29º Colóquio Brasileiro de Matemática

A New Method for the Inverse Potential Problem Based on the Topological Derivative Concept

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2. Topological Derivative Concept
3. Applications of the Topological Derivative
4. Second-Order Topological Derivative
5. Inverse Potential Problem
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Motivation

$$\inf_{\Omega \in \mathcal{E}} \psi(\Omega)$$

- $\psi(\Omega)$: shape functional
- Ω: geometrical domain
- $\mathcal{E}$: set of admissible domains

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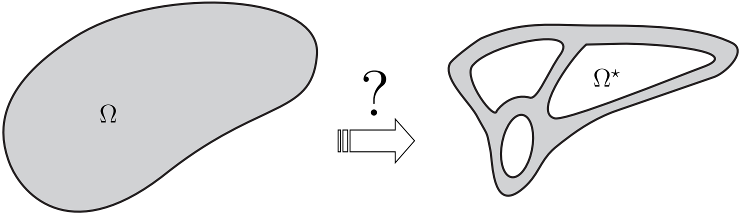
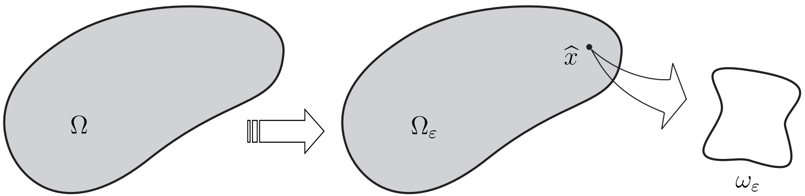
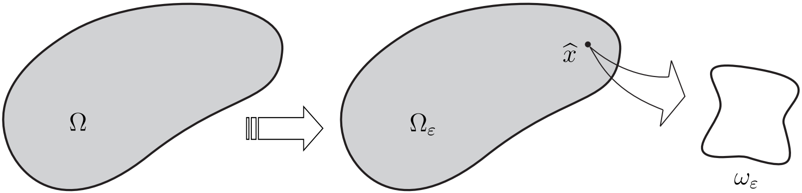


Topological Derivative Concept



Sokolowski & Zochowski, 1999

Topological Derivative Concept



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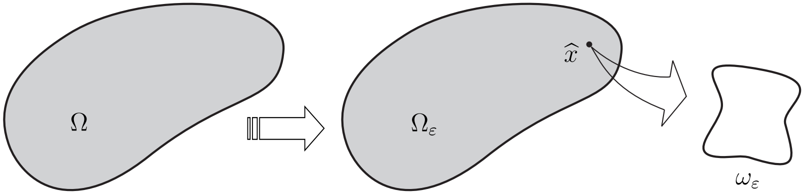
$$\psi(\Omega_\varepsilon(\widehat{x})) = \psi(\Omega) + f(\varepsilon)\mathcal{T}(\widehat{x}) + o(f(\varepsilon)) \ ,$$

where $\Omega_\varepsilon(\widehat{x}) = \Omega \setminus \overline{\omega_\varepsilon(\widehat{x})}$ and $f(\varepsilon) \to 0$, when $\varepsilon \to 0$.

Topological Derivative Concept



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$$\mathcal{T}(\widehat{x}) = \lim_{\varepsilon \to 0} \frac{\psi(\Omega_\varepsilon(\widehat{x})) - \psi(\Omega)}{f(\varepsilon)} \ .$$

In general, $f(\varepsilon) = |\omega_\varepsilon|$. It depends on the boundary condition on $\partial\omega_\varepsilon$.

