3D Tomography with Synchrotron Data

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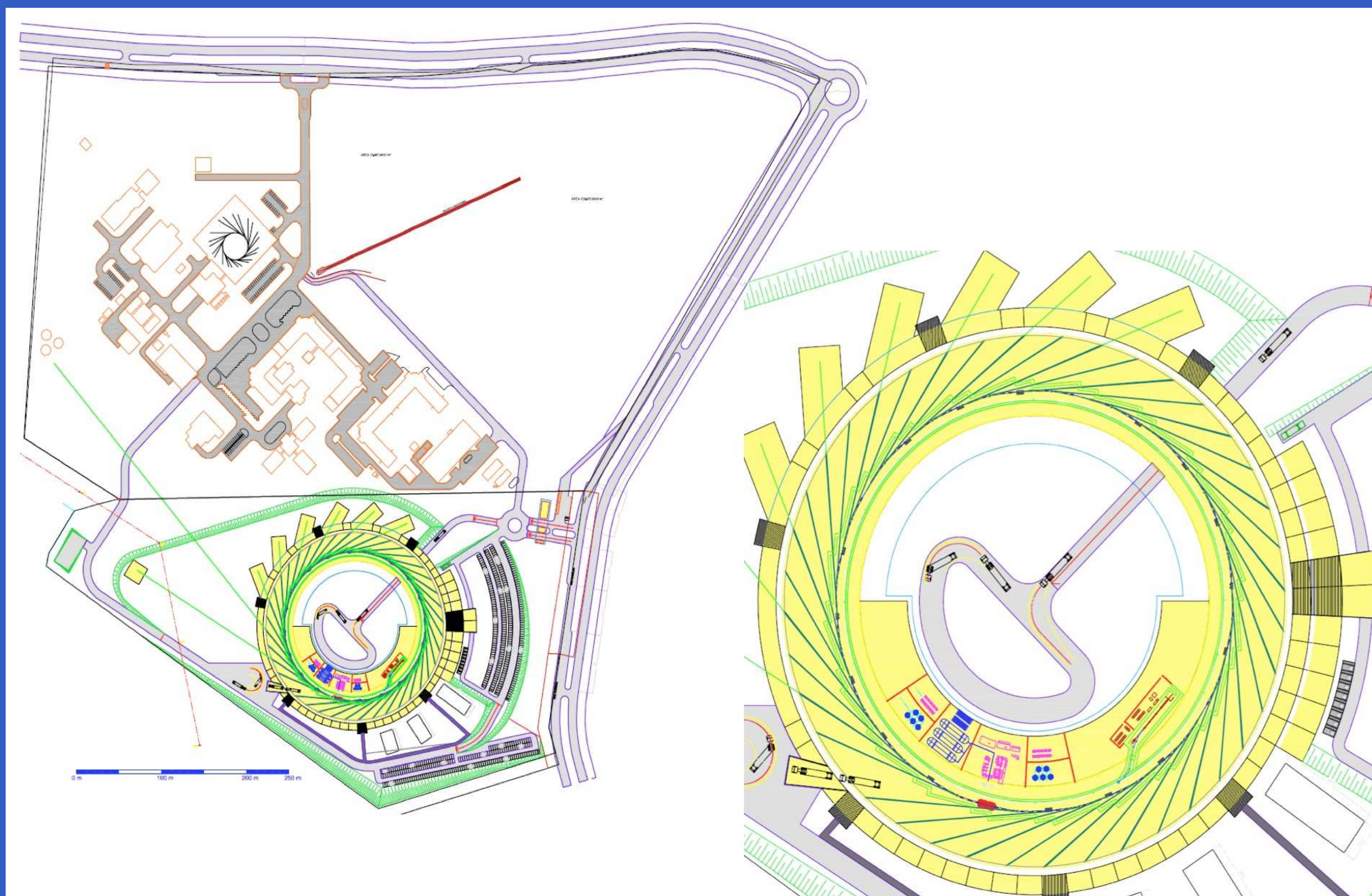
CNPEM - Brazilian Synchrotron Light Source

Campinas, SP - Brazil

Light Source: 2nd generation

Future: 3rd generation



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  - First approach: Stacking of 2D images

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  - $\mathbb{Q}$: your "favorite" reconstruction algorithm!

