INVERSE HEAT TRANSFER PROBLEMS

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OUTLINE

- **INTRODUCTION**
- **SOLUTION OF INVERSE PROBLEMS**
  - **General considerations**
  - **Bayesian framework: MCMC, PARTICLE FILTER**
    - **- Computational speed-up**
    - **- Improvement of solutions with reduced models**
- **CONCLUSIONS**

INTRODUCTION

Inverse heat transfer problems deal with the estimation of unknown quantities appearing in the mathematical formulation of physical processes in thermal sciences, by using measurements of temperature, heat flux, radiation intensities, etc.

INTRODUCTION

• Originally, inverse heat transfer problems have been associated with the estimation of an unknown boundary heat flux, by using temperature measurements taken below the boundary surface of a heat conducting medium.

• Recent technological advancements often require the use of involved experiments and indirect measurements, within the research paradigm of inverse problems.

• Nowadays, inverse analyses are encountered in single and multi-mode heat transfer problems, dealing with multi-scale phenomena.

• Applications range from the estimation of constant heat transfer parameters to the mapping of spatially and timely varying functions, such as heat sources, fluxes and thermophysical properties.

SOLUTION OF INVERSE PROBLEMS

General Considerations

Consider the mathematical formulation of a heat transfer problem, which, for instance, can be linear or non-linear, one or multi-dimensional, involve one single or coupled heat transfer modes, etc.

We denote the **vector of parameters** appearing in such formulation as:

$$\boxed{\mathbf{P}^T = [P_1, P_2, \ldots, P_N]}$$

where ***N* is the number of parameters**

- These parameters can possibly be thermal conductivity components, heat transfer coefficients, heat sources, boundary heat fluxes, etc.
- They can represent constant values of such quantities, or the parameters of the representation of a function in terms of known basis functions.

SOLUTION OF INVERSE PROBLEMS

General Considerations

Consider also that transient measurements are available within the medium, or at its surface, where the heat transfer processes are being mathematically formulated.

The vector containing the **measurements** is written as:

$$\boxed{\mathbf{Y}^T = \left(\vec{Y}_1 \; , \; \vec{Y}_2 \; , \ldots , \vec{Y}_I \right)}$$

$$\boxed{\vec{Y}_i = \left(Y_{i1} \; , \; Y_{i2} \; , \ldots , Y_{iM} \right)}$$

$$\left.\begin{array}{l} M = \# \text{ of sensors} \\ I = \# \text{ of transient measurements per sensor} \end{array}\right\} \; D = MI = \# \text{ of measurements}$$

• The measured data are not limited to temperatures, but could also include heat fluxes, radiation intensities, etc.

SOLUTION OF INVERSE PROBLEMS
Bayesian framework

The **statistical inversion approach** is based on the following principles (Jari P. Kaipio and Erkki Somersalo, *Computational and Statistical Methods for Inverse Problems*, Springer, 2004):

1. All variables included in the formulation are modeled as random variables.
2. The randomness describes the degree of information concerning their realizations.
3. The degree of information concerning these values is coded in the probability distributions.
4. The solution of the inverse problem is the posterior probability distribution.

SOLUTION OF INVERSE PROBLEMS
Bayesian framework

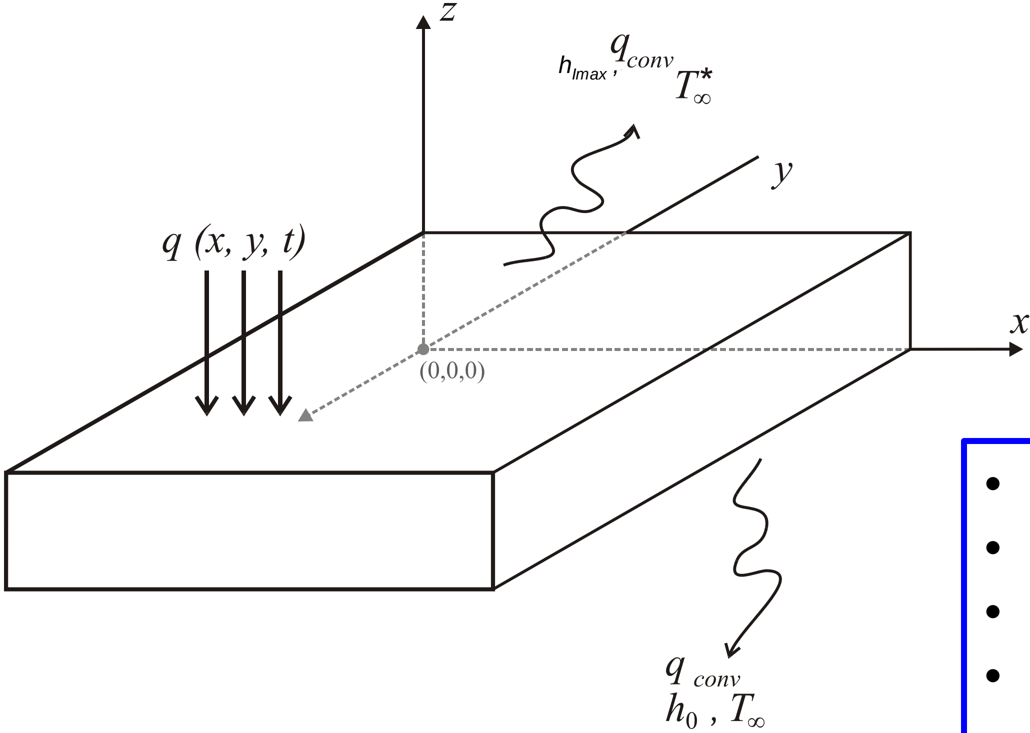
- In many cases, the Posterior Probability Distribution does not allow an analytical treatment.
- Draw samples from the set $\boldsymbol{\Omega}$ of all possible **P**'s, each sample with probability $\pi(\mathbf{P}|\mathbf{Y})$.
- Get a set $\Theta = \{\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_M\}$ of samples distributed like the posterior distribution.
- Inference on $\pi(\mathbf{P}|\mathbf{Y})$ becomes inference on $\Theta = \{\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_M\}$, for example the mean of the samples in Θ give us an estimation of the average values of $\pi(\mathbf{P}|\mathbf{Y})$.
- We generally need the constant that normalizes the probability distribution:

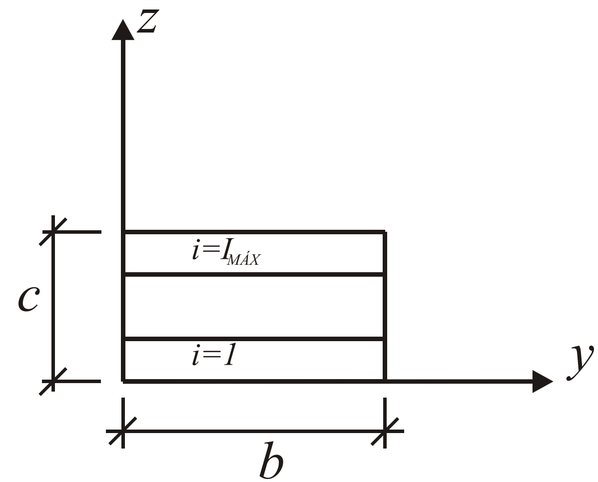
  MARKOV CHAIN MONTE-CARLO METHODS

  (Metropolis-Hastings Algorithm)
- Very time consuming.

SOLUTION OF INVERSE PROBLEMS
Bayesian framework

Example: Estimation of Contact Failures in Layered Composites





- Metropolis-Hastings algorithm
- 2 layers
- Simulated Measurements
- TV prior
- 22 hours

L. A. Abreu, H. R. B. Orlande, J. Kaipio, V. Kolehmainen, R. M. Cotta, J. N. N. Quaresma, Identification Of Contact Failures In Multi-layered Composites With The Markov Chain Monte Carlo Method, ASME *Journal of Heat Transfer*, (under review)

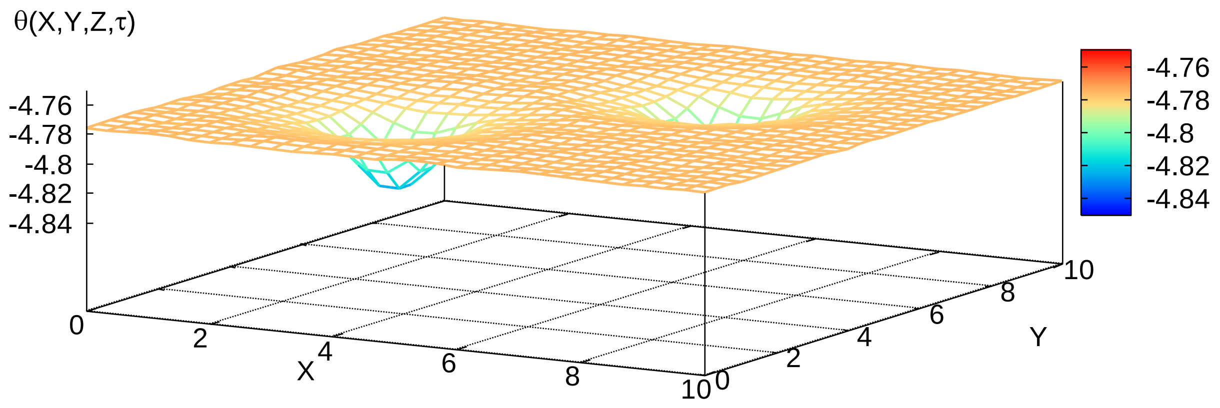


Figure 5.a Exact temperature distribution at $Z = 1$ and $\tau = 0.065$ – two square failures of size 0.005 m

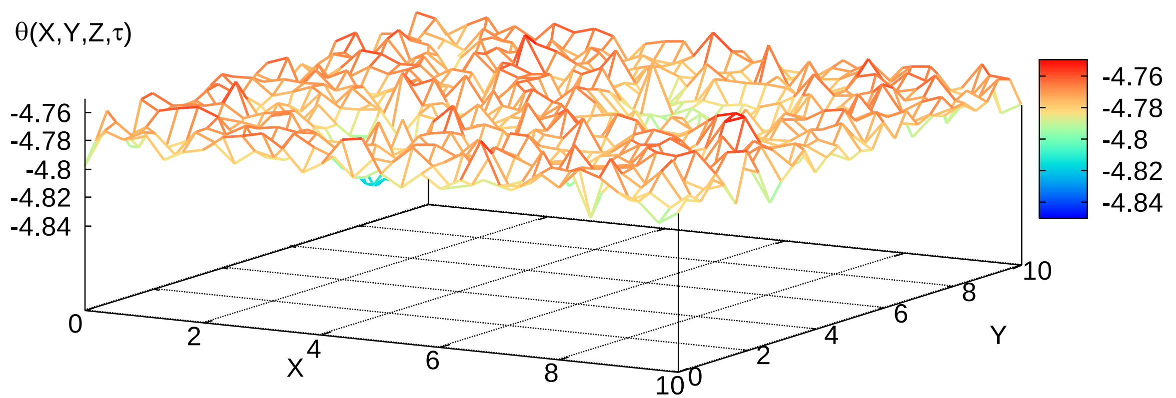
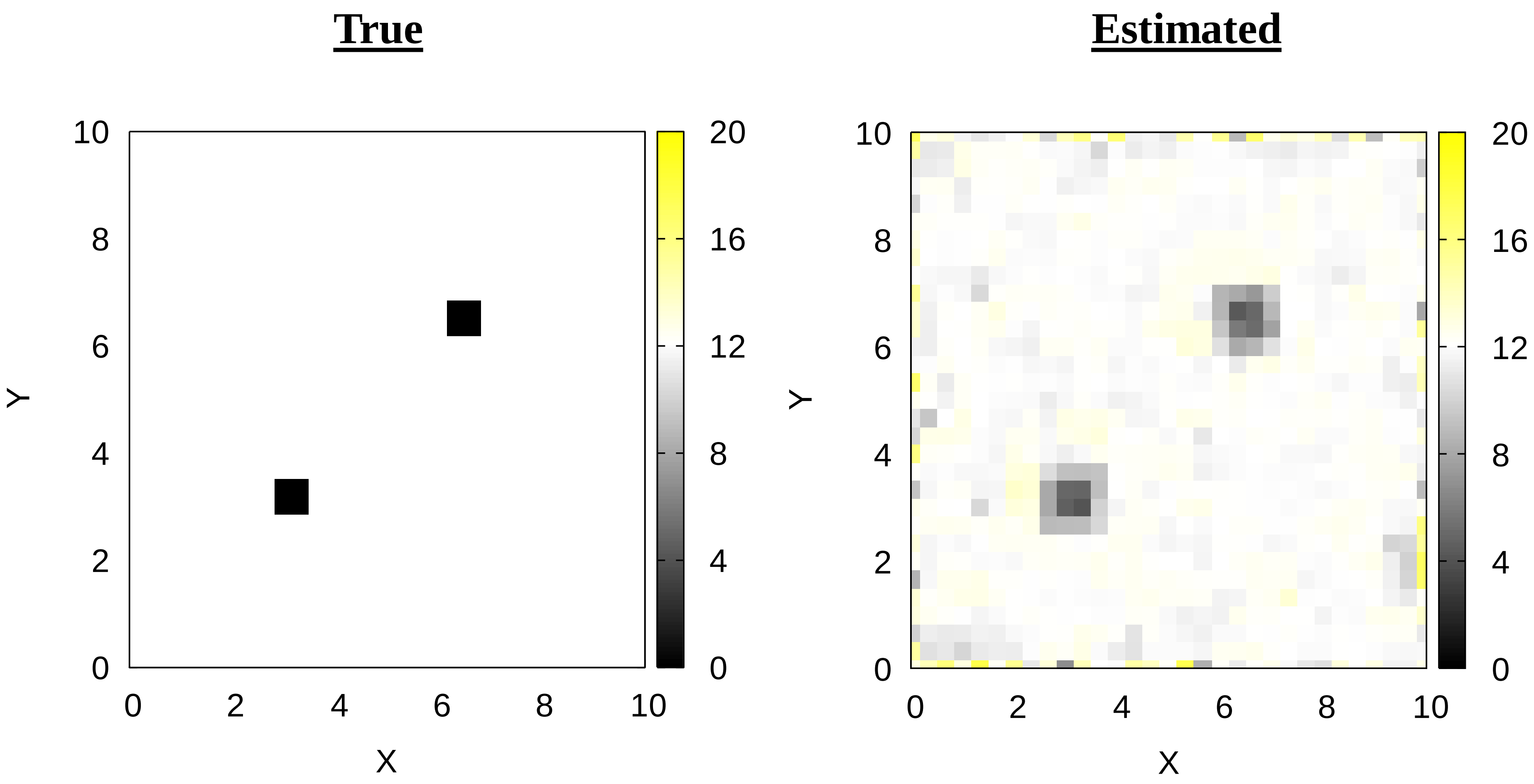


