Some inverse problems for dispersive partial differential equations.

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29 Colóquio Brasileiro de Matemática,
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Presentation of the problem

The Korteweg-de Vries (KdV) equation

$$y_t(t,x) + y_{xxx}(t,x) + y_x(t,x) + y(t,x)y_x(t,x) = 0,$$

is a nonlinear dispersive equation that serves as a mathematical model to study the propagation of long water waves in channels of relatively shallow depth and flat bottom. Here,

$$y(t,x) = \text{ surface elevation of the water wave at time } t \text{ and position } x.$$

The study of water waves moving over variable topography has been considered. If we denote $h = h(x)$ the variations in depth of the channel, then the proposed model becomes (after scaling)

$$y_t(t,x) + h^2(x)y_{xxx}(t,x) + (\sqrt{h(x)}y(t,x))_x + \frac{1}{\sqrt{h(x)}}y(t,x)y_x(t,x) = 0. \qquad (1)$$

Thus, we are led to consider variable coefficients KdV equations to model the water wave propagation in non-flat channels.

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We will deal with the KdV equation with non-constant coefficient $a = a(x)$ given by

$$\begin{cases} y_t + a(x)y_{xxx} + y_x + yy_x = g, & \forall (x,t) \in (0,L) \times (0,T), \\ y(t,0) = g_0(t), \quad y(t,L) = g_1(t), & \forall t \in (0,T), \\ \qquad\qquad\qquad\quad y_x(t,L) = g_2(t), & \forall t \in (0,T), \\ y(0,x) = y_0(x), & \forall x \in (0,L), \end{cases}$$

where the initial data y_0, the source term g, and the functions g_0, g_1, g_2 are assumed to be known.

In this context, the principal coefficient $a = a(x)$ represents the deepness of the bottom of the channel where the water wave propagates.

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Inverse Problem

Can we recover $a = a(x)$ *from some partial knowldege of* $y = y(x,t)$*?*

Inverse Problem (Uniqueness)

Given some boundary observations $Obs(y)$*, is there a unique* $a = a(x)$ *?*

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$$\|a - \tilde{a}\|_X \leq C \|Obs(y) - Obs(\tilde{y})\|_Y ?$$

Inverse Problem (Reconstruction)

Given some measurement $Obs(y)$, is it possible to reconstruct the coefficient $a = a(x)$?

In this talk, we are concerned with the **stability** of the inverse problem.

Remark: This kind of inverse problem is called a single-measurement IP

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We hope to get only boundary observations:

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The approach

1. The Bukhgeim-Klibanov-Malinsky method.
2. Carleman estimate for the linearized equation.

- Bukhgeim, Klibanov, 1981; Klibanov, Malinsky 1991: Inverse problems with Carleman estimates.
- Puel, Yamamoto 1996; Yamamoto, 1999; Imanuvilov, Yamamoto 2001: Wave equation.
- Imanuvilov, Yamamoto 1998, Benabdallah, Gaitan,Le Rousseau:2007 Parabolic equations.
- Baudouin, Puel 2002; Cardoulis,Cristofol, Gaitan 2008; Mercado,Osses, Rosier 2008: Schrödinger equation.
- Egger, Engl, Klibanov, 2005; Boulakia, Grandmont, Osses, 2009: Nonlinear equations.
- Bellassoued, Yamamoto 2006; Bellassoued, Choulli, 2009 : Logarithmic stability for the wave equation and the Schrödinger equation
- Isakov, *Inverse problems for partial differential equations*, Springer, 2006.

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BMK method

We follow ideas of Bukhgeim, Klibanov (1981), and Klibanov, Malinsky (1991).