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$f = \mathcal{B}[q]$ Expensive part

$q = \mathbb{F}[p]$ Non-expensive part (FFT)

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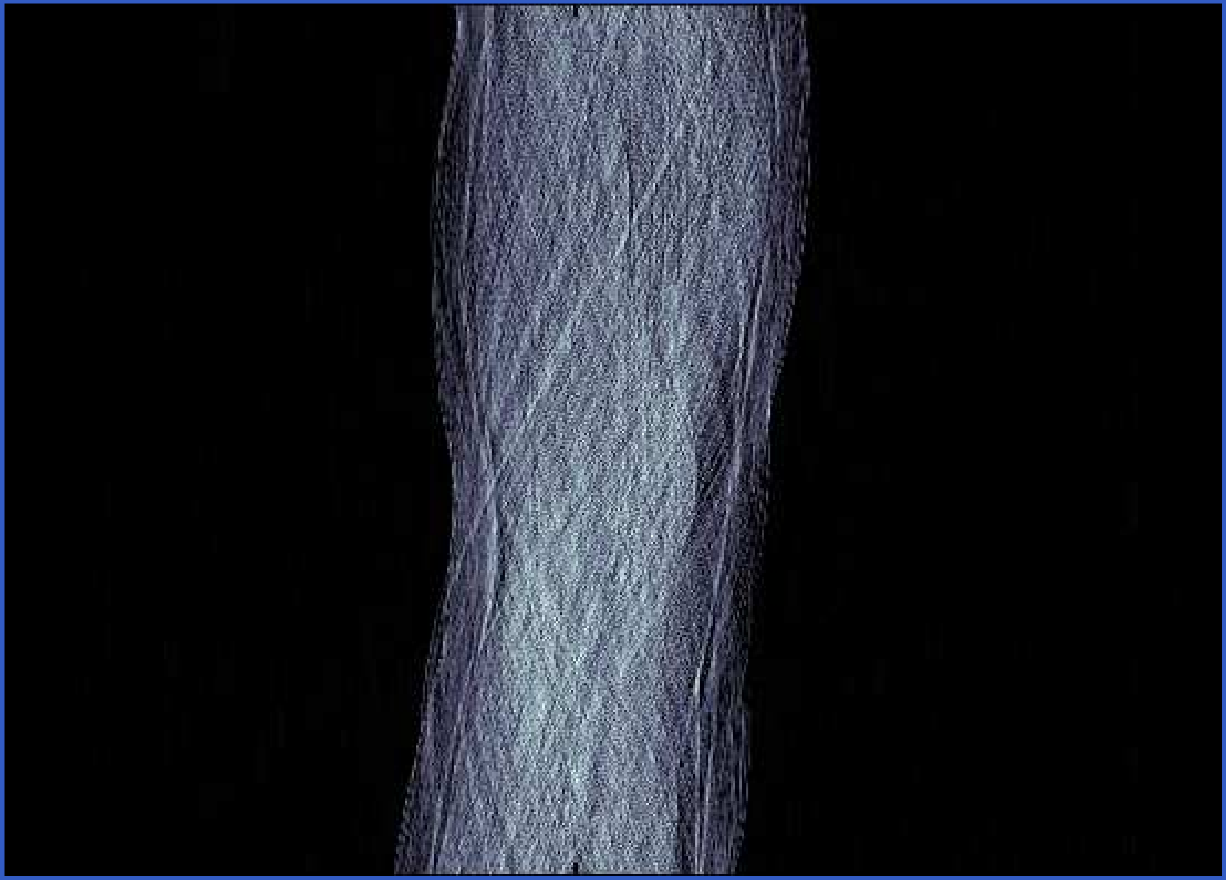
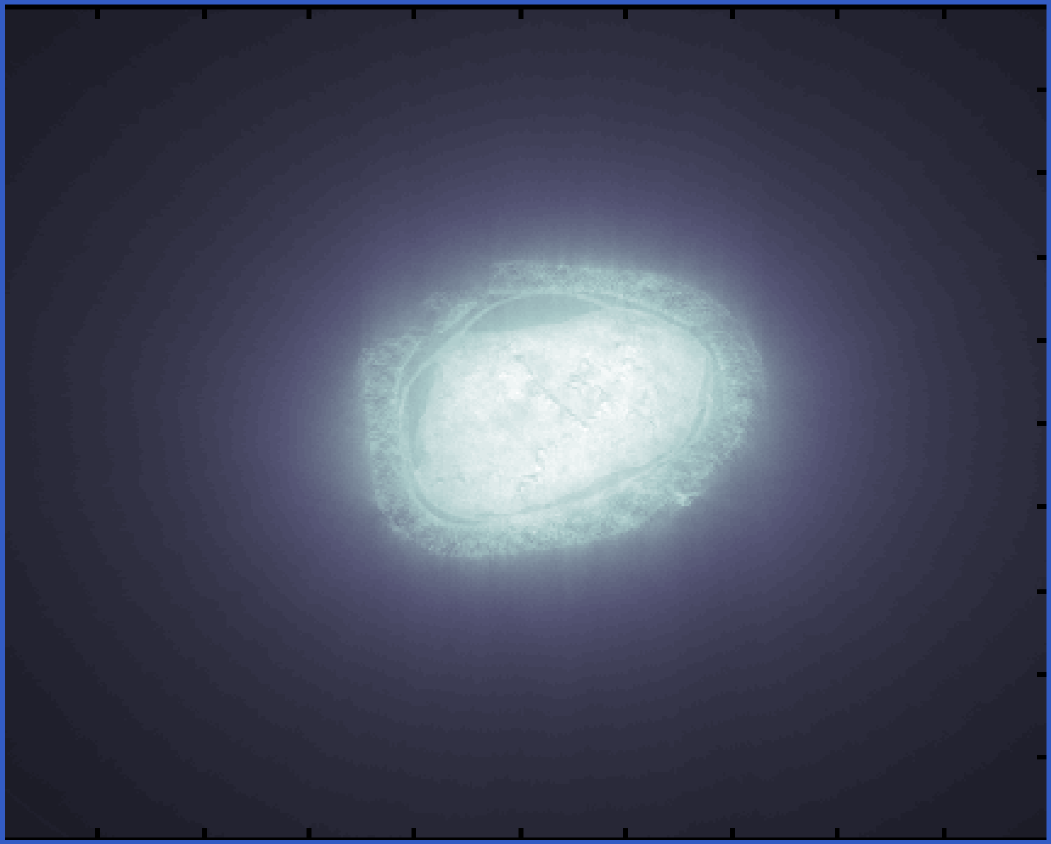
  N slices $\Rightarrow$ total cost: $O(N^4)$