Figure 5.b Simulated measurements at $Z = 1$ and $\tau = 0.065$ – two square failures of size 0.005



True
Estimated
10
8
6
4
2
0
Y
X
20
16
12
8
4
0

SOLUTION OF INVERSE PROBLEMS
Bayesian framework

Example: Estimation of Thermal Conductivity Components of Orthotropic Solids

$$k_1 \frac{\partial^2 T}{\partial x^2} + k_2 \frac{\partial^2 T}{\partial y^2} + k_3 \frac{\partial^2 T}{\partial z^2} = \frac{\partial T}{\partial t} \quad \text{in } 0 < x < a,\ 0 < y < b,\ 0 < z < c\ ;\ \ t > 0$$

$$T = 0 \text{ at } x = 0 \quad ; \quad k_1 \frac{\partial T}{\partial x} = q_1(t) \text{ at } x = a \quad , \quad \text{for } t > 0$$

$$T = 0 \text{ at } y = 0 \quad ; \quad k_2 \frac{\partial T}{\partial y} = q_2(t) \text{ at } y = b \quad , \quad \text{for } t > 0$$

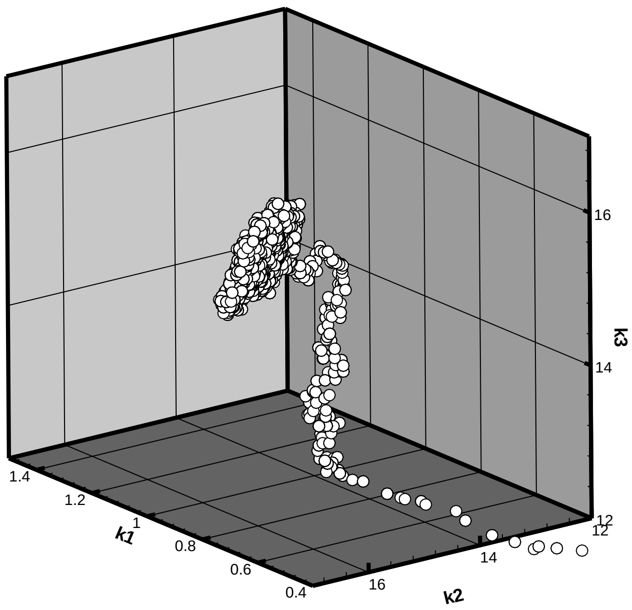
$$T = 0 \text{ at } z = 0 \quad ; \quad k_3 \frac{\partial T}{\partial z} = q_3(t) \text{ at } z = c \quad , \quad \text{for } t > 0$$

$$T = 0 \quad \text{for } t = 0\ ;\ \text{in } 0 < x < a\ ,\ 0 < y < b\ ,\ 0 < z < c$$

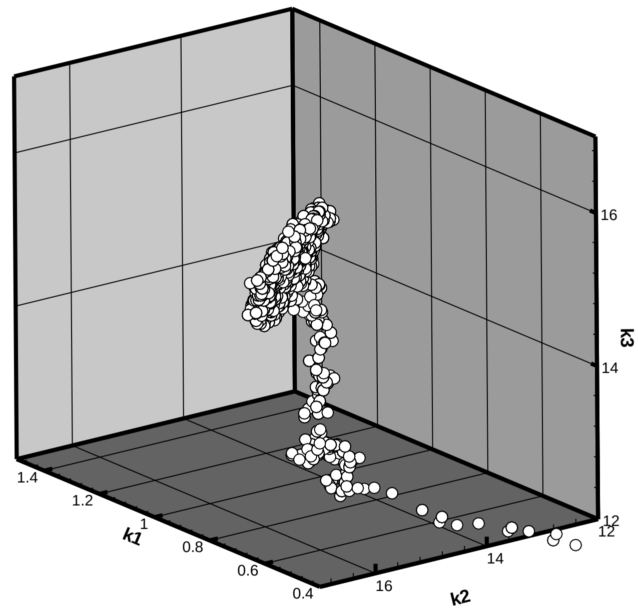
Orlande, H.R.B., Colaço, M., Dulikravich, G., Approximation of the likelihood function in the Bayesian technique for the solution of inverse problems, *Inverse Problems in Science and Engineering*, Vol. 16, pp. 677–692, 2008.

SOLUTION OF INVERSE PROBLEMS
Bayesian framework

Example: Estimation of Thermal Conductivity Components of Orthotropic Solids – Interpolation of the Likelihood with RBF's



Exact Likelihood (48 seconds)

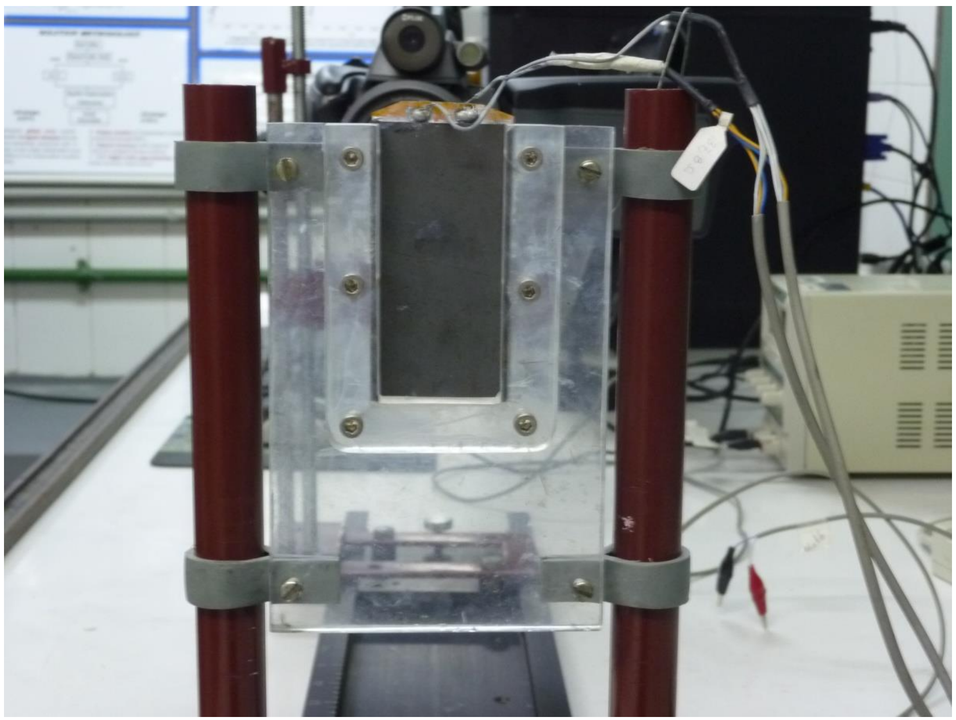


Interpolated Likelihood (1.8 seconds)

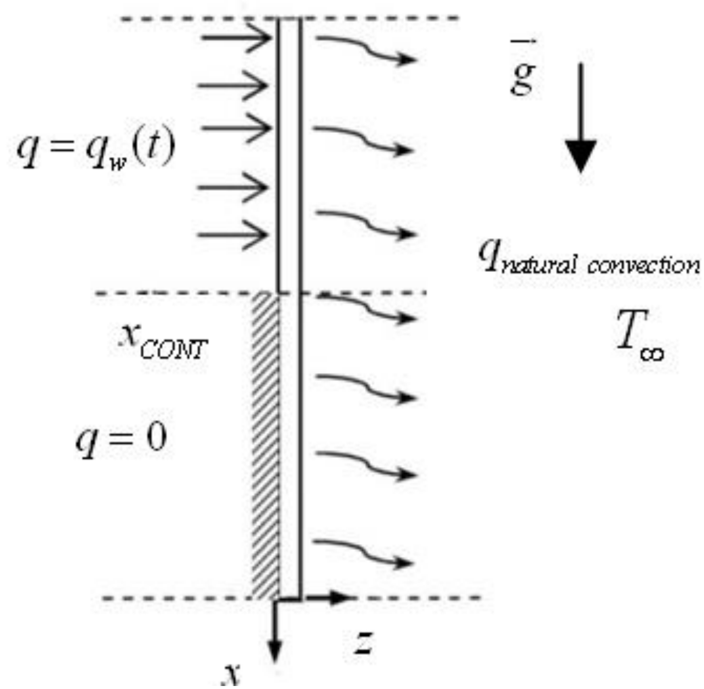
Orlande, H.R.B., Colaço, M., Dulikravich, G., Approximation of the likelihood function in the Bayesian technique for the solution of inverse problems, *Inverse Problems in Science and Engineering*, Vol. 16, pp. 677–692, 2008.

SOLUTION OF INVERSE PROBLEMS
Bayesian framework

Example: Characterization of Heterogeneous Media



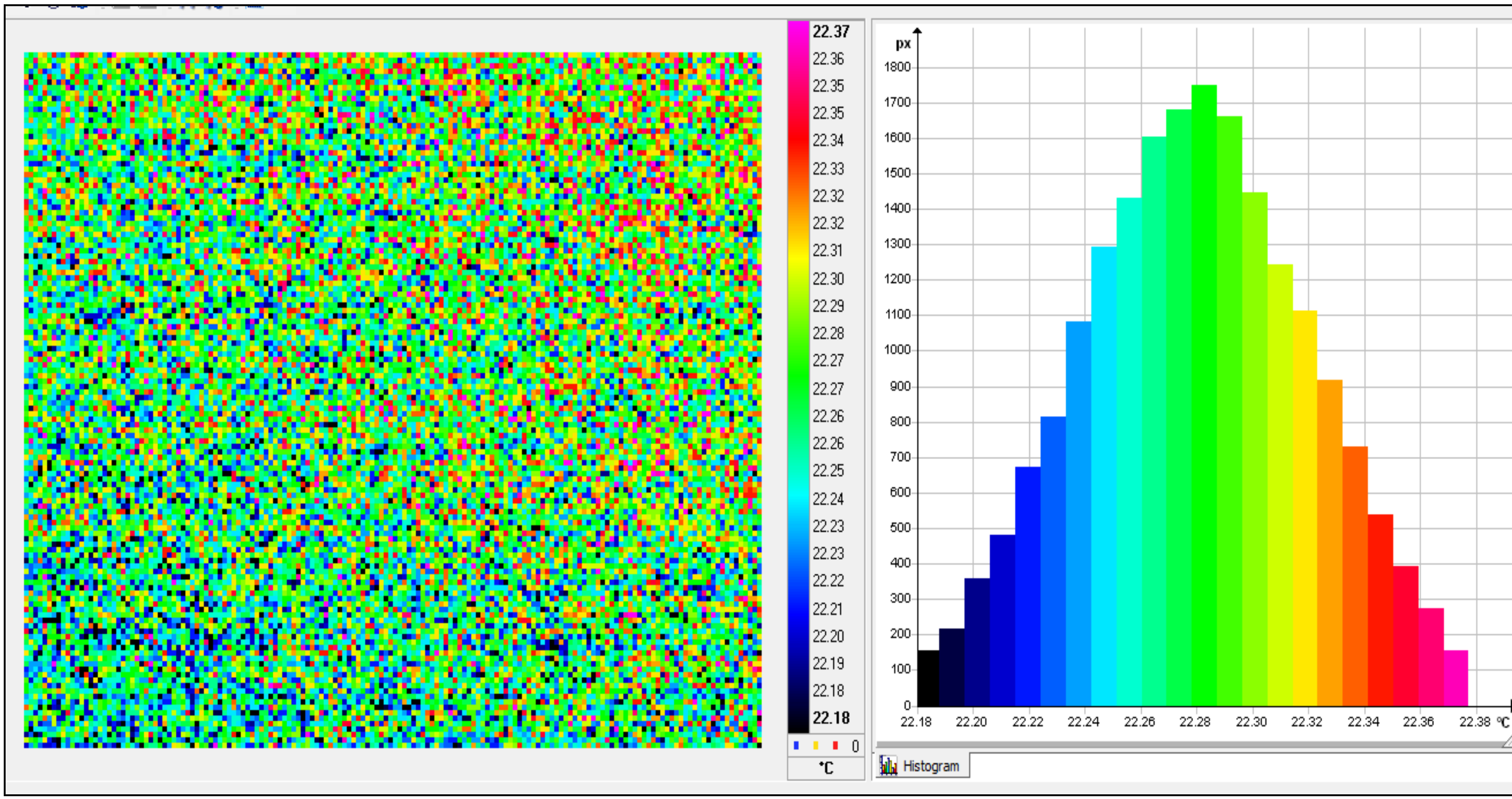
Thin plate: Lumped model in *z*



Orlande, H. R. B., Knupp, D., Naveira-cotta, C., Cotta, Renato, Experimental Identification of Thermophysical Properties in Heterogeneous Materials with Integral Transformation of Temperature Measurements from Infrared Thermography. *Experimental Heat Transfer*. , v.26, p.1 - 25, 2013.