Applications of the Topological Derivative

$$\boxed{\psi(\Omega_\varepsilon(\widehat{x})) = \psi(\Omega) + f(\varepsilon)\mathcal{T}(\widehat{x}) + o(f(\varepsilon))}$$

The topological sensitivity analysis gives the topological asymptotic expansion of a shape functional with respect to a singular domain perturbation, like the insertion of holes, inclusions or cracks. The first term of this expansion, called topological derivative, is now of common use for resolution of several problems, such as:

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- Image Processing: segmentation, restoration, denoising
- Mechanical Modeling: fracture and damage mechanics

Topology Optimization

Energy-Based Topological Derivative in Linear Elasticity

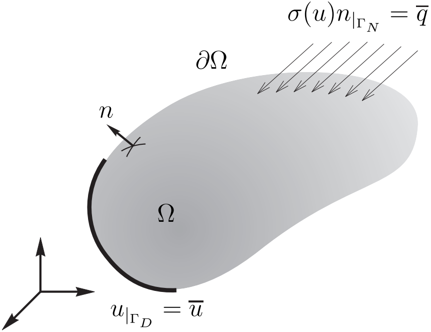


Figure : unperturbed problem defined in the domain Ω.

Topology Optimization

$$\psi(\Omega) := \mathcal{J}_\Omega(u) = \frac{1}{2}\int_\Omega \sigma(u)\cdot\nabla u^s - \int_{\Gamma_N} \overline{q}\cdot u \ ,$$

$$\begin{cases} \text{Find } u \text{, such that} \\ \begin{array}{rcll} -\mathrm{div}\sigma(u) & = & 0 & \text{in} \quad \Omega \ , \\ \sigma(u) & = & \mathbb{C}\nabla u^s & \\ u & = & \overline{u} & \text{on} \quad \Gamma_D \ , \\ \sigma(u)n & = & \overline{q} & \text{on} \quad \Gamma_N \ . \end{array} \end{cases}$$

$$\mathbb{C} = \frac{E}{1+\nu}\left(\mathbb{I} + \frac{\nu}{1-2\nu}\mathrm{I}\otimes\mathrm{I}\right) \ ,$$

Topology Optimization

Topological Derivative Calculation

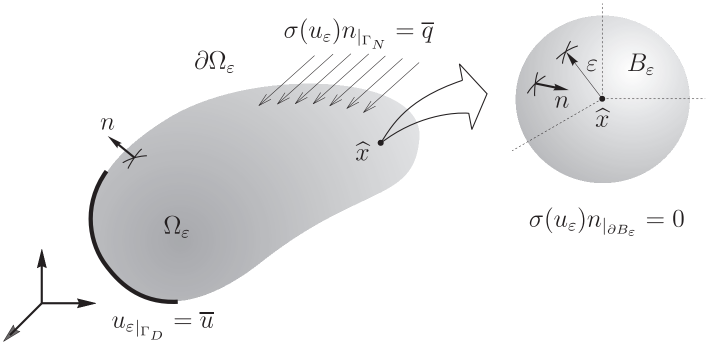


Figure : perturbed problem defined in the domain Ω_ε.

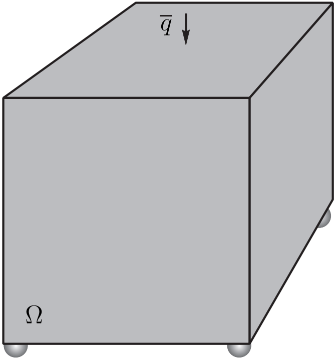
Topology Optimization

Topological Asymptotic Expansion

$$\psi(\Omega_\varepsilon(\widehat{x})) = \psi(\Omega) - \pi\varepsilon^3 \mathbb{P}\sigma(u(\widehat{x})) \cdot \nabla u^s(\widehat{x}) + o(\varepsilon^3) \ ,$$

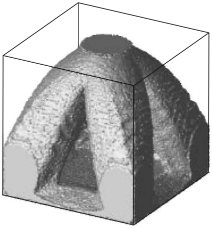
$$\mathbb{P} = \frac{3}{4}\frac{1-\nu}{7-5\nu}\left(10\mathbb{I} - \frac{1-5\nu}{1-2\nu}\mathrm{I}\otimes\mathrm{I}\right)$$