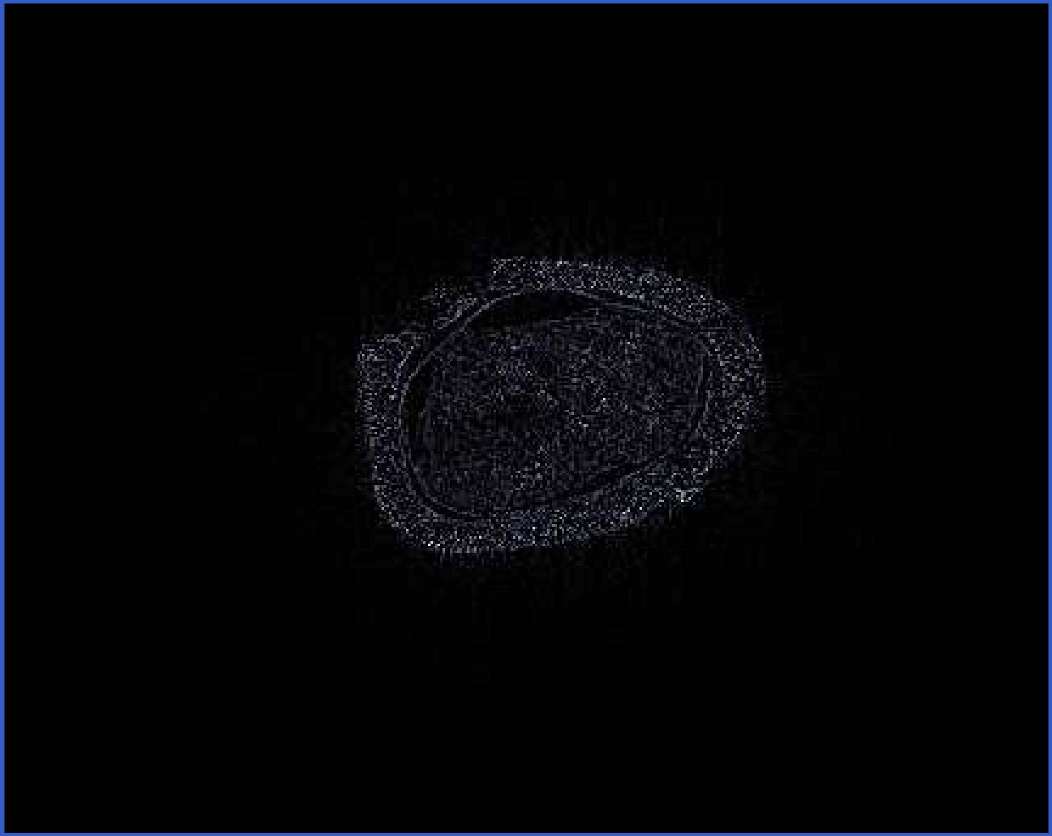
Real sinogram: Fiber



Backprojection



Filtered Backprojection



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$$\downarrow$$