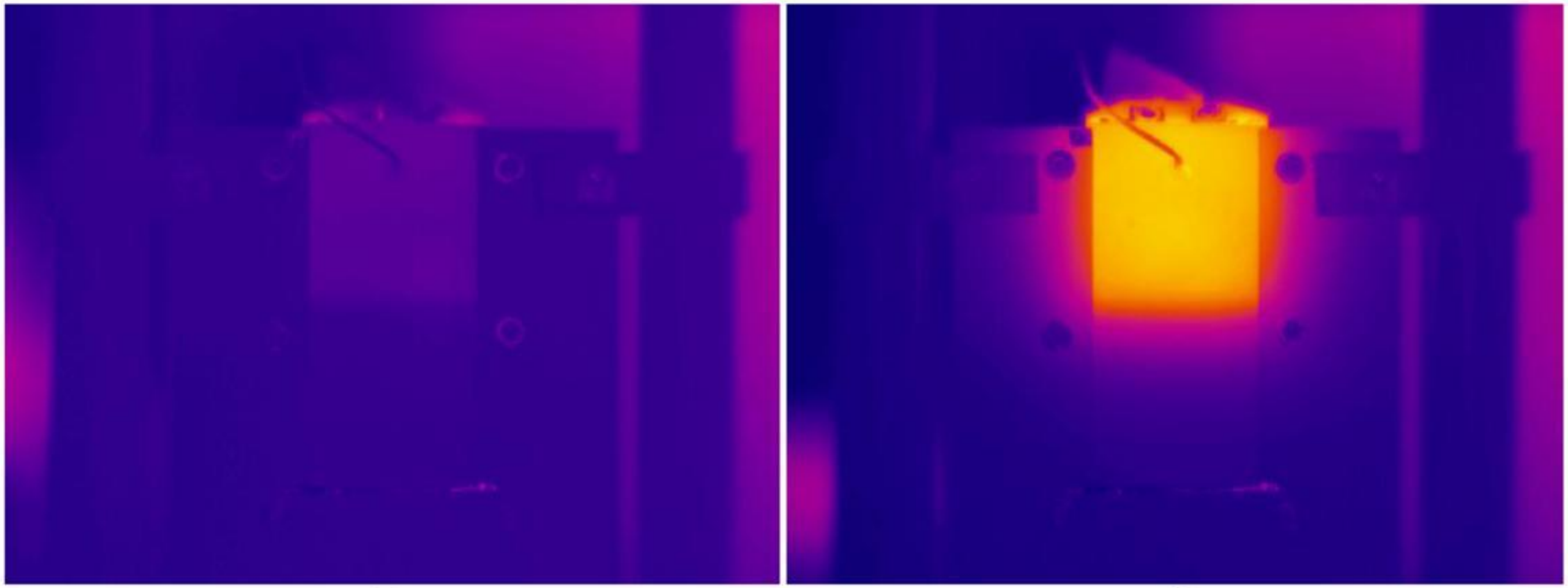
SOLUTION OF INVERSE PROBLEMS
Bayesian framework

YES! The likelihood is Gaussian!



SOLUTION OF INVERSE PROBLEMS
Bayesian framework

Example: Characterization of Heterogeneous Media

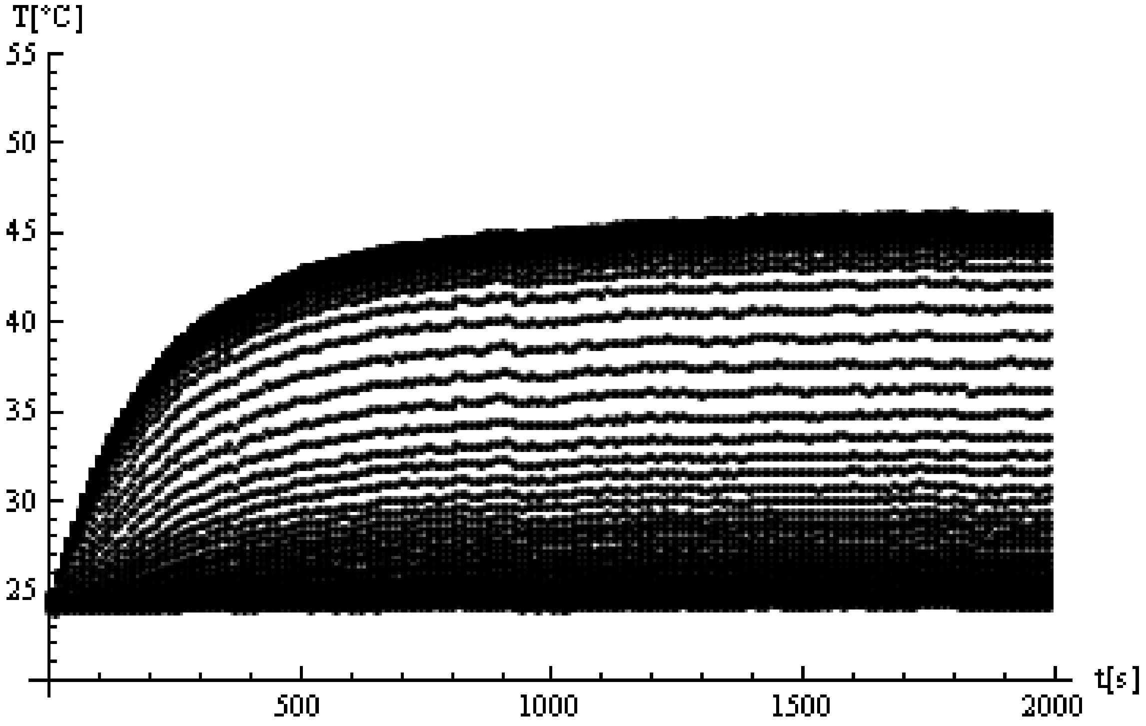


(*a*) (*b*)
Figure 7 – (a) Infrared camera image acquired at the moment the DC source is switched on. (b) Infrared camera image acquired after some elapsed time during heating period.

Orlande, H. R. B., Knupp, D., Naveira-cotta, C., Cotta, Renato, Experimental Identification of Thermophysical Properties in Heterogeneous Materials with Integral Transformation of Temperature Measurements from Infrared Thermography. *Experimental Heat Transfer*. , v.26, p.1 - 25, 2013.

SOLUTION OF INVERSE PROBLEMS
Bayesian framework

Example: Characterization of Heterogeneous Media



The number of pixels in the vertical direction for the configuration that has been tested provides the total number of **328 spatial measurements** along the 8 cm of the plate.

SOLUTION OF INVERSE PROBLEMS
Bayesian framework

Example: Characterization of Heterogeneous Media
Data Compression

Another advancement of the present study was the solution of the inverse problem in the transformed field, from the **integral transformation of the experimental temperature data, thus compressing the experimental measurements in the space variables into a few transformed fields**. Once the experimental temperature readings have been obtained, one proceeds to the integral transformation of the temperature field at each time through the integral transform pair below:

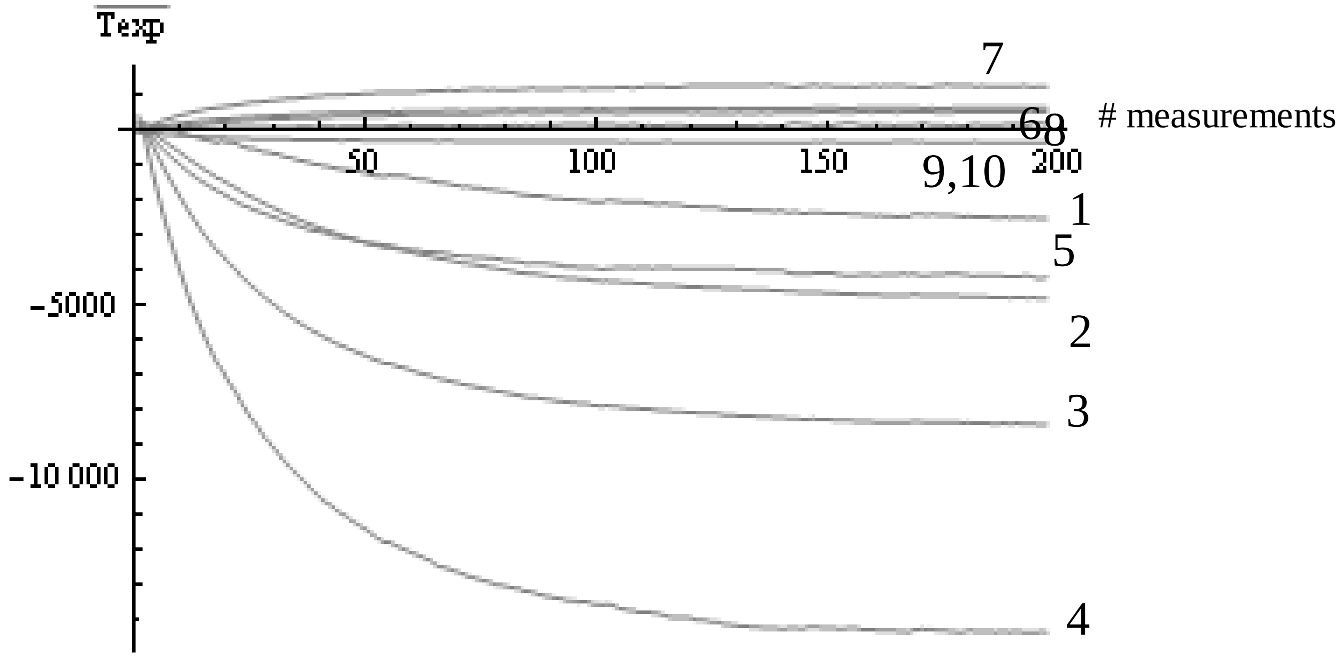
Transform $$\overline{T_{\exp,i}}(t) = \int_0^{Lx} w(x)\tilde{\psi}_i(x)[T_{\exp}(x,t) - T_\infty]dx$$

Inverse $$T_{\exp}(x,t) = T_\infty + \sum_{i=0}^{Ni} \tilde{\psi}_i(x)\overline{T_{\exp,i}}(t)$$

Orlande, H. R. B., Knupp, D., Naveira-cotta, C., Cotta, Renato, Experimental Identification of Thermophysical Properties in Heterogeneous Materials with Integral Transformation of Temperature Measurements from Infrared Thermography. *Experimental Heat Transfer*. , v.26, p.1 - 25, 2013.

SOLUTION OF INVERSE PROBLEMS
Bayesian framework

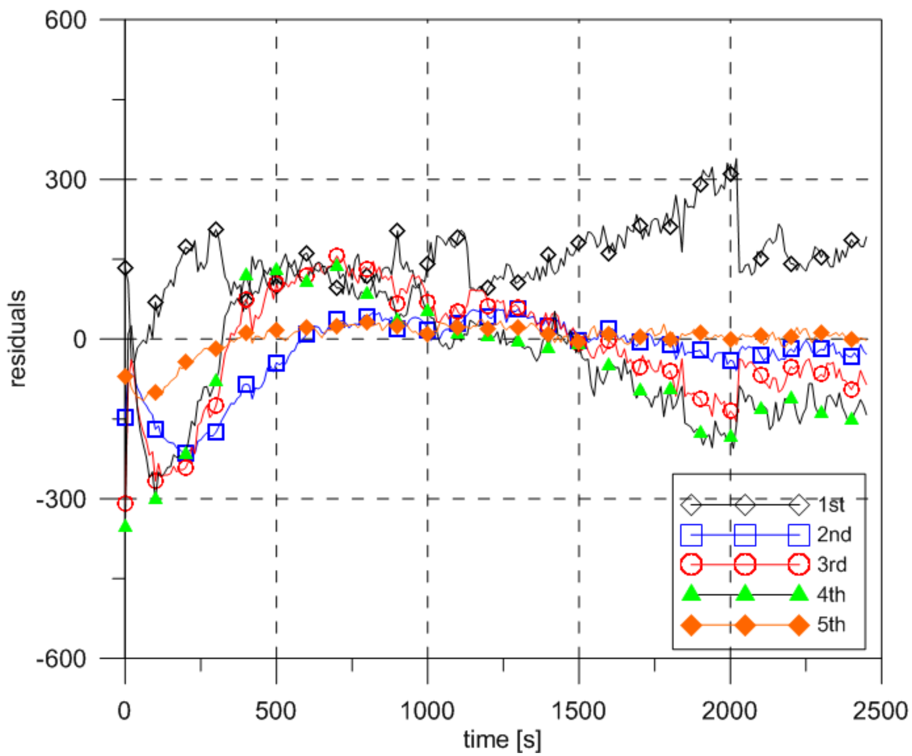
Example: Characterization of Heterogeneous Media
Data Compression



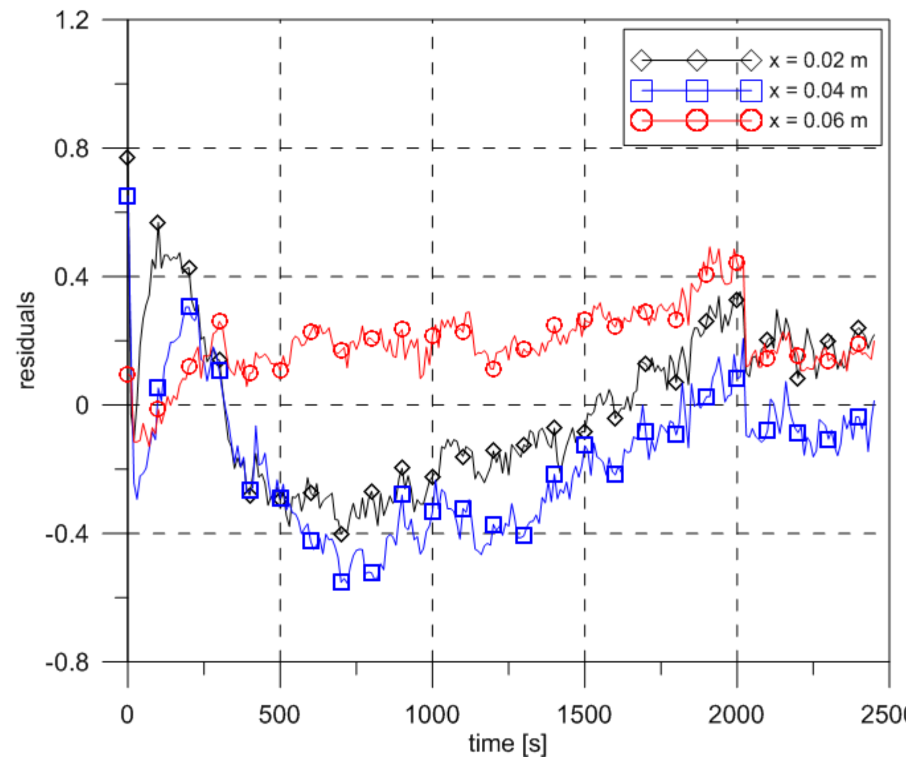
These are in fact the quantities that are employed in the inverse problem analysis. Therefore, a significant **data reduction of more than 95%** is achieved, as one chooses to solve the inverse problem in the transformed temperature domain.