A *benchmark* example in $3D$

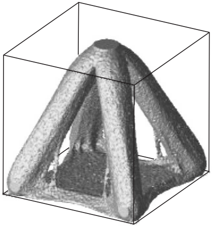


$$\Psi_\Omega(u) := -\mathcal{J}_\Omega(u) + \beta\,|\Omega| \;, \quad \mathcal{T} = \mathbb{P}\sigma(u)\cdot\nabla u^s - \beta \;.$$

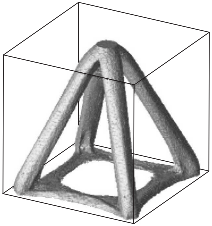
Topology Optimization



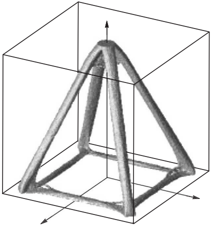
(a) iteration 13



(b) iteration 35

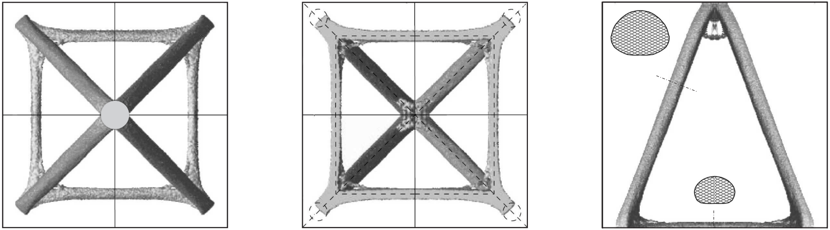


(c) iteration 52



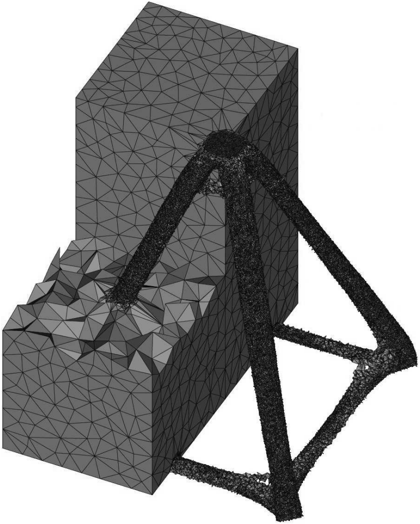
(d) iteration 76

Topology Optimization



(a) top (b) bottom (c) lateral

Topology Optimization



Second-Order Topological Derivative

$$\psi(\Omega_\varepsilon(\widehat{x})) = \psi(\Omega) + f(\varepsilon)\mathcal{T}(\widehat{x}) + f_2(\varepsilon)\mathcal{T}^2(\widehat{x}) + \mathcal{R}(f_2(\varepsilon)) \ ,$$

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where $f(\varepsilon) \to 0$ and $f_2(\varepsilon) \to 0$ with $\varepsilon \to 0$, and

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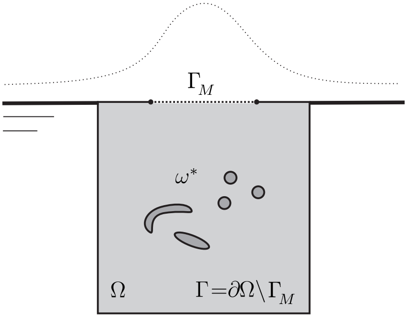
second order topological derivative

$$\mathcal{T}^2(\widehat{x}) := \lim_{\varepsilon\to 0}\frac{\psi(\Omega_\varepsilon(\widehat{x})) - \psi(\Omega) - f(\varepsilon)\mathcal{T}(\widehat{x})}{f_2(\varepsilon)} \ .$$