If we set:

- $u = y - \tilde{y}$ and
- $\sigma = \tilde{a} - a$

then u solves the following KdV equation:

$$\begin{cases} u_t + a(x)u_{xxx} + (1+\tilde{y})u_x + \tilde{y}_x u + uu_x = \sigma\tilde{y}_{xxx}, & \forall (x,t) \in (0,L)\times(0,T), \\ u(t,0) = 0, \quad u(t,L) = 0, \quad u_x(t,L) = 0 & \forall t \in (0,T), \\ u(x,0) = 0, & \forall x \in (0,L). \end{cases}$$

Then $z = u_t$ satisfies the following equation:

$$\begin{cases} z_t + a(x)z_{xxx} + (1+y)z_x + y_x z = f_\sigma, & \forall (x,t) \in (0,L)\times(0,T), \\ z(t,0) = 0, \quad z(t,L) = 0, \quad z_x(t,L) = 0 & \forall t \in (0,T), \\ z(x,0) = \sigma(x)y_{0,xxx}(x), & \forall x \in (0,L), \end{cases}$$

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- We would like to have an estimate like

$$\|z(x,0)\|_X \leq C \|f_\sigma\|_Y + \text{ (boundary terms)}$$

  where:

- We shall need $y_{0,xxx}(x)$ bounded by below by a positive constant.
- The constant C can be chosen small enough.

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Remark: This kind of inequality is called observability in control theory.

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  where:

- We shall need $y_{0,xxx}(x)$ bounded by below by a positive constant.
- The constant C can be chosen small enough.

- We will use Carleman estimates.

Remark: This kind of inequality is called observability in control theory

Carleman inequalities.

Carleman inequalities were introduced by Trosten Carleman in 1939 in the study of uniqueness for some PDE's.

Since then, Carleman inequalities have been widely used in the study of :

- Unique continuation properties.
- Control problems of equations with non-regular lower order terms.
- Control problems of semi-linear equations.
- Some **inverse problems.**

Lebeau-Robianno (1995), Fursikov-Imanuvilov (1996), Tataru (1996).

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Carleman estimates. An example.

Consider $L = \Delta$ for functions $w \in C_c^\infty(\Omega)$.

We define

$$L_\phi w = e^{-\lambda\phi} L(e^{\lambda\phi} w)$$

$$\Delta(e^{\lambda\phi} w) = e^{\lambda\phi} \left(\lambda^2 |\nabla\phi|^2 w + \lambda\Delta\phi w + 2\lambda\nabla\phi\cdot\nabla w + \Delta w\right)$$

If $\phi(x) = \alpha\cdot x$ with $\alpha \in \mathbb{R}^n \setminus \{0\}$ then:

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A is self-adjoint.
B is anti-adjoint.

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$$L_\phi w = \underbrace{\lambda^2|\alpha|^2 w + \Delta w}_{Aw} + \underbrace{2\lambda\alpha \cdot \nabla w}_{Bw}$$

We have

$$\|L_\phi w\|_{L^2}^2 = \|Aw\|_{L^2}^2 + \|Bw\|_{L^2}^2 + 2\langle Aw, Bw\rangle_{L^2} \tag{2}$$

A is self-adjoint and B is anti-adjoint (and both have constant coefficients), we get

$$2\langle Aw, Bw\rangle_{L^2} = \langle [A,B]w, w\rangle_{L^2} = 0, \quad \forall\, w \in C_c^\infty(\Omega)$$

Thus

$$\|L_\phi w\|_{L^2} \geq 2\lambda \|\alpha \cdot \nabla w\|_{L^2} \tag{3}$$

$$\geq \lambda\delta \|w\|_{L^2} \tag{4}$$

Which means that

$$\lambda \|e^{-\lambda\phi} u\|_{L^2} \leq C\|e^{-\lambda\phi}\Delta u\|_{L^2} \tag{5}$$

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$$\begin{aligned} \|L_\phi w\|_{L^2} &\geq 2\lambda\|\alpha \cdot \nabla w\|_{L^2} && (3) \\ &\geq \lambda\delta\|w\|_{L^2} && (4) \end{aligned}$$

Which means that

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Carleman estimates. An example.

In other cases:

$$L_\phi w = \underbrace{\lambda^2 |\nabla \phi|^2 w + \Delta w}_{Aw} + \underbrace{2\lambda \nabla \phi \cdot \nabla w}_{Bw}$$

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$$\|L_\phi w\|_{L^2}^2 = \|Aw\|_{L^2}^2 + \|Bw\|_{L^2}^2 + 2\langle Aw, Bw\rangle_{L^2} \tag{6}$$

$2\langle Aw, Bw\rangle_{L^2} = \langle [A,B]w, w\rangle_{L^2} =$ lower order + boundary terms, $\forall\, w \in C_c^\infty(\Omega)$

Thus, we need to prove an estimate

$$\langle [A,B]w, w\rangle_{L^2} \geq \lambda\delta \|w\|_{H^k} - Obs(w) \tag{7}$$

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Carleman inequalities.