SOLUTION OF INVERSE PROBLEMS
Bayesian framework

Transformed Potentials



Temperature (ºC)



Orlande, H. R. B., Knupp, D., Naveira-cotta, C., Cotta, Renato, Experimental Identification of Thermophysical Properties in Heterogeneous Materials with Integral Transformation of Temperature Measurements from Infrared Thermography. *Experimental Heat Transfer*. , v.26, p.1 - 25, 2013.

SOLUTION OF INVERSE PROBLEMS
Bayesian framework

Example: Characterization of Heterogeneous Media – Nodal Approach

Thin plate: Lumped model in *z*

$$C(x,y)\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left[k(x,y)\frac{\partial T}{\partial x}\right]+\frac{\partial}{\partial y}\left[k(x,y)\frac{\partial T}{\partial y}\right]-h(x,y)\left(T-T_{\infty}\right)+g(x,y)$$

By writing the equation above in non-conservative form:

$$\frac{\partial T}{\partial t}=a(x,y)\nabla^2 T+\frac{1}{C(x,y)}\left[\frac{\partial k(x,y)}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial k(x,y)}{\partial y}\frac{\partial T}{\partial y}\right]-H(x,y)\left(T-T_{\infty}\right)+G(x,y)$$

$$a(x,y)=\frac{k(x,y)}{C(x,y)} \qquad H(x,y)=\frac{h(x,y)}{C(x,y)} \qquad G(x,y)=\frac{g(x,y)}{C(x,y)}$$

Massard, H., Fudym, O., Orlande, H. R. B., Batsale, J. C., Nodal predictive error model and Bayesian approach for thermal diffusivity and heat source mapping, *Comptes Rendus Mécanique* , v.338, p.434 - 449, 2010

SOLUTION OF INVERSE PROBLEMS
Bayesian framework

Example: Characterization of Heterogeneous Media – Nodal Approach

$$\boxed{\mathbf{Y}_{ij} = \mathbf{J}_{ij}\mathbf{P}_{ij}}$$

$$\mathbf{J}_{ij} = \begin{bmatrix} L_{i,j}^{1} & Dx_{i,j}^{1} & Dy_{i,j}^{1} & -\Delta t(T_{i,j}^{1} - T_{\infty}) & \Delta t \\ L_{i,j}^{2} & Dx_{i,j}^{2} & Dy_{i,j}^{2} & -\Delta t(T_{i,j}^{2} - T_{\infty}) & \Delta t \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{i,j}^{n_t} & Dx_{i,j}^{n_t} & Dy_{i,j}^{n_t} & -\Delta t(T_{i,j}^{n_t} - T_{\infty}) & \Delta t \end{bmatrix}$$

$$\mathbf{Y}_{ij} = \begin{bmatrix} Y_{i,j}^{1} \\ Y_{i,j}^{2} \\ \vdots \\ Y_{i,j}^{n_t} \end{bmatrix}$$

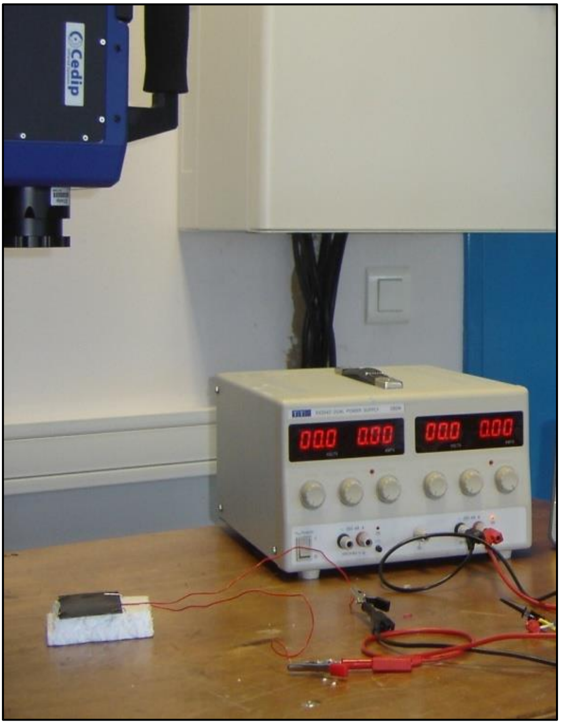
$$\mathbf{P}_{ij} = \begin{bmatrix} a_{i,j} \\ \delta_{i,j}^{x} \\ \delta_{i,j}^{y} \\ H_{i,j} \\ G_{i,j} \end{bmatrix}$$

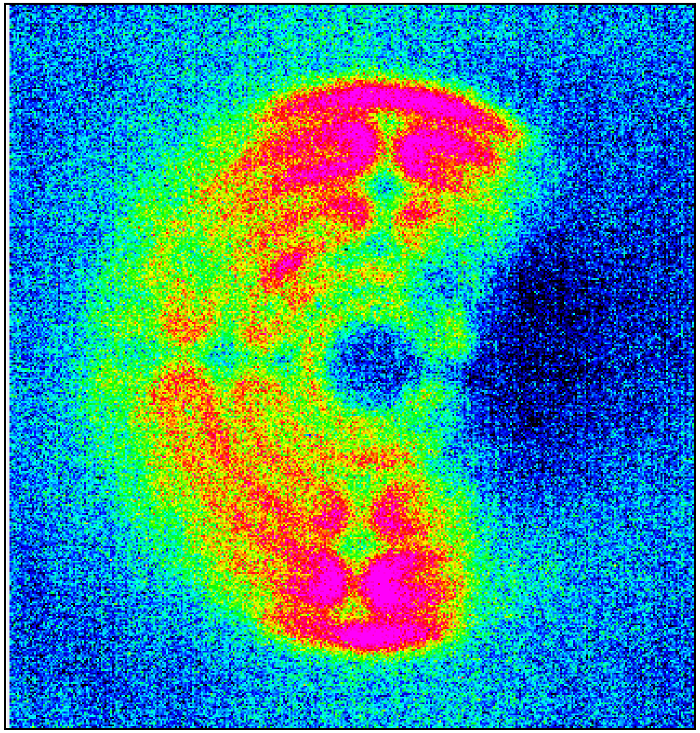
In the nodal strategy, the sensitivity matrix is approximately computed with the measurements:

$$\boxed{\pi\left(\mathbf{P},\mathbf{J}\middle|\mathbf{Y}\right) \propto \pi\left(\mathbf{Y}\middle|\mathbf{P},\mathbf{J}\right)\pi\left(\mathbf{P}\right)\pi\left(\mathbf{J}\right)}$$

Massard, H., Fudym, O., Orlande, H. R. B., Batsale, J. C., Nodal predictive error model and Bayesian approach for thermal diffusivity and heat source mapping, *Comptes Rendus Mécanique* , v.338, p.434 - 449, 2010

Example: Characterization of Heterogeneous Media – Nodal Approach

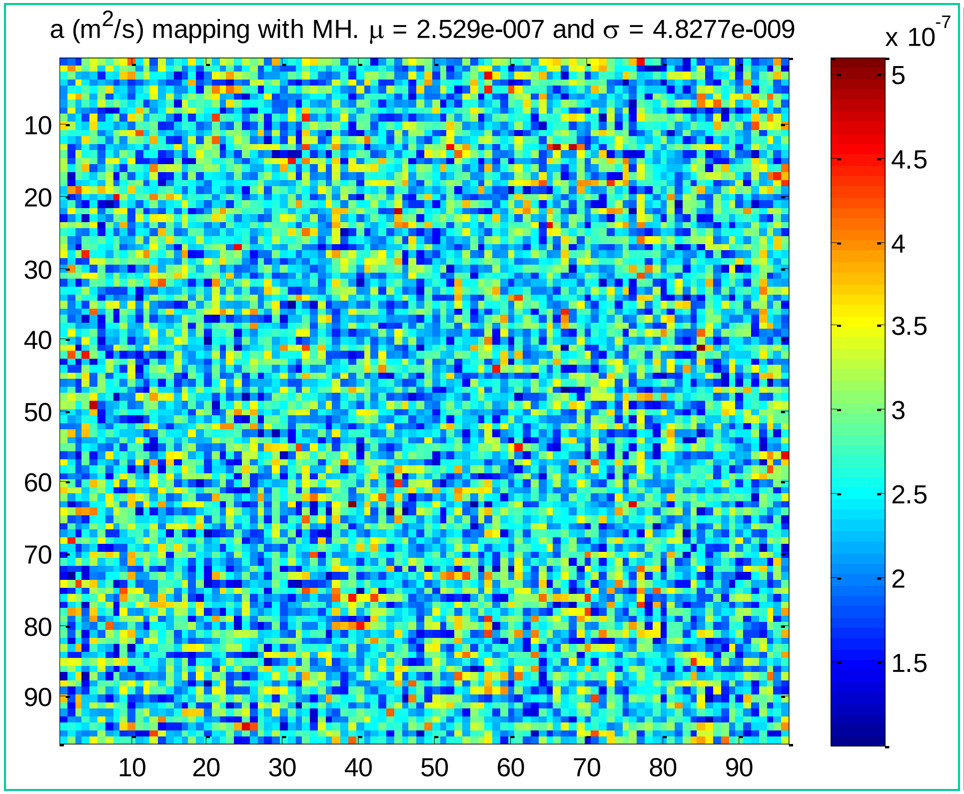


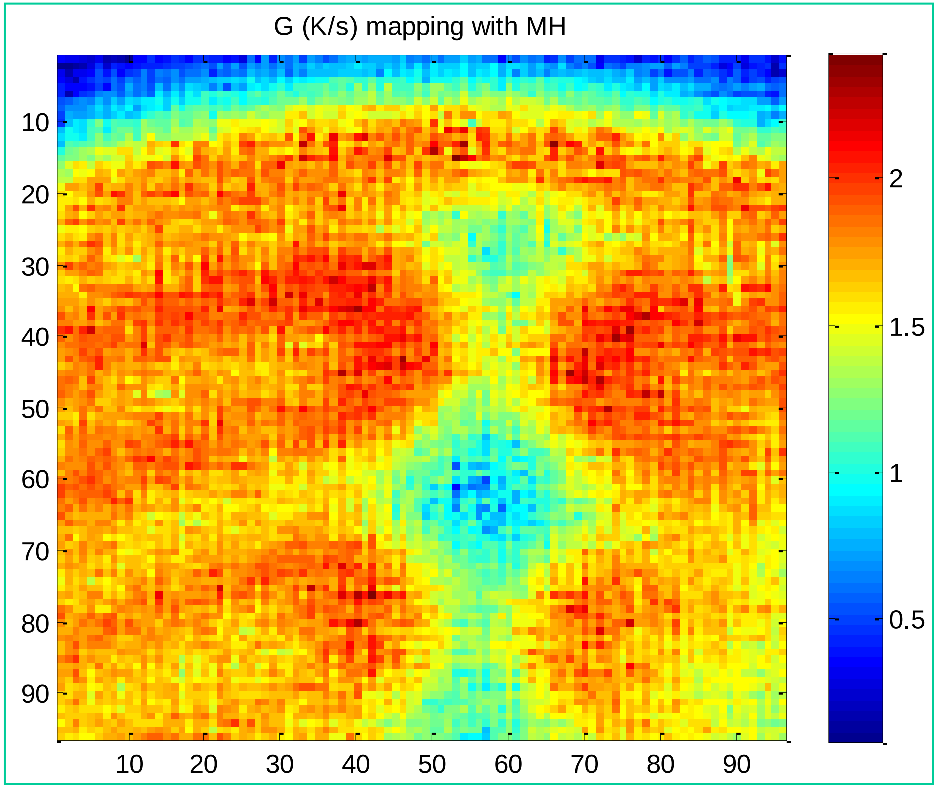


SOLUTION OF INVERSE PROBLEMS

Bayesian framework

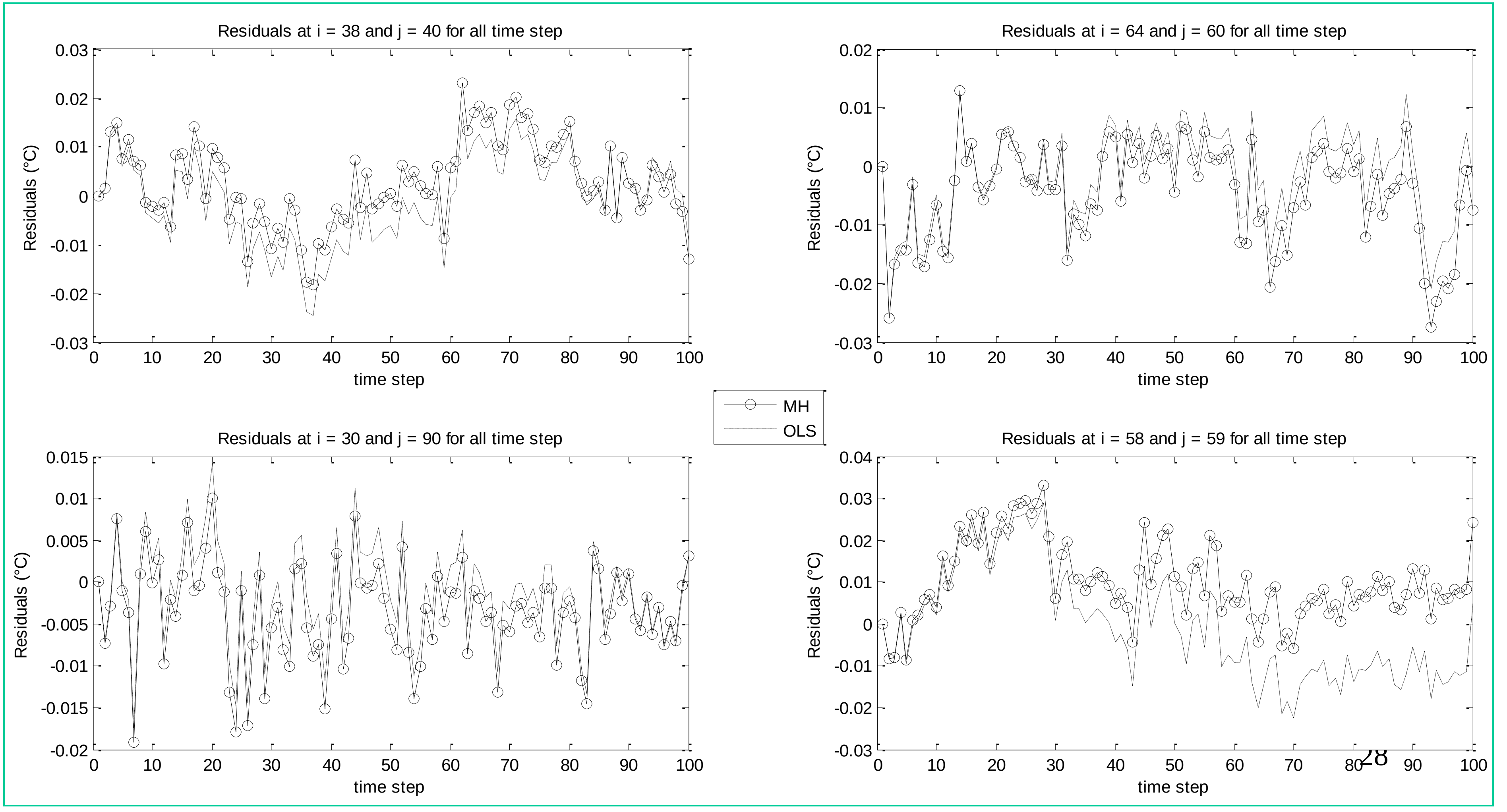
Example: Characterization of Heterogeneous Media – Nodal Approach