Inverse Potential Problem

Problem Formulation: Gravimetry Inverse Problem



$$\left\{ \begin{array}{l} \text{Find } b^*, \text{ such that} \\ \begin{array}{rcll} -\Delta u & = & b^* & \text{in} \quad \Omega\,, \\ \left.\begin{array}{r} u \\ -\partial_n u \end{array}\right. & \begin{array}{c} = \\ = \end{array} & \left.\begin{array}{l} u^* \\ q^* \end{array}\right\} & \text{on} \quad \Gamma_M\,. \end{array} \end{array} \right.$$

$$b^* = \gamma\chi_{\omega^*} \in PC_\gamma(\Omega)\,,$$

$$PC_\gamma(\Omega) := \{b \in L^\infty(\Omega) \ : \ b = \gamma\chi_\omega, \ \omega \subset \Omega \text{ is measurable}\}\,,$$

Inverse Potential Problem

$$u[b^*](x) = \int_\Omega K(x,y) b^*(y)\, dy \ ,$$

$$K(x,y) = \begin{cases} \dfrac{1}{4\pi|x-y|} & \text{for} \quad n = 3 \ , \\ -\dfrac{1}{2\pi} \ln|x-y| & \text{for} \quad n = 2 \ . \end{cases}$$

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$$u^* := u[b^*]|_{\Gamma_M} \text{ and } q^* := -\partial_n u[b^*]|_{\Gamma_M} \ .$$

Inverse Potential Problem

Difficulties

- The problem is over determined and highly ill-posed;
- Additional measurements do not provide extra information;
- Lack of uniqueness if the intensity γ and the region ω^* are unknown.

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$\triangleright$ We assume that the intensity γ is known

Inverse Potential Problem

Theorem (Uniqueness Result)

Let ω_1 and ω_2 be two star-shaped domains with respect to their centers of gravity. If $u_1 = u_2$ and $\partial_n u_1 = \partial_n u_2$ on Γ_M, with $|\Gamma_M| \neq 0$, then $\omega_1 = \omega_2$.

V. Isakov. Inverse Source Problems. American Mathematical Society, Providence, Rhode Island, 1990.

Inverse Potential Problem

Kohn-Vogelius Criterion

$$\min_{b \in PC_\gamma(\Omega)} J(b) := \frac{1}{2} \int_\Omega \left(u^D[b] - u^N[b] \right)^2 \ ,$$

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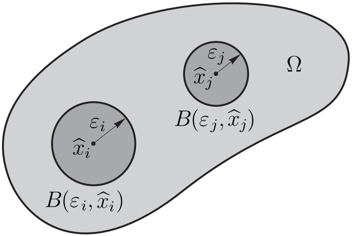
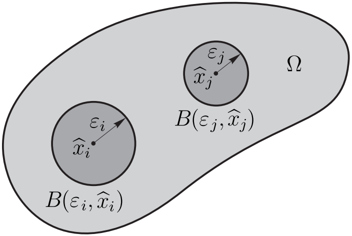
$$u^T[b] = \int_\Omega K(x,y) b(y)\, dy \ .$$

Inverse Potential Problem



Inverse Potential Problem



$$b_{\mathbf{e},\hat{\mathbf{x}}} = b + \gamma \sum_{i \in \mathcal{I}} \chi_{B(\varepsilon_i, \widehat{x}_i)} \, .$$

$$\varpi_{\mathbf{e},\hat{\mathbf{x}}} = \cup_{i \in \mathcal{I}} B(\varepsilon_i, \widehat{x}_i) \, , \quad \text{with} \quad \mathcal{I} = \{1, ..., m\}$$

$$\mathbf{e} := \{\varepsilon_i\}_{i \in \mathcal{I}} \quad \hat{\mathbf{x}} := \{\hat{x}_i\}_{i \in \mathcal{I}} \, , \quad \text{with} \quad \varepsilon_i > 0 \, , \quad \widehat{x}_i \in \Omega$$