In general, given a differential operator P and a smooth function ϕ, we define

$$P_\phi = e^{\lambda\phi} P e^{-\lambda\phi}$$

Remark that $P_\phi = p(x, D + i\lambda\nabla\phi)$

For instance, ϕ is pseudoconvex if:

- For $P = \partial_t - \Delta$ if $|\nabla\phi| \neq 0$
- For $P = \partial_t^2 - \Delta$ if ϕ is convex.
- For $P = i\partial_t - \Delta$ si ϕ is convex.

Boundary condition: Usually is required $\dfrac{\partial\phi}{\partial\nu} < 0$ in $\partial\Omega$.

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Theorem (Carleman inequalities)

If ϕ is pseudoconvex with respect to P then

$$\|v\|_{H^m_\lambda} \leq C\|P_\phi v\|_{L^2}$$

for λ large enough.

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If the previous properties are not satisfied in a set $\omega \subset \Omega$ or $\omega \subset \partial\Omega$, then

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If ϕ is pseudoconvex with respect to P then

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In the original variable, we get:

$$\|e^{-\lambda\phi} w\|_{H^m} \leq C\|e^{-\lambda\phi} P w\|_{L^2} + \underbrace{\|e^{-\lambda\phi} w\|_{H^m(\omega)}}_{observation}$$

BMK method - Wave equation

$$\begin{cases} z_{tt} - a(x) z_{xx} = f_\sigma, & \forall (x,t) \in (0,L) \times (0,T), \\ z(t,0) = 0, \quad z(t,L) = 0 & \forall t \in (0,T), \\ z(x,0) = \sigma(x) y_{0,xx}(x), & \forall x \in (0,L), \end{cases}$$

What happens for wave equation?

- Extend the solution to $(-T,T)$ by using the symmetry under the change of variable $t \to (T-t)$.
- Use Carleman inequalities on $(-T,T)$.
- The time $t=0$ is not singular and you get

$$\|z(x,0)\|_X \leq C\|f_\sigma\|_Y + \text{ (boundary terms)},$$

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$$\begin{cases} iz_t + a(x)z_{xx} = f_\sigma, & \forall (x,t) \in (0,L)\times(0,T), \\ z(t,0)=0, \quad z(t,L)=0 & \forall t \in (0,T), \\ z(x,0) = \sigma(x) y_{0,xx}(x), & \forall x \in (0,L), \end{cases}$$

What happens for the Schrödinger equation?

- The Carleman weight is singular at $t=0$ and $t=T$.
- Extend the solution to $(-T,T)$ by defining the solution for negative time as

$$z(x,t) := -\bar{z}(x,-t), \quad f_\sigma(x,t) = -\bar{f}_\sigma(x,-t)$$

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Remark: The method needs $\mathrm{Re}(y_0(x))$ to be zero.

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BMK method - Heat equations

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What happens for the heat equation?

- Observability

$$\|z(x,0)\|_X \leq C\|f_\sigma\|_Y + \text{ (boundary terms)},$$

can not be proved for parabolic equation.

- Instead, one gets $\|z(x,T_0)\|_X \leq C\|f_\sigma\|_Y + $ (boundary terms).
- We use the equation

$$\|z(x,T_0)\| = \|u_t(x,T_0)\| = \|\sigma R(x,T_0) + a(x)u_{xx}(x,T_0)\| \\ \geq \|\sigma R(x,T_0)\| - \|a(x)u_{xx}(x,T_0)\|$$

and we have to add an observation like $\|y_{xx}(x,T_0) - \tilde{y}_{xx}(x,T_0)\|$!

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z(t,0) = 0, \quad z(t,L) = 0 & \forall t \in (0,T), \\
z(x,0) = \sigma(x) y_{0,xx}(x), & \forall x \in (0,L),
\end{cases}
$$

What happens for the heat equation?