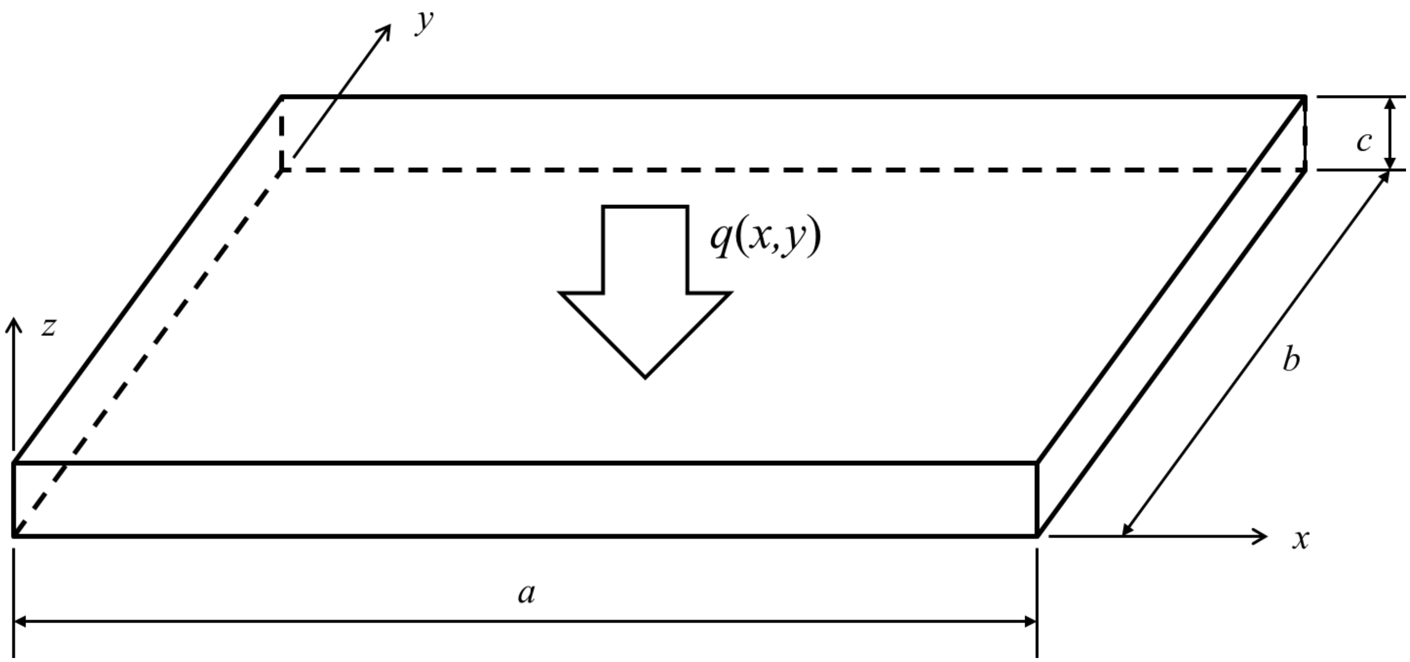
SOLUTION OF INVERSE PROBLEMS
Bayesian framework

Example: Characterization of Heterogeneous Media – Nodal Approach

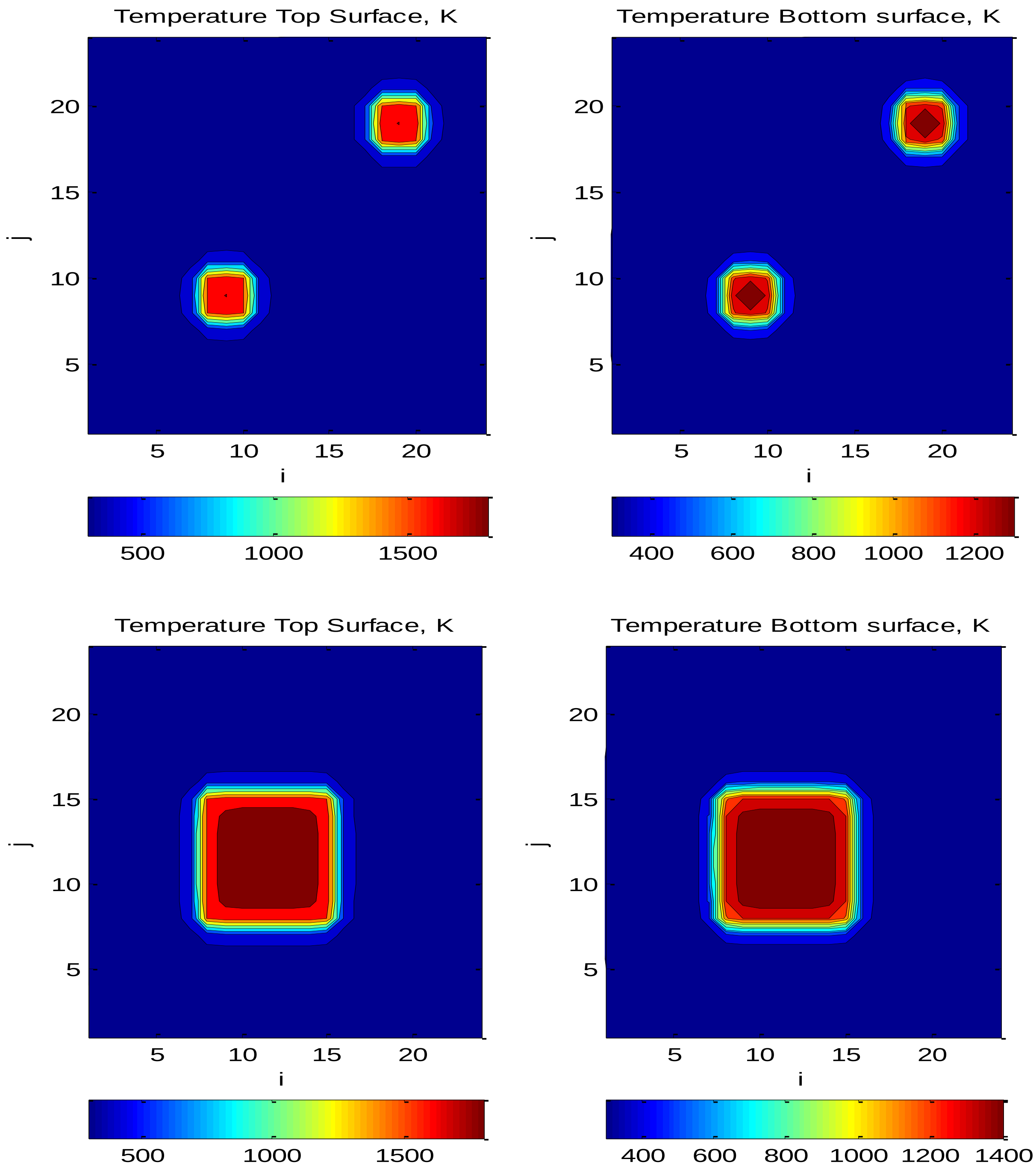


SOLUTION OF INVERSE PROBLEMS

Example: Non linear 3D heat conduction
Estimation of $q(x,y)$ with measurements of $T(x,y,0,t)$



Helcio R. B. Orlande, George S. Dulikravich, Markus Neumayer, Daniel Watzenig, Marcelo J. Colaço, Accelerated Bayesian Inference For The Estimation Of Spatially Varying Heat Flux In A Heat Conduction Problem, *Numerical Heat Transfer – Part A*, In press


Temperature Top Surface, K
Temperature Bottom surface, K
j
i
5
10
15
20
500
1000
1500
400
600
800
1000
1200
Temperature Top Surface, K
Temperature Bottom surface, K
j
i
500
1000
1500
400
600
800
1000
1200
1400

SOLUTION OF INVERSE PROBLEMS

Complete model

$$C(T_c)\frac{\partial T_c(x,y,z,t)}{\partial t}=\frac{\partial}{\partial x}\left[k(T_c)\frac{\partial T_c}{\partial x}\right]+\frac{\partial}{\partial y}\left[k(T_c)\frac{\partial T_c}{\partial y}\right]+\frac{\partial}{\partial z}\left[k(T_c)\frac{\partial T_c}{\partial z}\right]$$

in $0 < x < a$, $0 < y < b$, $0 < z < c$, for $t > 0$

$$\frac{\partial T_c}{\partial x}=0$$ at $x = 0$ and $x = a$, $0 < y < b$, $0 < z < c$, for $t > 0$

$$\frac{\partial T_c}{\partial y}=0$$ at $y = 0$ and $y = b$, $0 < x < a$, $0 < z < c$, for $t > 0$

$$\frac{\partial T_c}{\partial z}=0$$ at $z = 0$, $0 < x < a$, $0 < y < b$, for $t > 0$

$$k(T_c)\frac{\partial T_c}{\partial z}=q(x,y)$$ at $z = c$, $0 < x < a$, $0 < y < b$, for $t > 0$

$$T_c = T_0$$ for $t = 0$, in $0 < x < a$, $0 < y < b$, $0 < z < c$

SOLUTION OF INVERSE PROBLEMS

Reduced models: Linear problem with properties at T^*

$$C^* \frac{\partial \bar{T}(x,y,t)}{\partial t} = \frac{\partial}{\partial x}\left[k^* \frac{\partial \bar{T}}{\partial x}\right] + \frac{\partial}{\partial y}\left[k^* \frac{\partial \bar{T}}{\partial y}\right] + \frac{q(x,y)}{c}$$

in $0 < x < a$, $0 < y < b$, for $t > 0$

$$\frac{\partial \bar{T}}{\partial x} = 0$$ at $x = 0$ and $x = a$, $0 < y < b$, for $t > 0$

$$\frac{\partial \bar{T}}{\partial y} = 0$$ at $y = 0$ and $y = b$, $0 < x < a$, for $t > 0$

$$\bar{T} = T_0$$ for $t = 0$, in $0 < x < a$, $0 < y < b$

where

$$\bar{T}(x,y,t) = \frac{1}{c}\int_{z=0}^{c} T(x,y,z,t)\,dz$$

SOLUTION OF INVERSE PROBLEMS

Reduced models: Linear problem with properties at T^*

Classical Lumped Formulation:

Temperature gradients across the thickness of the plate are fully neglected.

$$\boxed{T(x,y,0,t) = T(x,y,c,t) = \bar{T}(x,y,t)}$$

Helcio R. B. Orlande, George S. Dulikravich, Markus Neumayer, Daniel Watzenig, Marcelo J. Colaço, Accelerated Bayesian Inference For The Estimation Of Spatially Varying Heat Flux In A Heat Conduction Problem, *Numerical Heat Transfer – Part A*, In press

SOLUTION OF INVERSE PROBLEMS

Improved Lumped Formulation:

Temperature gradients across the thickness of the plate are not neglected, but taken into account in an approximate form (Cotta, R.M., Mikhailov, M.D., *Heat Conduction: Lumped Analysis, Integral Transforms*, Symbolic Computation, Wiley-Interscience, New York, USA, 1997.).

$H_{1,1}$ formula (correct trapezoidal rule):
$$\bar{T}(x,y,t) \approx \frac{1}{2}\left[T(x,y,0,t)+T(x,y,c,t)\right]+\frac{c}{12}\left[\left.\frac{\partial T}{\partial z}\right|_{z=0} - \left.\frac{\partial T}{\partial z}\right|_{z=c}\right]$$

$H_{0,0}$ formula (trapezoidal rule):
$$\int_{z=0}^{c} \frac{\partial T(x,y,z,t)}{\partial z} dz = T(x,y,c,t) - T(x,y,0,t) \approx \frac{c}{2}\left[\left.\frac{\partial T}{\partial z}\right|_{z=0} + \left.\frac{\partial T}{\partial z}\right|_{z=c}\right]$$

$$\boxed{T(x,y,0,t) = \bar{T}(x,y,t) - \frac{c}{6k^*} q(x,y)}$$

$$\boxed{T(x,y,c,t) = \bar{T}(x,y,t) + \frac{c}{3k^*} q(x,y)}$$

In general, the direct problem solution with the **complete model took around 7.2 s**, while the solution with the **reduced model took around 0.09 s** of CPU time.