Inverse Potential Problem

$$\boxed{J(b_{\mathbf{e},\hat{\mathbf{x}}}) = J(b) - \int_\Omega (u^D[b] - u^N[b]) \sum_{i\in\mathcal{I}} a_i h_i + \frac{1}{2}\int_\Omega \left(\sum_{i\in\mathcal{I}} a_i h_i\right)^2}$$

where $a_i := |B(\varepsilon_i, \widehat{x}_i)|$ and

$$\left\{\begin{array}{rcll} -\Delta h_i &=& 0 & \text{in} \quad \Omega \,, \\ -\partial_n h_i &=& g_i & \text{on} \quad \Gamma_M \,, \\ h_i &=& 0 & \text{on} \quad \Gamma \,, \end{array}\right.$$

with $g_i = \partial_n v_i$ on Γ_M and

$$\left\{\begin{array}{rcll} -\Delta v_i &=& \gamma\delta(x - \widehat{x}_i) & \text{in} \quad \Omega \,, \\ v_i &=& 0 & \text{on} \quad \Gamma_M \,. \\ v_i &=& \gamma K(x, \widehat{x}_i) & \text{on} \quad \Gamma \,. \end{array}\right.$$

Inverse Potential Problem

$$J(b_{\mathbf{e},\hat{\mathbf{x}}}) = J(b) - \int_\Omega (u^D[b] - u^N[b]) \sum_{i\in\mathcal{I}} a_i h_i + \frac{1}{2}\int_\Omega \left(\sum_{i\in\mathcal{I}} a_i h_i\right)^2$$

Minimization with respect to a_i yields

$$\boxed{H_{ij} a_j = f_i}$$

$$H_{ij} = \int_\Omega h_i h_j \qquad \text{and} \qquad f_i = \int_\Omega (u^D - u^N) h_i \ ,$$

Inverse Potential Problem

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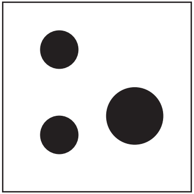
$$J(b_{\mathbf{e},\hat{\mathbf{x}}}) = J(b) - \frac{1}{2} a_i f_i$$

Inverse Potential Problem

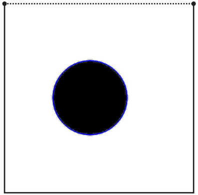
Example 1: Looking for three anomalies



target

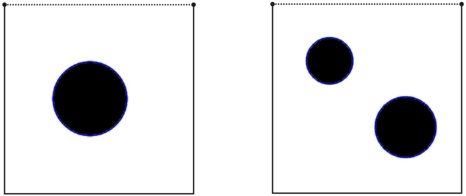
Inverse Potential Problem

Inverse Potential Problem



(a) one ball

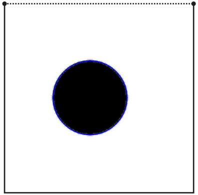
Inverse Potential Problem



(a) one ball

(b) two balls

Inverse Potential Problem



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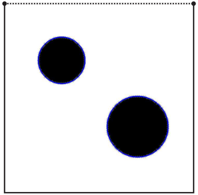
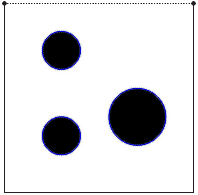
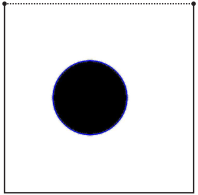


(b) two balls



(c) three balls

Inverse Potential Problem



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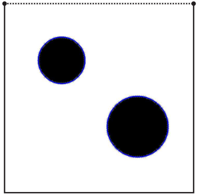
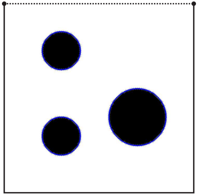
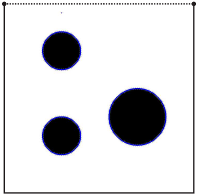