- Observability
$$\|z(x,0)\|_X \leq C \|f_\sigma\|_Y + \text{ (boundary terms)},$$
can not be proved for parabolic equation.
- Instead, one gets $\|z(x,T_0)\|_X \leq C\|f_\sigma\|_Y +$ (boundary terms).
- We use the equation

$$
\begin{aligned}
\|z(x,T_0)\| = \|u_t(x,T_0)\| &= \|\sigma R(x,T_0) + a(x) u_{xx}(x,T_0)\| \\
&\geq \|\sigma R(x,T_0)\| - \|a(x) u_{xx}(x,T_0)\|
\end{aligned}
$$

and we have to add an observation like $\|y_{xx}(x,T_0) - \tilde{y}_{xx}(x,T_0)\|$!

BMK method - KdV equation

$$\begin{cases} z_t + a(x) z_{xxx} + (1+y) z_x + y_x z = f_\sigma, & \forall (x,t) \in (0,L) \times (0,T), \\ z(t,0) = 0, \quad z(t,L) = 0, \quad z_x(t,L) = 0 & \forall t \in (0,T), \\ z(x,0) = \sigma(x) y_{0,xxx}(x), & \forall x \in (0,L), \end{cases}$$

What happens for KdV equation?

- Not parabolic neither hyperbolic.
- From a control point of view, in some cases it is parabolic and in others hyperbolic.
- KdV has only one time-derivative and so the change $t \to T - t$ is not adequate.
- But it has the symmetry $t \to T - t$ and $x \to L - x$, which allows to define the solution for negative times.
- Carleman estimate on $(-T,T) \times (0,L)$.
- Time $t = 0$ is not singular any more for Carleman and therefore

$$\|z(x,0)\|_X \leq C \|f_\sigma\|_Y + \text{ (boundary terms)},$$

is obtained with C small.

Remark: Some symmetry conditions have to be imposed.

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BMK method - Extension for negative time

Symmetric extension to $(0,L)\times(-T,T)$ of g defined on $(0,L)\times(0,T)$:

$$g^s(x,t)=\begin{cases} g(x,t) & \text{if } x\in[0,L],\ t\in[0,T],\\ g(L-x,-t) & \text{if } x\in[0,L],\ t\in[-T,0).\end{cases}$$

Anti-symmetric extension to $(0,L)\times(-T,T)$ of g defined on $(0,L)\times(0,T)$:

$$g^a(x,t)=\begin{cases} g(x,t) & \text{if } x\in[0,L],\ t\in[0,T],\\ -g(L-x,-t) & \text{if } x\in[0,L],\ t\in[-T,0).\end{cases}$$

Defining $v=z^s$, we obtain:

$$\begin{cases} v_t+a(x)v_{xxx}+(1+y^s)v_x+(y_x)^a v=f^a_\sigma, & \forall x\in(0,L),\, t\in(-T,T),\\ v(t,0)=0,\quad v(t,L)=0, & \forall t\in(-T,T),\\ v_x(t,L)=\begin{cases}0, & \forall t\in(0,T),\\ -z_x(0,-t), & \forall t\in(-T,0).\end{cases} & \\ v(x,0)=\sigma(x)y_{0,xxx}(x), & \forall x\in(0,L).\end{cases}$$

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The solution of

$$\begin{cases} v_t + a(x)v_{xxx} + (1+y^s)v_x + (y_x)^a v = f_\sigma^a, & \forall x \in (0,L),\ t \in (-T,T), \\ v(t,0) = 0, \quad v(t,L) = 0, & \forall t \in (-T,T), \\ v_x(t,L) = \begin{cases} 0, & \forall t \in (0,T), \\ -z_x(0,-t), & \forall t \in (-T,0). \end{cases} & \\ v(x,0) = \sigma(x) y_{0,xxx}(x), & \forall x \in (0,L). \end{cases}$$

satisfies a Carleman estimate which allows to prove

$$\|v(x,0)\|_X \leq C\|f_\sigma\|_Y + \text{ (boundary terms)}$$

with C small.