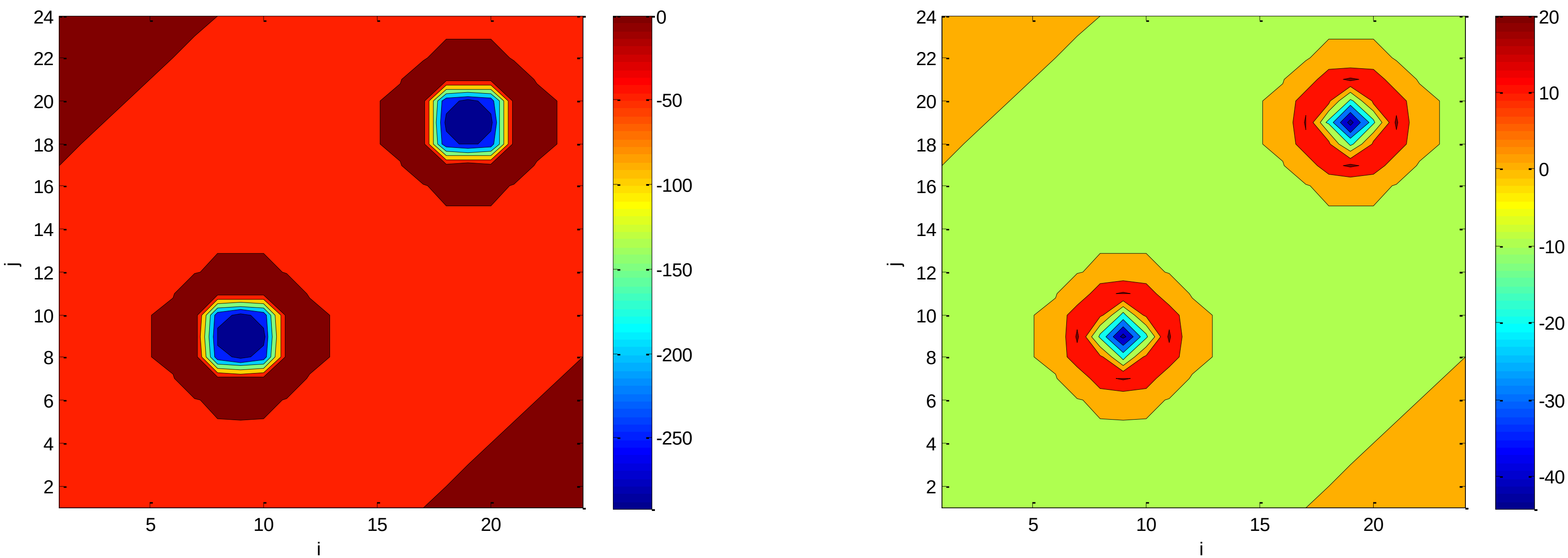
SOLUTION OF INVERSE PROBLEMS
Effects of reduced models

Error of the Direct Problem Solution at the final time:

$$q(x_i, y_j) = \begin{cases} 10^7\,Wm^{-2} \;, & \text{for } 8 \le i \le 10 \text{ and } 8 \le j \le 10 \\ 10^7\,Wm^{-2} \;, & \text{for } 18 \le i \le 20 \text{ and } 18 \le j \le 20 \\ 0 \;, & \text{elsewhere} \end{cases}$$

Classical Lumped Model

Improved Lumped Model



Orlande, H.R.B., Dulikravich, G., Inverse Heat Transfer Problems and their Solutions within the Bayesian Framework, ECCOMAS Special Interest Conference, Numerical Heat Transfer 2012, 4-6 September 2012, Gliwice-Wrocław, Poland

SOLUTION OF INVERSE PROBLEMS

Improvement of solutions with reduced models

DELAYED ACCEPTANCE METROPOLIS-HASTINGS ALGORITHM

(Christen, J. and Fox, C., Markov chain Monte Carlo Using an Approximation, *Journal of Computational and Graphical Statistics*, vol. 14, no. 4, pp. 795–810, 2005)

1. Sample a *Candidate Point* $\mathbf{P}^*$ from a proposal distribution $p(\mathbf{P}^*,\mathbf{P}^{(t-1)})$.
2. Calculate the acceptance factor with the surrogate model:

$$\alpha = \min\left[1, \frac{\pi(\mathbf{P}^* \mid \mathbf{Y})\, p(\mathbf{P}^{(t-1)}, \mathbf{P}^*)}{\pi(\mathbf{P}^{(t-1)} \mid \mathbf{Y})\, p(\mathbf{P}^*, \mathbf{P}^{(t-1)})}\right]$$

3. Generate a random value U that is uniformly distributed on (0,1).
4. If $U \leq \alpha$, proceed to step 5. Otherwise, return to step 1.
5. Calculate a new acceptance factor with the complete model:

$$\alpha_c = \min\left[1, \frac{\pi_c(\mathbf{P}^* \mid \mathbf{Y})\, p(\mathbf{P}^{(t-1)}, \mathbf{P}^*)}{\pi_c(\mathbf{P}^{(t-1)} \mid \mathbf{Y})\, p(\mathbf{P}^*, \mathbf{P}^{(t-1)})}\right]$$

6. Generate a new random value U_c which is uniformly distributed on (0,1).
7. If $U_c \leq \alpha_c$, set $\mathbf{P}^{(t)} = \mathbf{P}^*$. Otherwise, set $\mathbf{P}^{(t)} = \mathbf{P}^{(t-1)}$.
8. Return to step 1.

where $\pi(\mathbf{P} \mid \mathbf{Y})$ and $\pi_c(\mathbf{P} \mid \mathbf{Y})$ are the posterior distributions with the likelihoods computed with the surrogate model and with the complete model, respectively.

SOLUTION OF INVERSE PROBLEMS

PRIOR DISTRIBUTIONS

Total variation non-informative prior

$$\boxed{\pi(\mathbf{P}) \propto \exp\left[-\alpha\, TV(\mathbf{P})\right]}$$

$$TV(\mathbf{P}) = \sum_{i=2}^{I-1}\sum_{j=2}^{J-1} \Delta y\left[\left|q(x_i, y_j) - q(x_{i+1}, y_j)\right| + \left|q(x_i, y_j) - q(x_{i-1}, y_j)\right|\right] + $$
$$+\Delta x\left[\left|q(x_i, y_j) - q(x_i, y_{j+1})\right| + \left|q(x_i, y_j) - q(x_i, y_{j-1})\right|\right]$$

Helcio R. B. Orlande, George S. Dulikravich, Markus Neumayer, Daniel Watzenig, Marcelo J. Colaço, Accelerated Bayesian Inference For The Estimation Of Spatially Varying Heat Flux In A Heat Conduction Problem, *Numerical Heat Transfer – Part A*, In press

APPROXIMATION ERROR MODEL

- Kaipio, J. and Somersalo, E., *Statistical and Computational Inverse Problems*, Applied Mathematical Sciences 160, Springer-Verlag, 2004
- Kaipio, J., and Somersalo, E., Statistical Inverse Problems: Discretization, Model Reduction and Inverse Crimes, *Journal of Computational and Applied Mathematics*, vol. 198, pp. 493–504, 2007.

In the approximation error model (AEM) approach, the statistical model of the approximation error is constructed and then represented as additional noise in the measurement model [1,19-23]. With the hypotheses that the measurement errors are additive and independent of the parameters **P**, one can write

$$\mathbf{Y} = \mathbf{T}_c(\mathbf{P}) + \mathbf{e} \tag{16}$$

where $\mathbf{T}_c(\mathbf{P})$ is the sufficiently accurate solution of the complete model given by equations (1.a-h). The vector of measurement errors, $\mathbf{e}$ are assumed here to be Gaussian, with zero mean and known covariance matrix **W**.

APPROXIMATION ERROR MODEL

$$\mathbf{Y} = \mathbf{T}(\mathbf{P}) + [\mathbf{T}_c(\mathbf{P}) - \mathbf{T}(\mathbf{P})] + \mathbf{e}$$

By defining the error between the complete and the surrogate model solutions as

$$\boldsymbol{\varepsilon} = [\mathbf{T}_c(\mathbf{P}) - \mathbf{T}(\mathbf{P})]$$

equation (17) can be written as

$$\mathbf{Y} = \mathbf{T}(\mathbf{P}) + \boldsymbol{\eta}$$

where

$$\boldsymbol{\eta} = \boldsymbol{\varepsilon} + \mathbf{e}$$

Helcio R. B. Orlande, George S. Dulikravich, Markus Neumayer, Daniel Watzenig, Marcelo J. Colaço, Accelerated Bayesian Inference For The Estimation Of Spatially Varying Heat Flux In A Heat Conduction Problem, *Numerical Heat Transfer – Part A*, In press

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APPROXIMATION ERROR MODEL

$\boldsymbol{\eta}$ is modeled as a Gaussian variable

$$\tilde{\pi}(\gamma,\mathbf{P}|\mathbf{Y}) \propto \gamma^{(IJ+2)/2}\exp\left\{-\frac{1}{2}[\mathbf{Y}-\mathbf{T}(\mathbf{P})-\overline{\boldsymbol{\eta}}]^T\tilde{\mathbf{W}}^{-1}[\mathbf{Y}-\mathbf{T}(\mathbf{P})-\overline{\boldsymbol{\eta}}]-\frac{1}{2}\gamma(\mathbf{P}-\boldsymbol{\mu})^T\boldsymbol{\Gamma}^{-1}(\mathbf{P}-\boldsymbol{\mu})-\frac{1}{2}\left(\frac{\gamma}{\gamma_0}\right)^2\right\}$$

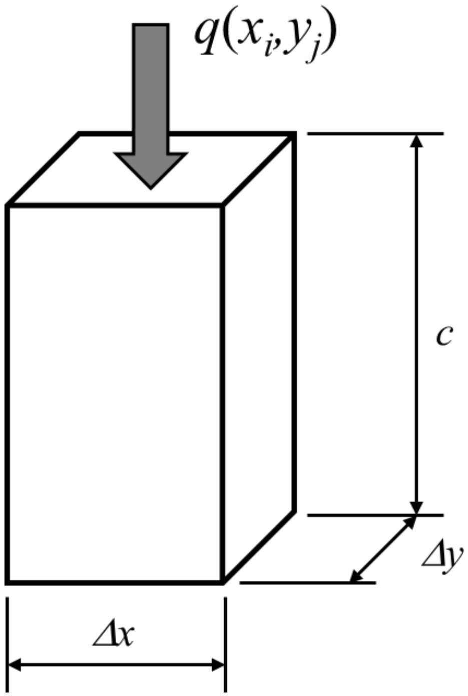
Enhanced error model:

$$\begin{cases}\overline{\boldsymbol{\eta}} \approx \overline{\boldsymbol{\varepsilon}} \\ \tilde{\mathbf{W}} \approx \mathbf{W}_{\boldsymbol{\varepsilon}} + \mathbf{W}\end{cases}$$

Helcio R. B. Orlande, George S. Dulikravich, Markus Neumayer, Daniel Watzenig, Marcelo J. Colaço, Accelerated Bayesian Inference For The Estimation Of Spatially Varying Heat Flux In A Heat Conduction Problem, *Numerical Heat Transfer – Part A*, In press

SOLUTION OF INVERSE PROBLEMS

Gaussian prior



Energy Balance: $$q(x_i, y_j) = C^* c \frac{d\,T(x_i, y_j)}{d\,t}$$

In order to generate this physically motivated Gaussian prior, and at the same time not violate the Bayesian principle that the prior is the information for the unknowns (coded in the form of probability distribution functions) that is available before the measurements are taken, we assume here that another kind of measurements is also available. Such other kind of measurements is only used to generate the prior, and is considered independent of the temperature measurements used in the inverse analysis, that is, for the computation of the likelihood.

Helcio R. B. Orlande, George S. Dulikravich, Markus Neumayer, Daniel Watzenig, Marcelo J. Colaço, Accelerated Bayesian Inference For The Estimation Of Spatially Varying Heat Flux In A Heat Conduction Problem, *Numerical Heat Transfer – Part A*, In press

Test case	Flux	Prior	Approach
1	A	TV	-
2	B	TV	-
3	C	TV	-
4	A	TV	DAMH
5	B	TV	DAMH
6	C	TV	DAMH
7	A	Gaussian	-
8	B	Gaussian	-
9	C	Gaussian	-
10	A	Gaussian	AEM
11	B	Gaussian	AEM
12	C	Gaussian	AEM