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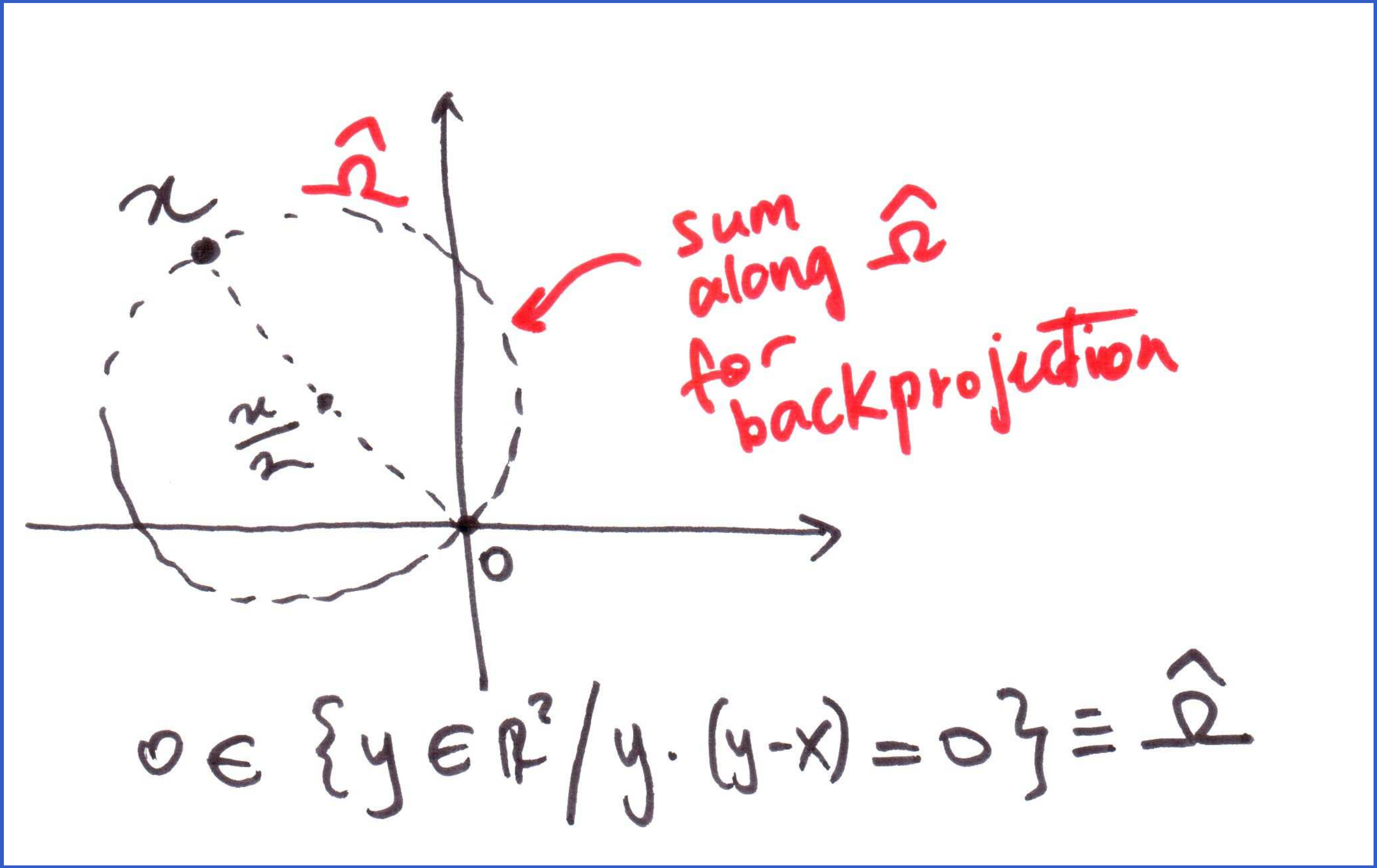
- $y \cdot (y - x) = 0 \quad \Leftrightarrow \quad \|y - \frac{x}{2}\|_2 = \frac{1}{2}\|x\|_2$