Carleman estimates - $Lv = v_t + av_{xxx} = f$

Any $v \in L^2(-T,T;H^3 \cap H_0^1(0,L))$ and a weight function $\phi(x,t) = \frac{\beta(x)}{(T+t)(T-t)}$.

$$w = e^{-\lambda\phi}v, \quad \text{and} \quad L_\phi w = e^{-\lambda\phi}L(e^{\lambda\phi}w)$$

where λ is a large parameter to be chosen later.

The obtained Carleman estimate is an inequality like

$$\lambda^5\|w\|^2_{L^2_\phi} + \lambda^3\|w_x\|^2_{L^2_\phi} + \lambda\|w_{xx}\|^2_{L^2_\phi} + \frac{1}{\lambda}\|w_t\|^2_{L^2_\phi} \le C\,\|L_\phi w\|^2_{L^2_\phi} + B.D.(w)$$

Note that $w(-T,0)=0$, and therefore

$$\|w(0,x)\|^2_{L^2_\phi} = 2\int_{-T}^{0}\int ww_t \le \left(\lambda\int\int|w|^2\right)^{1/2}\left(\frac{1}{\lambda}\int\int|w_t|^2\right)^{1/2}$$

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Carleman estimates for KdV.

- Rosier [2004]. Null control of the surface of a water wave by means of a wavemaker at the left end-point.
- Glass-Guerrero [2008]. Cost of the null control of KdV by means of a control at the left end-point.
- Both papers prove Carleman estimates with one parameter $\lambda > 0$.
- For us, it is important a second parameter. Look at one dominating term:

$$\lambda^5 \iint \phi_x^4(-a_x\phi_x - 5a\phi_{xx} + 4a^2\phi_{xx})|w|^2$$

  This impose bad conditions of kind $\|a_x/a\|_{L^\infty} \leq M$.
- Solution is to choose ϕ such that $\phi_{xx} \approx s^2\varphi$ with a second parameter $s > 0$.

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- Solution is to choose ϕ such that $\phi_{xx} \approx s^2\varphi$ with a second parameter $s > 0$.

Main Result.

$$\begin{cases} y_t + a(x)y_{xxx} + y_x + yy_x = g, & \forall (x,t) \in (0,L)\times(0,T), \\ y(t,0) = g_0(t), \quad y(t,L) = g_1(t), & \forall t \in (0,T), \\ \qquad\qquad\qquad\quad y_x(t,L) = g_2(t), & \forall t \in (0,T), \\ y(0,x) = y_0(x), & \forall x \in (0,L). \end{cases}$$

Data (g, g_k, y_0) fixed and regular enough!

Theorem (M, Baudouin, Cerpa, Crepeau; JIIP 2013)

Let $|y_{0,xxx}(x)| \geq \delta > 0$*, symmetric wrt* $L/2$*. Let*

$$\Sigma = \left\{ a \text{ symmetric wrt } L/2 \middle/ a \geq a_0 > 0, \|a\|_{W^{3,\infty}} \leq M_1, \text{ and } \|y(a)\|_{W^{1,\infty}(Q)} \leq M_2 \right\}$$

There exists a constant $C = C(L, T, a_0, M_1, M_2, \delta) > 0$ *such that for any* $a, \tilde{a} \in \Sigma$*:*

$$C\|a - \tilde{a}\|_{L^2(0,L)} \leq \|y_x(t,0) - \tilde{y}_x(t,0)\|_{H^1(0,T)} + \|y_{xx}(t,0) - \tilde{y}_{xx}(t,0)\|_{H^1(0,T)} + \|y_{xx}(t,L) - \tilde{y}_{xx}(t,L)\|_{H^1(0,T)}$$

Future work

- Deal with the original model:

$$y_t(t,x) + h^2(x)y_{xxx}(t,x) + (\sqrt{h(x)}y(t,x))_x + \frac{1}{\sqrt{h(x)}}y(t,x)y_x(t,x) = f. \qquad (8)$$

- Remove the symmetry hypothesis.
- **Reconstruction**: Follow ideas of a work of Baudouin-de Buhan-Ervedoza, where is proposed a constructive algorithm to rebuild the potential in a wave equation.

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Muito obrigado!