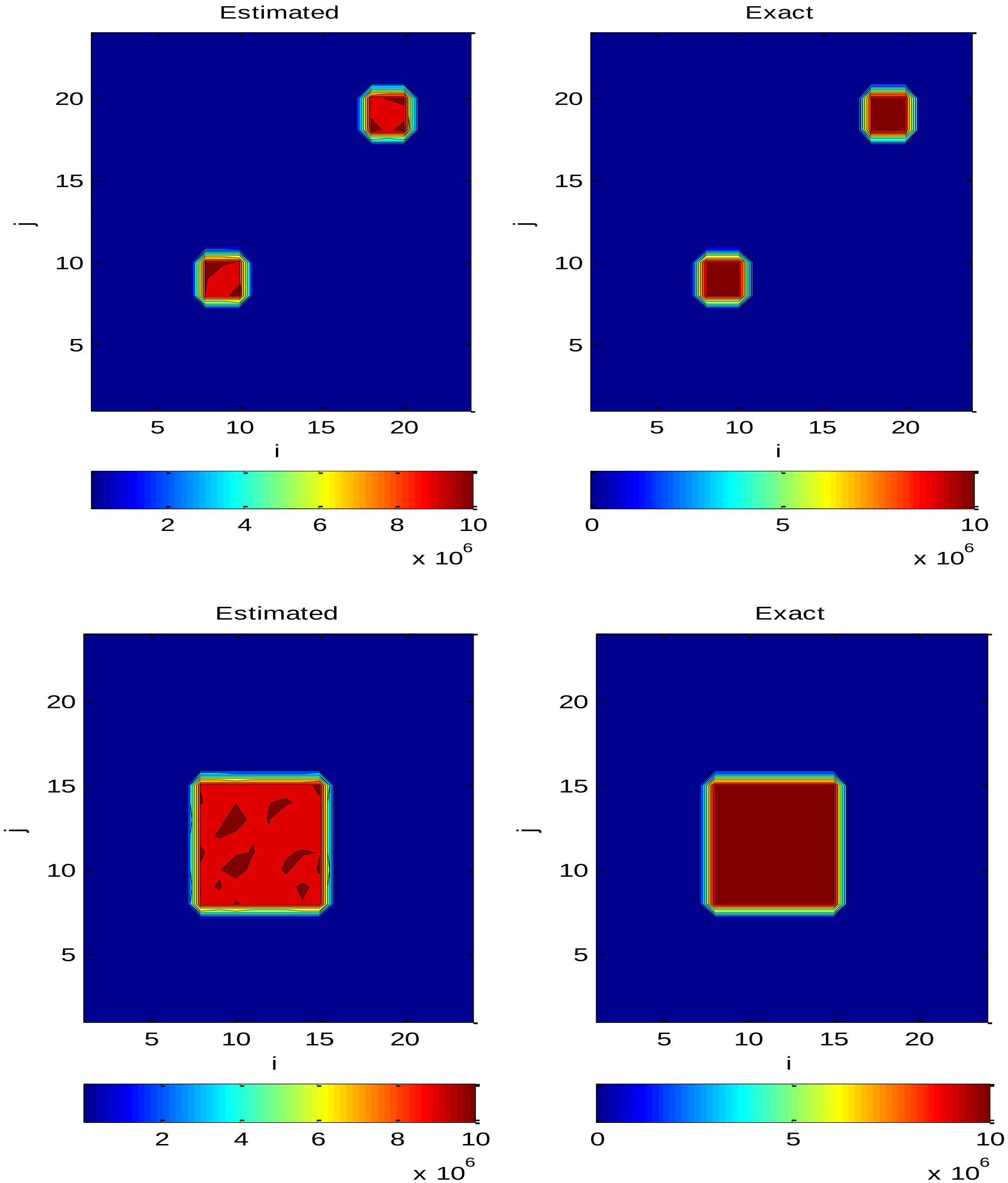
$\sigma = 0.02$ K

Test case	CPU Time (h)	Acceptance ratio (%)	RMS Error (W/m^2)
1	2.7	10.9	9.3x10^4
2	2.8	9.0	6.6x10^4
3	2.6	9.9	1.1x10^5
4	114.2	46.7 – 5.3	9.8x10^4
5	113.0	47.9 – 4.2	5.9x10^4
6	98.3	40.8 – 5.9	1.4x10^5
7	2.6	11.3	9.3x10^4
8	2.8	9.3	6.6x10^4
9	2.7	10.2	1.1x10^5
10	44.5	12.8	4.1x10^4
11	44.2	11.0	2.6x10^4
12	42.5	11.2	8.5x10^4

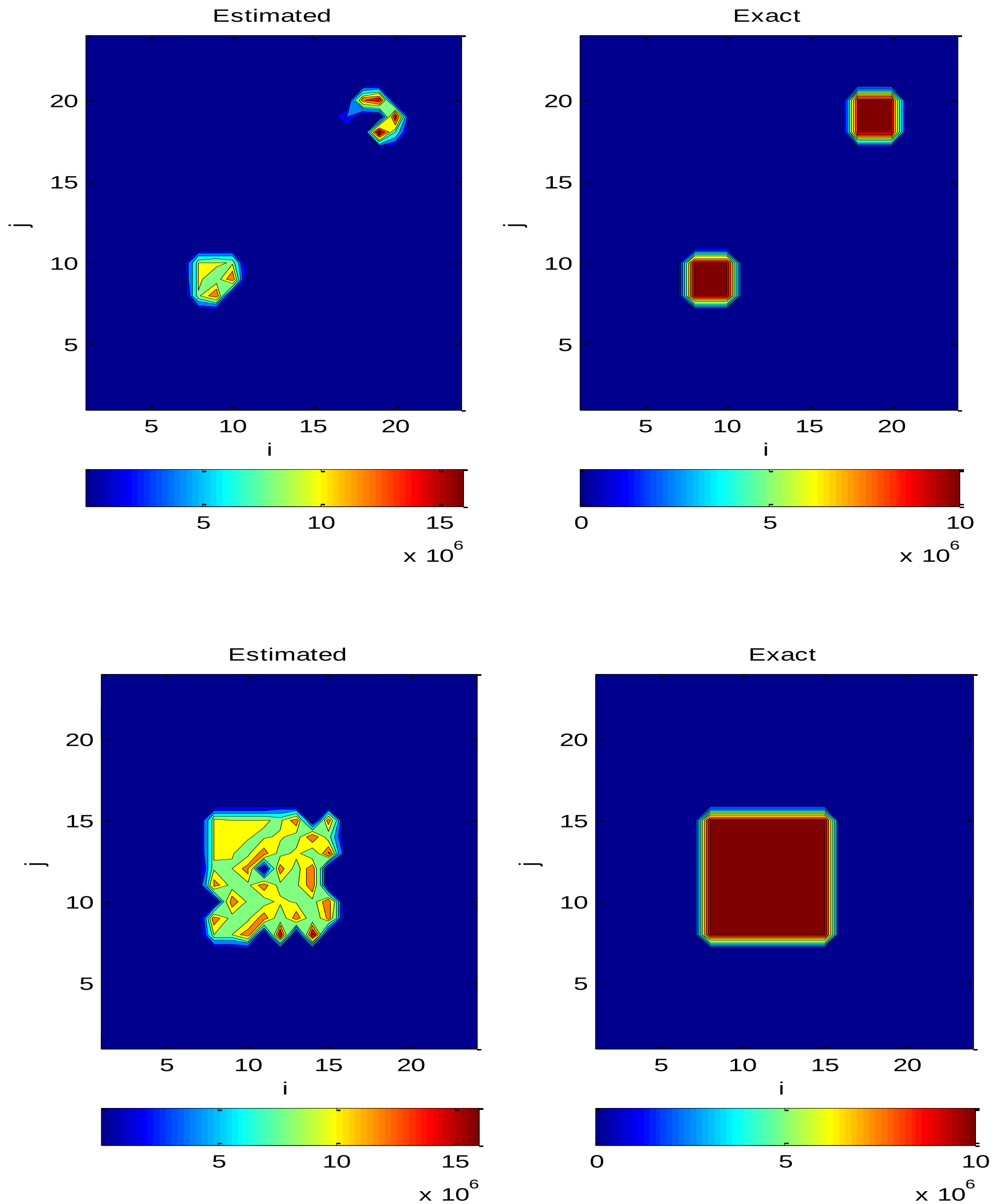
$\sigma = 1.25$ K

Test case	CPU Time (h)	Acceptance ratio (%)	RMS Error (W/m^2)
1	2.6	9.1	1.1x10^6
2	2.6	7.5	1.0x10^6
3	2.6	9.1	1.8x10^6
4	98.7	41.9 – 6.8	1.1x10^6
5	93.5	44.6 – 5.7	6.9x10^5
6	64.5	34.6 – 5.4	1.4x10^6
7	2.7	9.1	1.1x10^6
8	2.6	8.3	9.8x10^5
9	2.6	9.4	1.3x10^6
10	42.9	9.8	1.2x10^6
11	43.3	11.9	1.2x10^6
12	43.1	8.7	2.0x10^6

Gaussian prior + AEM - $\sigma = 0.02$ K

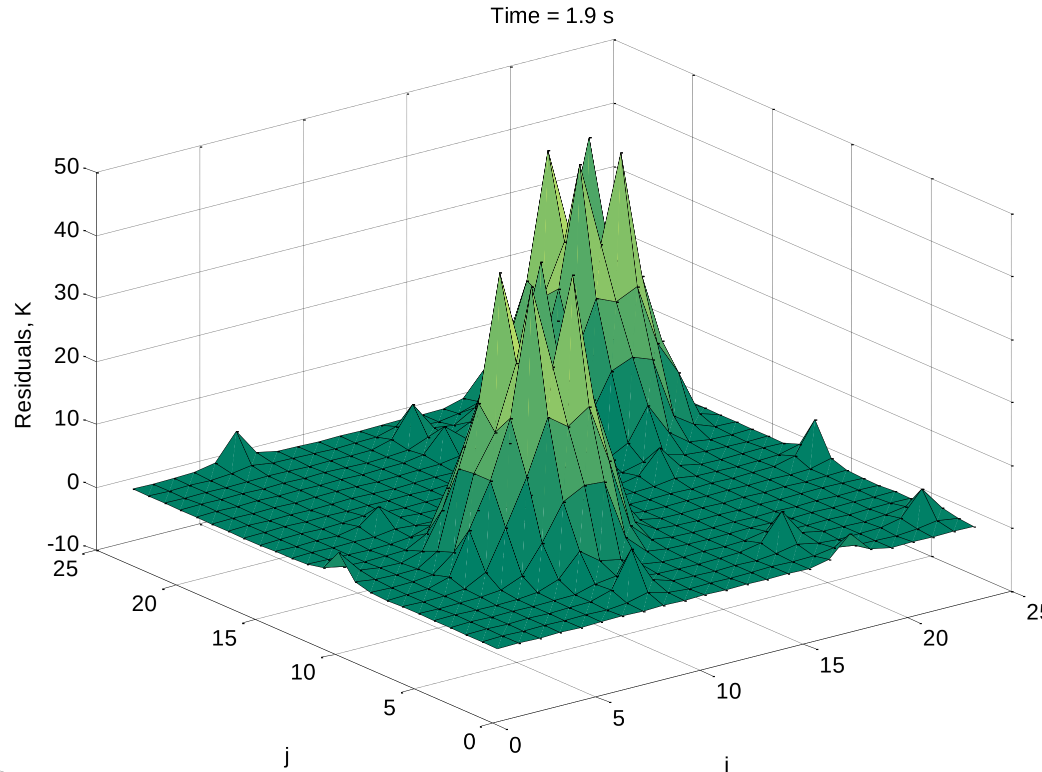


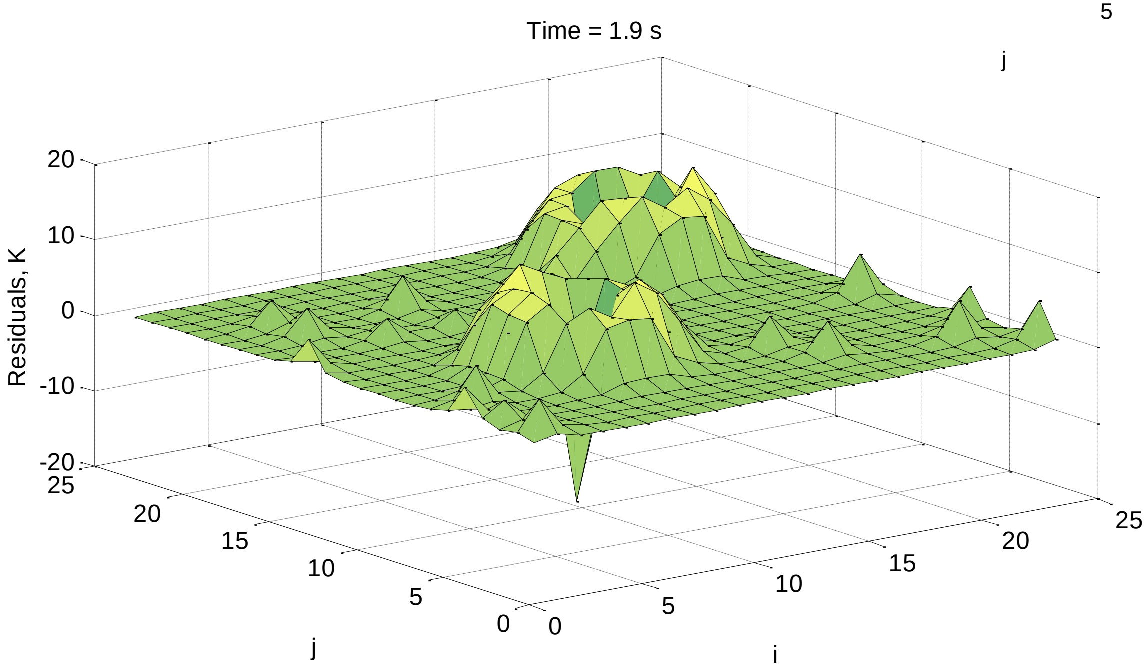
TV prior + DAMH - $\sigma = 1.25$ K



$\sigma = 1.25$ K

Gaussian prior

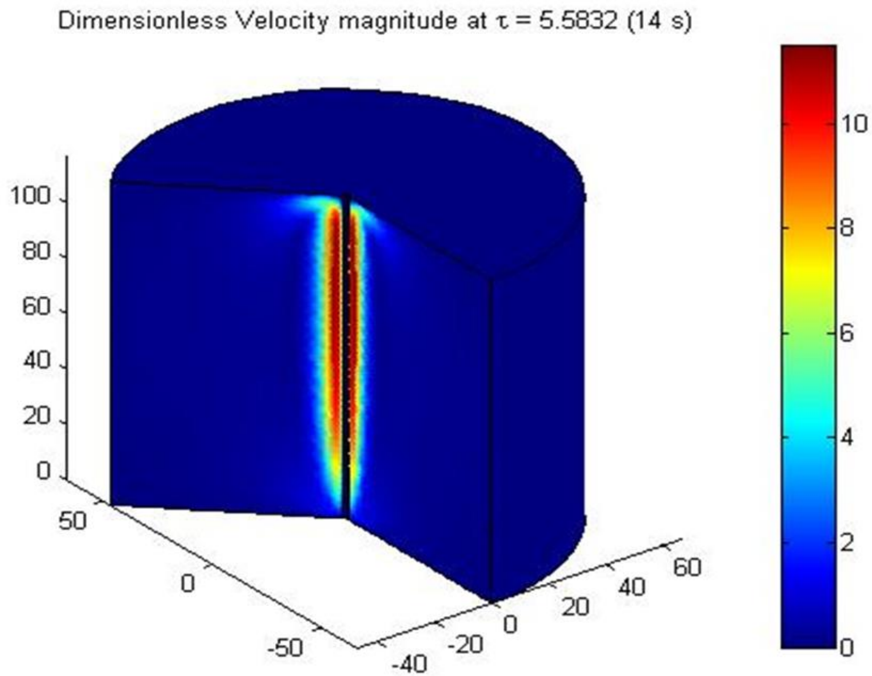




Gaussian prior + AEM

SOLUTION OF INVERSE PROBLEMS
Improvement of solutions with reduced models

CONVECTIVE EFFECTS IN LIQUIDS CHARACTERIZED BY THE LINE HEAT SOURCE PROBE



Bernard Lamien, Helcio R. B. Orlande, Approximation Error Model To Account For Convective Effects In Liquids Characterized By The Line Heat Source Probe, *4th Inverse Problems, Design and Optimization Symposium (IPDO-2013)*, Albi, France, June 26-28, 2013

