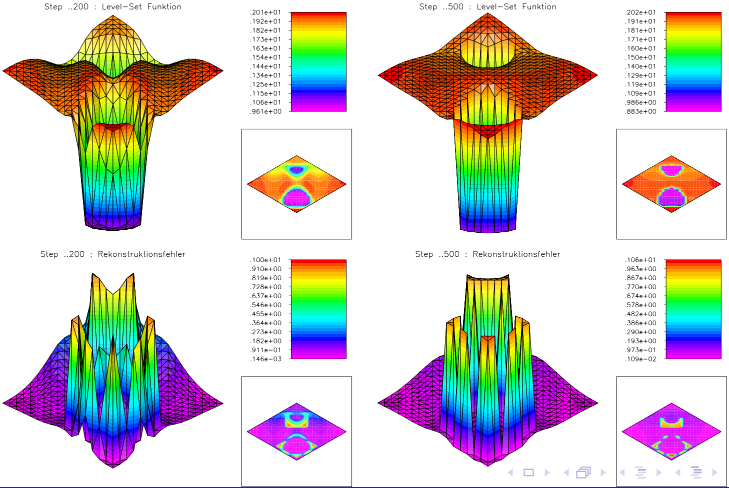
Penalty method - $\mu = constant$ (exact data)



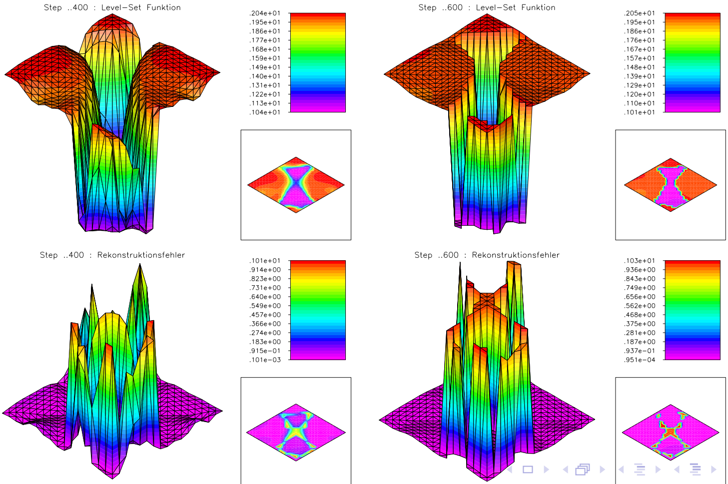
Penalty method - μ non constant (exact data)



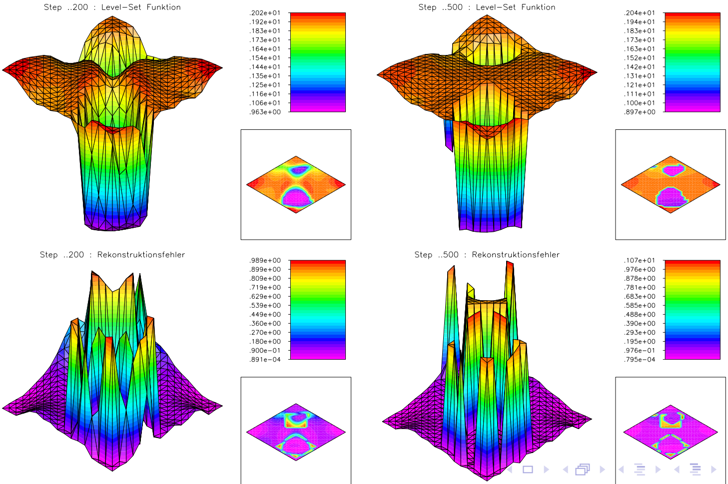
Augmented Lagrangian (exact data)



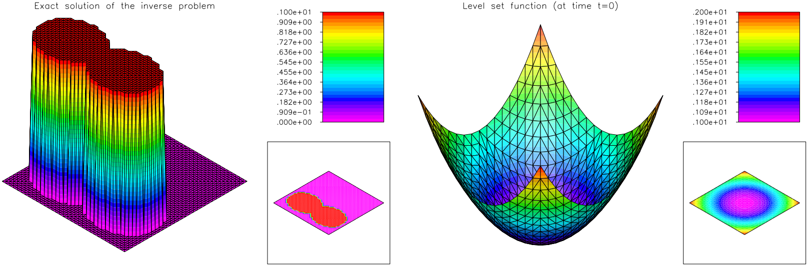
Penalty method - $\mu = non - constant$ $(\delta = 10\%)$



Augmented Lagrangian ($\delta = 10\%$)



exact solution - non-convex inclusions



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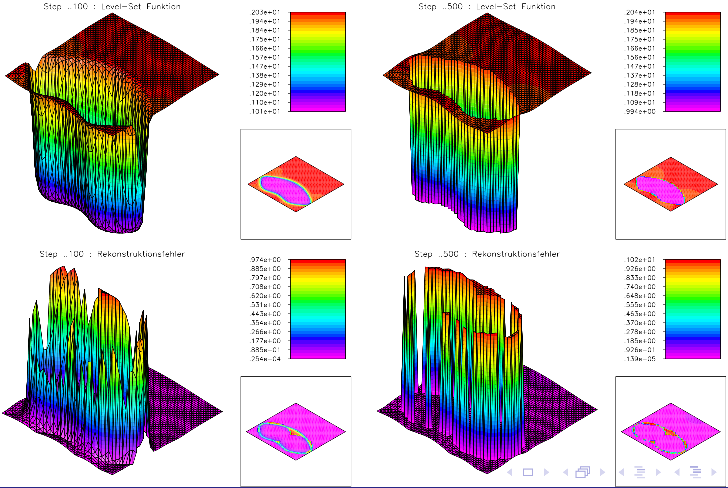

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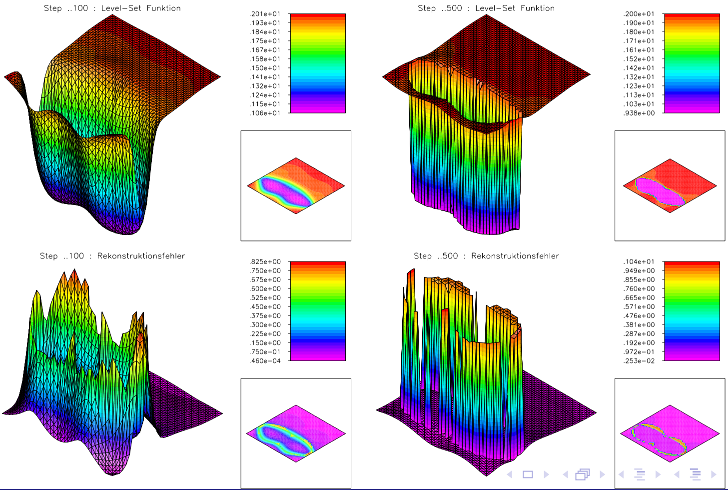

Conclusions and future investigations

- convergence analysis for Penalty and AL approaches.
- the penalty method with non-constant μ generates a faster numerical algorithm, with solutions with the ”same”quality.
- the quality of approx. solutions using the AL approach are clearly better than the penalty approach.
- the performance of AL are compared with the non-constant choice of μ in the penalty method.
- Future works: Investigate the so-called **sub-optimal path** for the duality scheme and analyze the convergence properties.
- In Burachik et. al. the authors proves that every cluster point of a sub-optimal path related to the dual problem is a primal solution.

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