(b) two balls



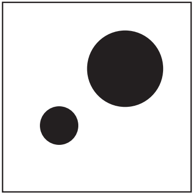
(c) three balls



(d) four balls

Inverse Potential Problem

Example 2: Partial boundary measurement



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Inverse Potential Problem

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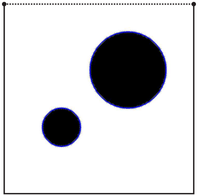
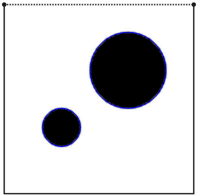
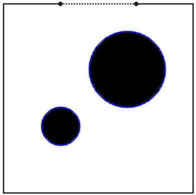


(a) $|\Gamma_M| = 1.0$

Inverse Potential Problem

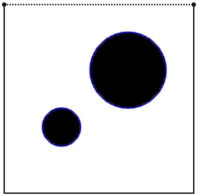


(a) $|\Gamma_M| = 1.0$

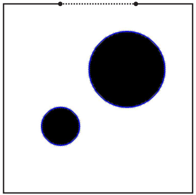


(b) $|\Gamma_M| = 0.4$

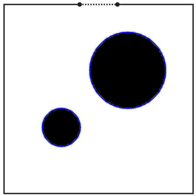
Inverse Potential Problem



(a) $|\Gamma_M| = 1.0$

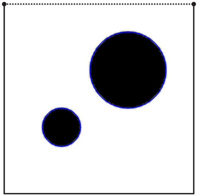


(b) $|\Gamma_M| = 0.4$

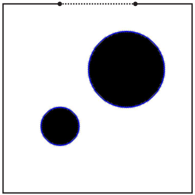


(c) $|\Gamma_M| = 0.2$

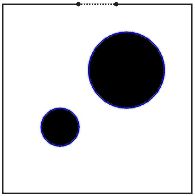
Inverse Potential Problem



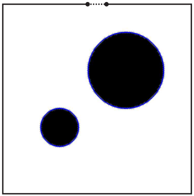
(a) $|\Gamma_M| = 1.0$



(b) $|\Gamma_M| = 0.4$



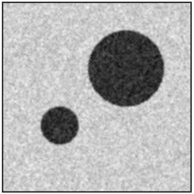
(c) $|\Gamma_M| = 0.2$



(d) $|\Gamma_M| = 0.1$

Inverse Potential Problem

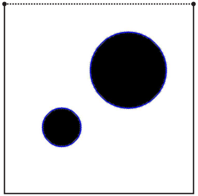
Example 3: Noisy data



target

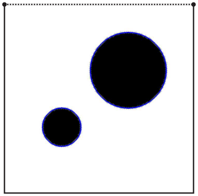
Inverse Potential Problem

Inverse Potential Problem

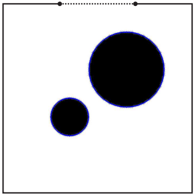


(a) $|\Gamma_M| = 1.0$

Inverse Potential Problem

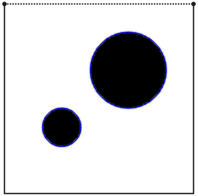


(a) $|\Gamma_M| = 1.0$

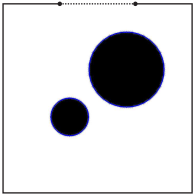


(b) $|\Gamma_M| = 0.4$

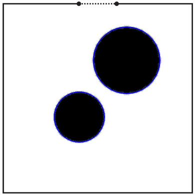
Inverse Potential Problem



(a) $|\Gamma_M| = 1.0$

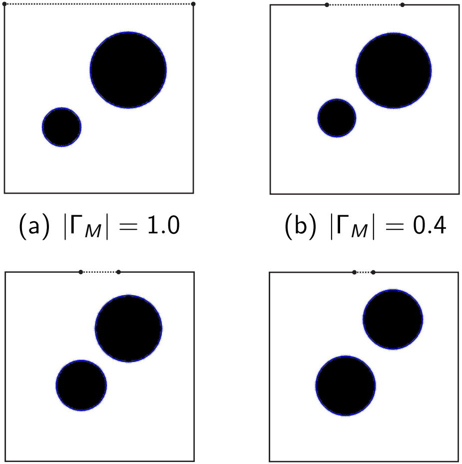


(b) $|\Gamma_M| = 0.4$



(c) $|\Gamma_M| = 0.2$

Inverse Potential Problem



(a) $|\Gamma_M| = 1.0$ (b) $|\Gamma_M| = 0.4$

(c) $|\Gamma_M| = 0.2$ (d) $|\Gamma_M| = 0.1$

Inverse Potential Problem

Example 4: Shape and topology reconstruction

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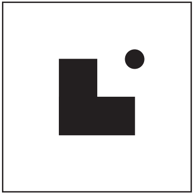
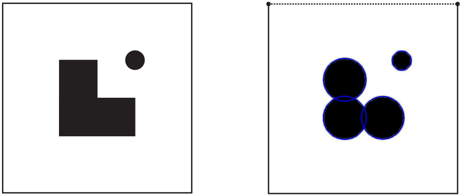


(a) target

Inverse Potential Problem

Example 4: Shape and topology reconstruction