Geometrically...



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- $\hat{\hat{p}}(\mu,\beta)$: sinogram @ coordinates (μ,β)

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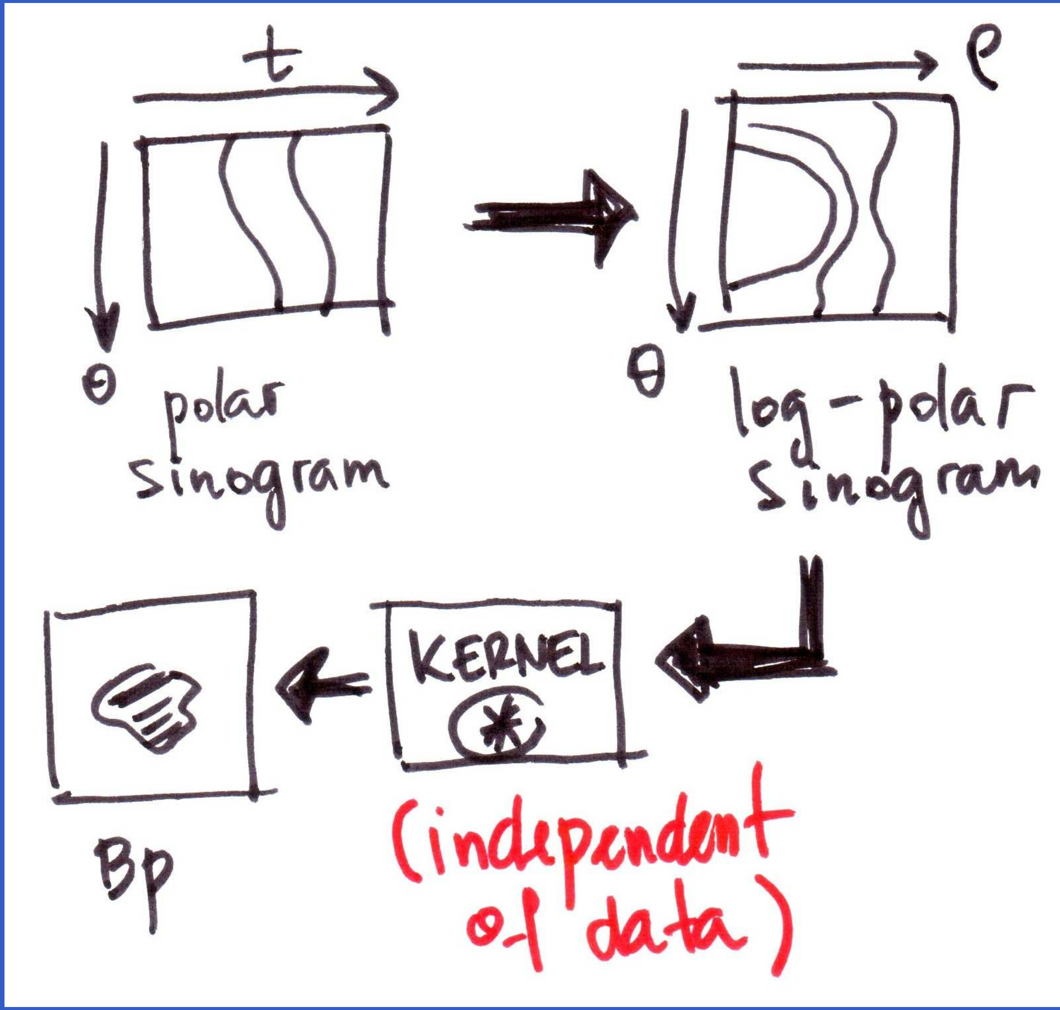
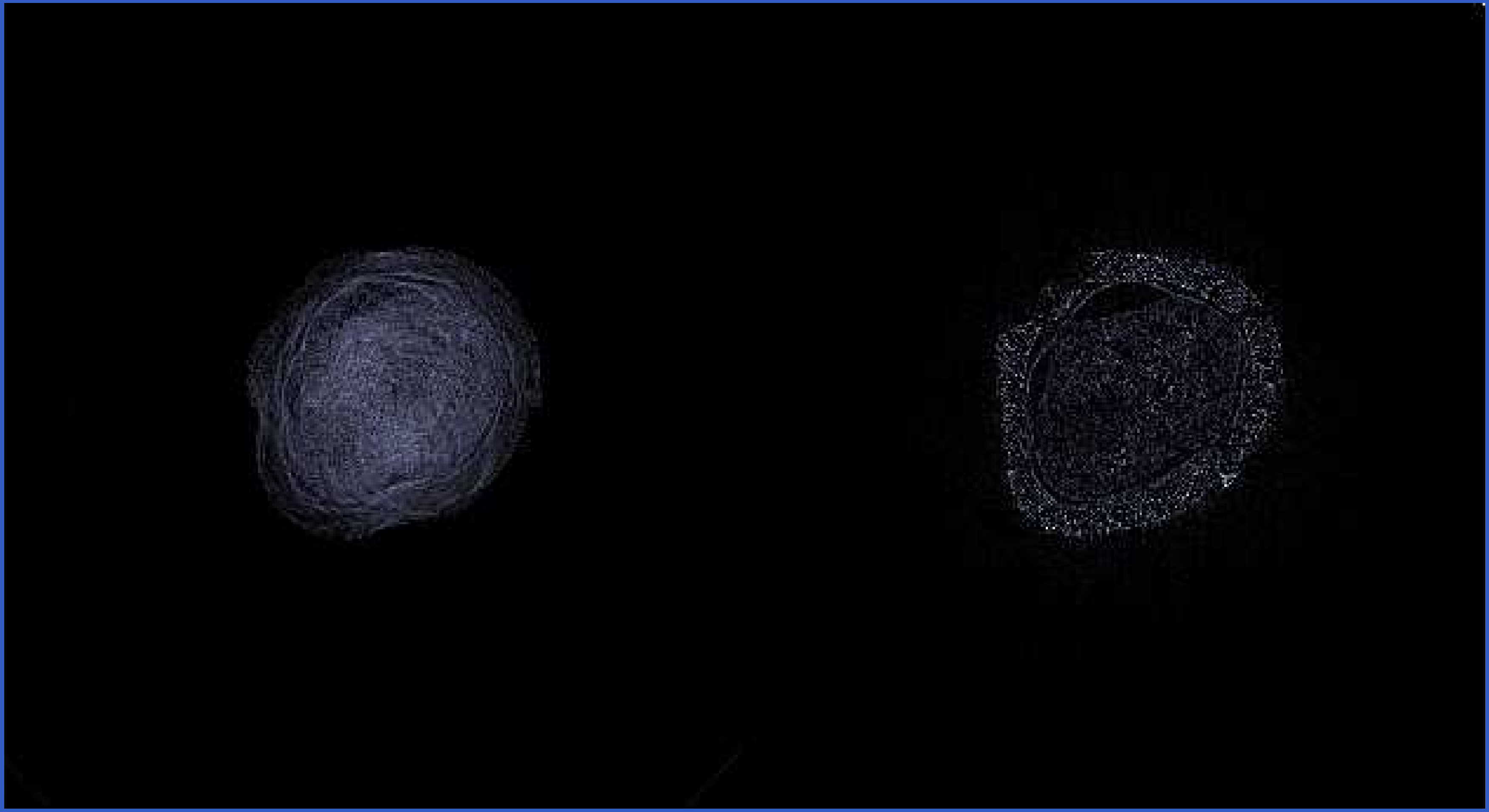
- ℓ, z are *log-polar* coordinates

New paradigm for $\mathcal{B}$

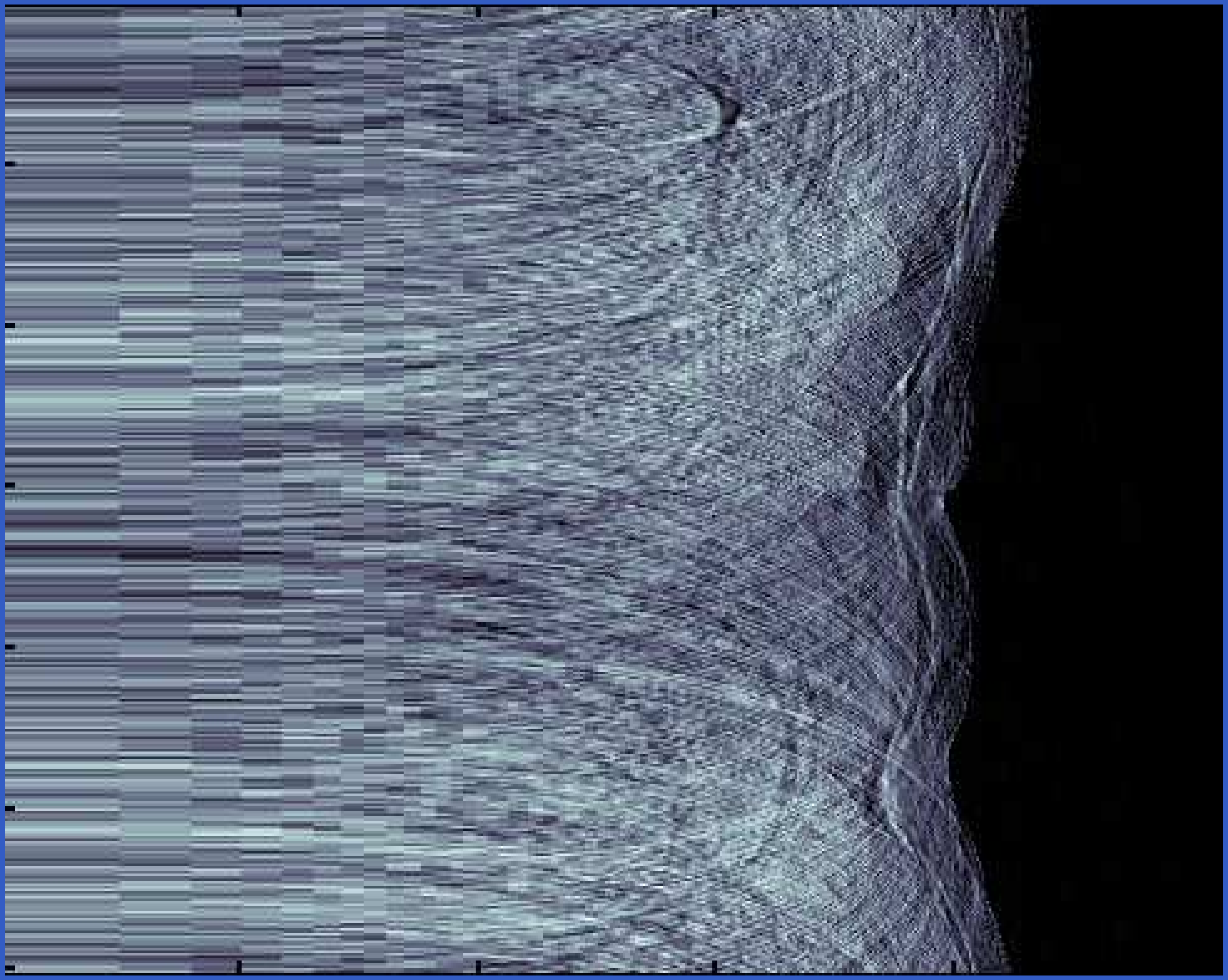


Example: cartesian sinogram

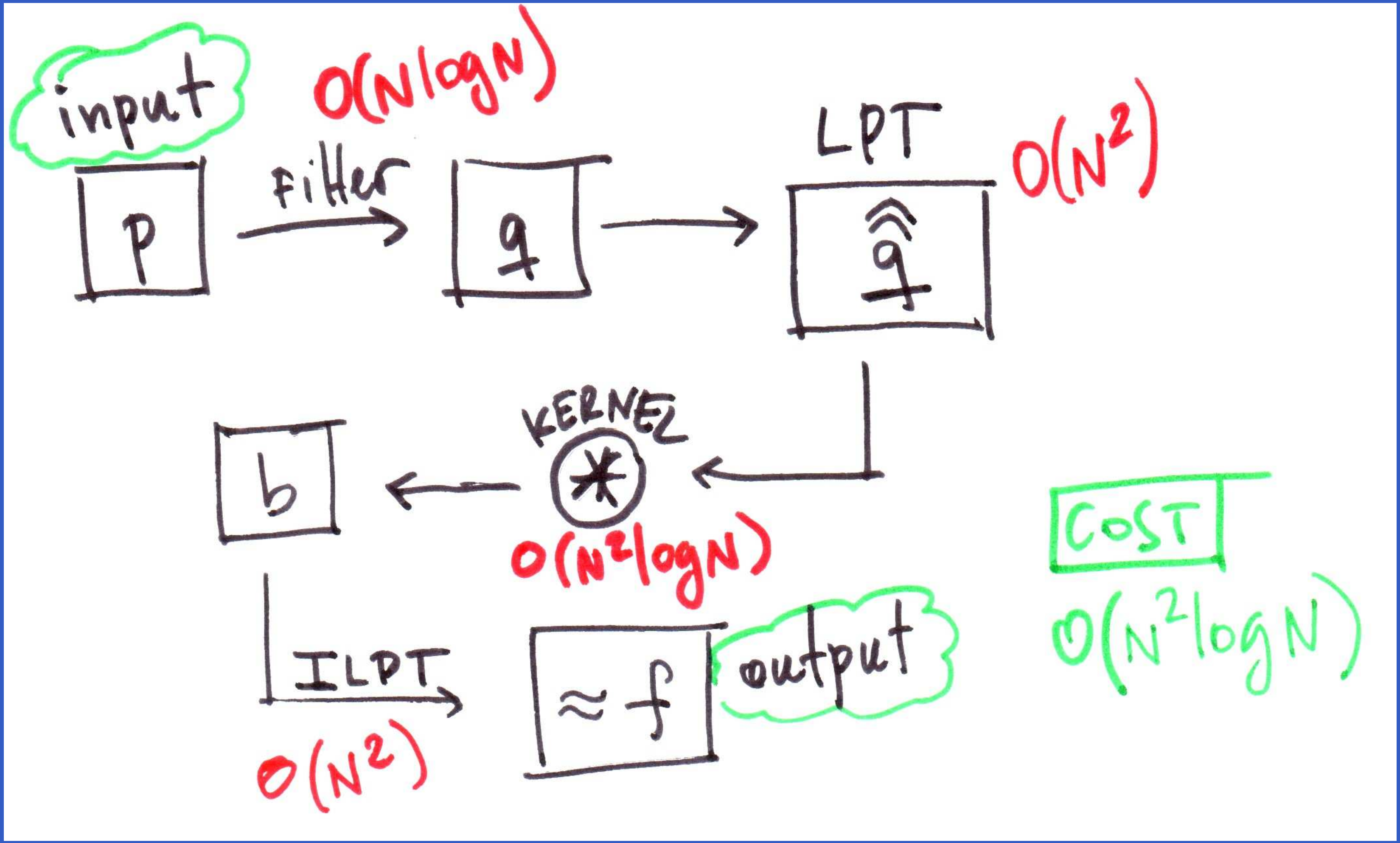
Example: cartesian sinogram $\times$ FBP



Example: log-polar sinogram



Fast 2D reconstruction



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  - Others

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fast and accurate?

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