(a) target (b) result

Inverse Potential Problem

Example 5: Two anomalies far from each other

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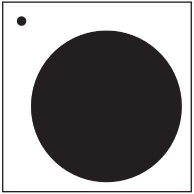
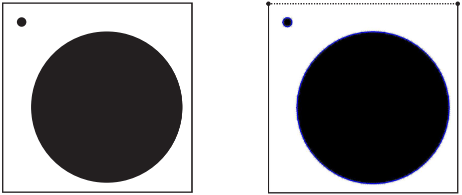


(a) target

Example 5: Two anomalies far from each other



(a) target (b) result

Example 6: Two anomalies close to each other

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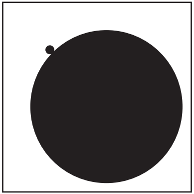
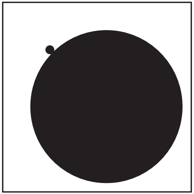
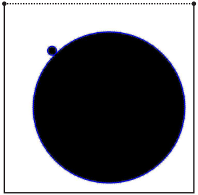


(a) target

Example 6: Two anomalies close to each other



(a) target



(b) result

Example 7: Hidden object

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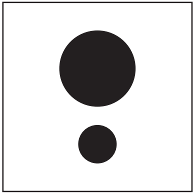
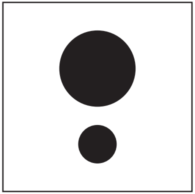


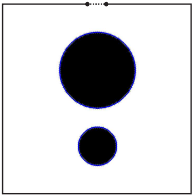
(a) target

Inverse Potential Problem

Example 7: Hidden object



(a) target



(b) result

Conclusions

- The number of unknown anomalies can be found after some trials.
- Due to the combinatorial nature of the search procedure, the problem is tractable only in the case of small number of unknown measures.
- Completely hidden anomalies can be detected from very few information (single partial boundary measurement).
- Corrupted measurements with a high level of noise can be reconstructed with acceptable precision.
- The characterization of the biggest set $PC_\gamma(\Omega)$ seems to be an open problem.

Muito Obrigado!

A.A. Novotny (joint with A. Canelas and A. Laurain). A Non-Iterative Method for the Inverse Potential Problem Based on the Topological Derivative. Oberwolfach Seminars vol. 47. Geometries, Shapes and Topologies in PDE-based Applications. Edited by M. Hintermüller, G. Leugering, J. Sokolowski. To appear.

M. Hintermüller, A. Laurain, A.A. Novotny, Second-order topological expansion for electrical impedance tomography, Advances in Computational Mathematics 36(2):235–265, 2012.

A.A. Novotny & J. Sokołowski. *Topological Derivatives in Shape Optimization*. Mechanics and Mathematics Iteraction Series. 432p. Springer, 2013.