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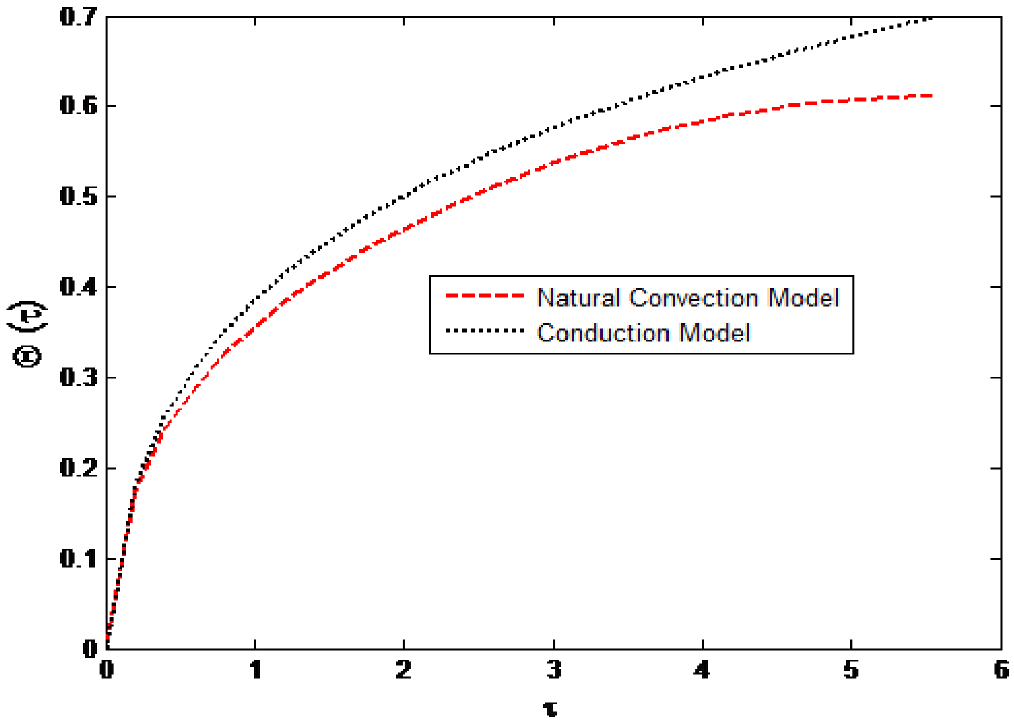
CONVECTIVE EFFECTS IN LIQUIDS CHARACTERIZED BY THE LINE HEAT SOURCE PROBE



Bernard Lamien, Helcio R. B. Orlande, Approximation Error Model To Account For Convective Effects In Liquids Characterized By The Line Heat Source Probe, *4th Inverse Problems, Design and Optimization Symposium (IPDO-2013)*, Albi, France, June 26-28, 2013

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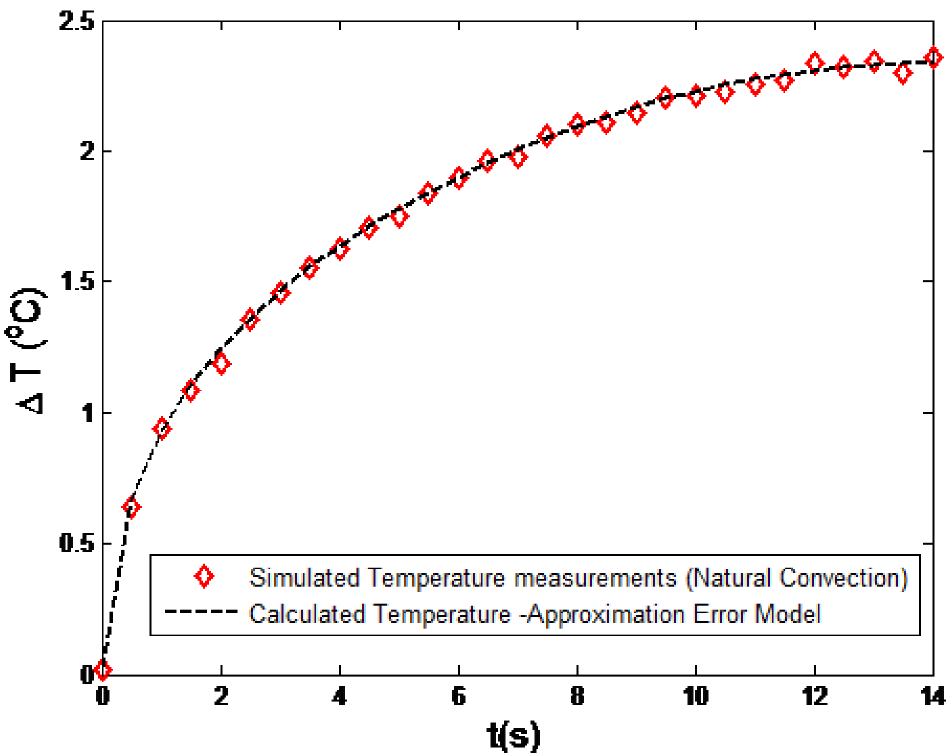
CONVECTIVE EFFECTS IN LIQUIDS CHARACTERIZED BY THE LINE HEAT SOURCE PROBE

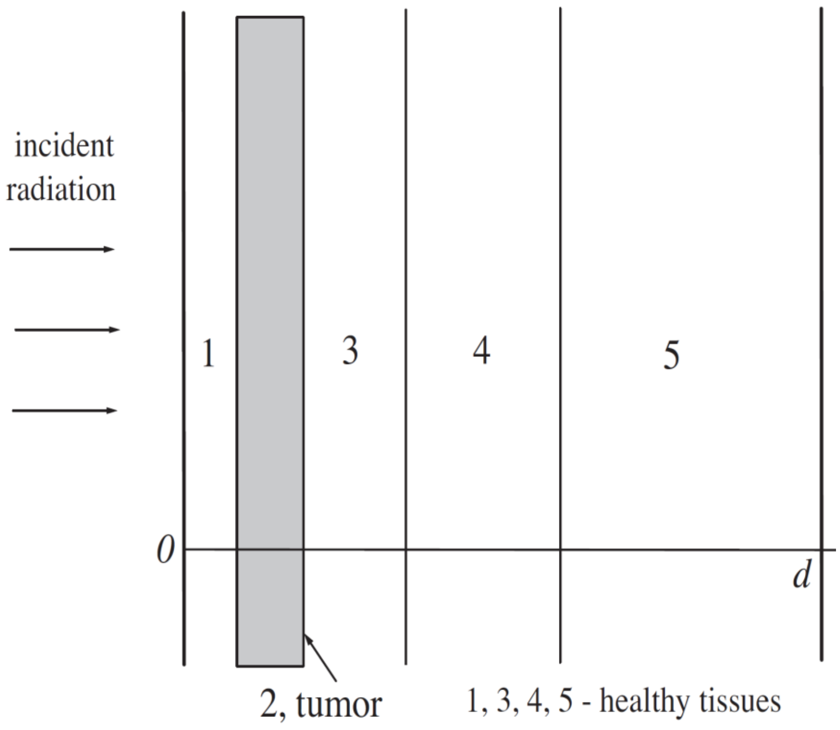


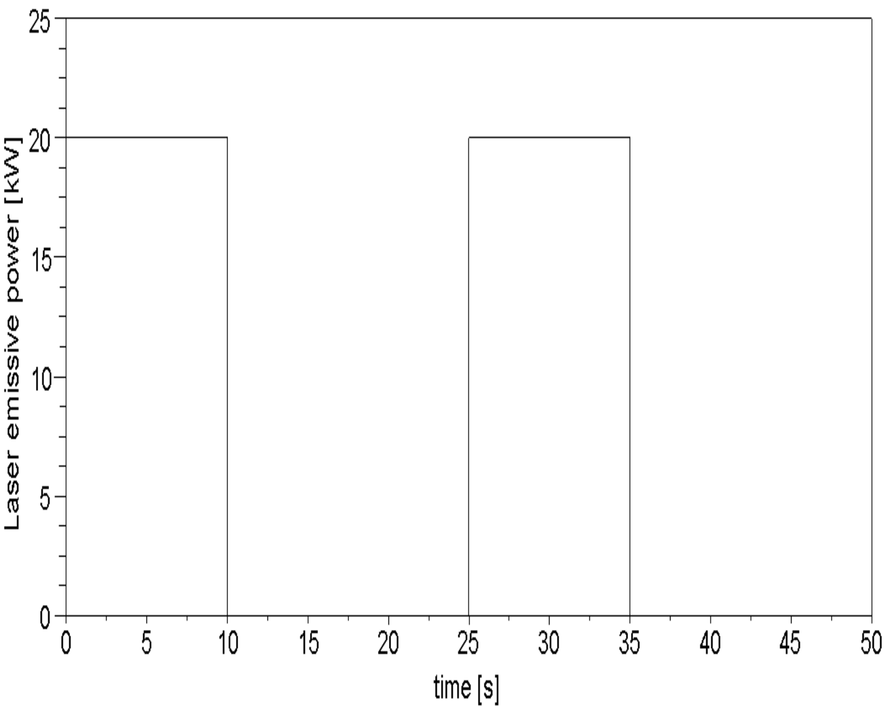
Bernard Lamien, Helcio R. B. Orlande, Approximation Error Model To Account For Convective Effects In Liquids Characterized By The Line Heat Source Probe, *4th Inverse Problems, Design and Optimization Symposium (IPDO-2013)*, Albi, France, June 26-28, 2013

SOLUTION OF INVERSE PROBLEMS

Improvement of solutions with reduced models

HYPERTHERMIA TREATMENT OF CANCER - NANOPARTICLES





Leonid A. Dombrovsky, Victoria Timchenko, Michael Jackson, Guan H. Yeoh, A combined transient thermal model for laser hyperthermia of tumors with embedded gold nanoshells, *International Journal of Heat and Mass Transfer*, Volume 54, Issues 25–26, December 2011, Pages 5459–5469

SOLUTION OF INVERSE PROBLEMS
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COMPLETE MODEL FOR THE FLUENCE RATE

$$\nabla \cdot \left[-D^*(\mathbf{r})\nabla\Phi_d(\mathbf{r}) + \frac{\sigma_s^*(\mathbf{r})g^*(\mathbf{r})}{\mu_{tr}^*(\mathbf{r})}\Phi_c(\mathbf{r})\mathbf{i} \right] + \sigma_a(\mathbf{r})\Phi_d(\mathbf{r}) = \sigma_s^*(\mathbf{r})\Phi_c(\mathbf{r})$$

$$\mathbf{n} \cdot \left(-D(\mathbf{r})\nabla\Phi_d(\mathbf{r}) + \frac{\sigma_s^*(\mathbf{r})g^*(\mathbf{r})}{\mu_{tr}^*(\mathbf{r})}\Phi_c(\mathbf{r})\mathbf{i} \right) + \frac{1}{2A}\Phi_d(\mathbf{r}) = 0$$

Where $\Phi_c(\mathbf{r})$ *is given by Beer Law*

$$\sigma_s^* = \sigma_s\left(1-g^2\right) \qquad g^* = \frac{g}{1+g}$$

$$\mu_{tr} = \sigma_a + \sigma_s(1-g) \qquad D = 1/3\mu_{tr} \qquad A = \frac{1-r_{id}}{1+r_{id}}$$

SOLUTION OF INVERSE PROBLEMS
Improvement of solutions with reduced models

REDUCED MODEL FOR THE FLUENCE RATE

Semi-infinite medium irradiated by a wide collimated beam with refractive index mismatched boundaries - Welch, (2011)

$$F_+(z) = \frac{S_+ - r_{id} S_-}{(\mu_t^2 - \mu_{eff}^2)(1 - r_{id})} \exp(-\mu_{eff} z) - \frac{S_+}{(\mu_t^2 - \mu_{eff}^2)} \exp(-\mu_t z)$$

$$F_-(z) = \frac{q(S_+ - r_{id} S_-)}{(\mu_t^2 - \mu_{eff}^2)(1 - r_{id})} \exp(-\mu_{eff} z) - \frac{S_-}{(\mu_t^2 - \mu_{eff}^2)} \exp(-\mu_t z)$$

$$Where\ S_+ = (\mu_s / 4)\left[(5 + 9g)\mu_a + 5\mu_s\right]$$

$$S_- = (\mu_s / 4)\left[(1 - 3g)\mu_a + \mu_s\right]$$

$$q = \left(\mu_{eff} - 2\mu_a\right) / \left(\mu_{eff} + 2\mu_a\right)$$

$$The\ total\ fluence\ rate\ is\ given\ as:$$

$$\Phi_t(z) = 2\left[F_+(z) + F_-(z)\right] + \Phi_c(z)$$

SOLUTION OF INVERSE PROBLEMS
Improvement of solutions with reduced models

BIOHEAT TRANSFER EQUATION

$$\rho(\mathbf{r},t)c(\mathbf{r},t)\frac{\partial T(\mathbf{r},t)}{\partial t} = k_c(\mathbf{r},t)\nabla T(\mathbf{r},t) + Q_{bio}(\mathbf{r},t) + Q_{laser}(\mathbf{r},t)$$

$$T(r,t) = T_b$$

$$k_c(\mathbf{r},t)\frac{\partial T(\mathbf{r},t)}{\partial \mathbf{r}} + hT(\mathbf{r},t) = hT_b$$

$$T(r,t) = T_b,\ t > 0$$

$$Q_{laser}(\mathbf{r},t) = \sigma_a(r)\Phi_t(r,t)$$

$$Q_{bio}(\mathbf{r},t) = \rho_b(\mathbf{r},t)c_b(\mathbf{r},t)v_b(\mathbf{r},t)\left[T(\mathbf{r},t) - T_b(\mathbf{r},t)\right] + Q_m(\mathbf{r},t)$$

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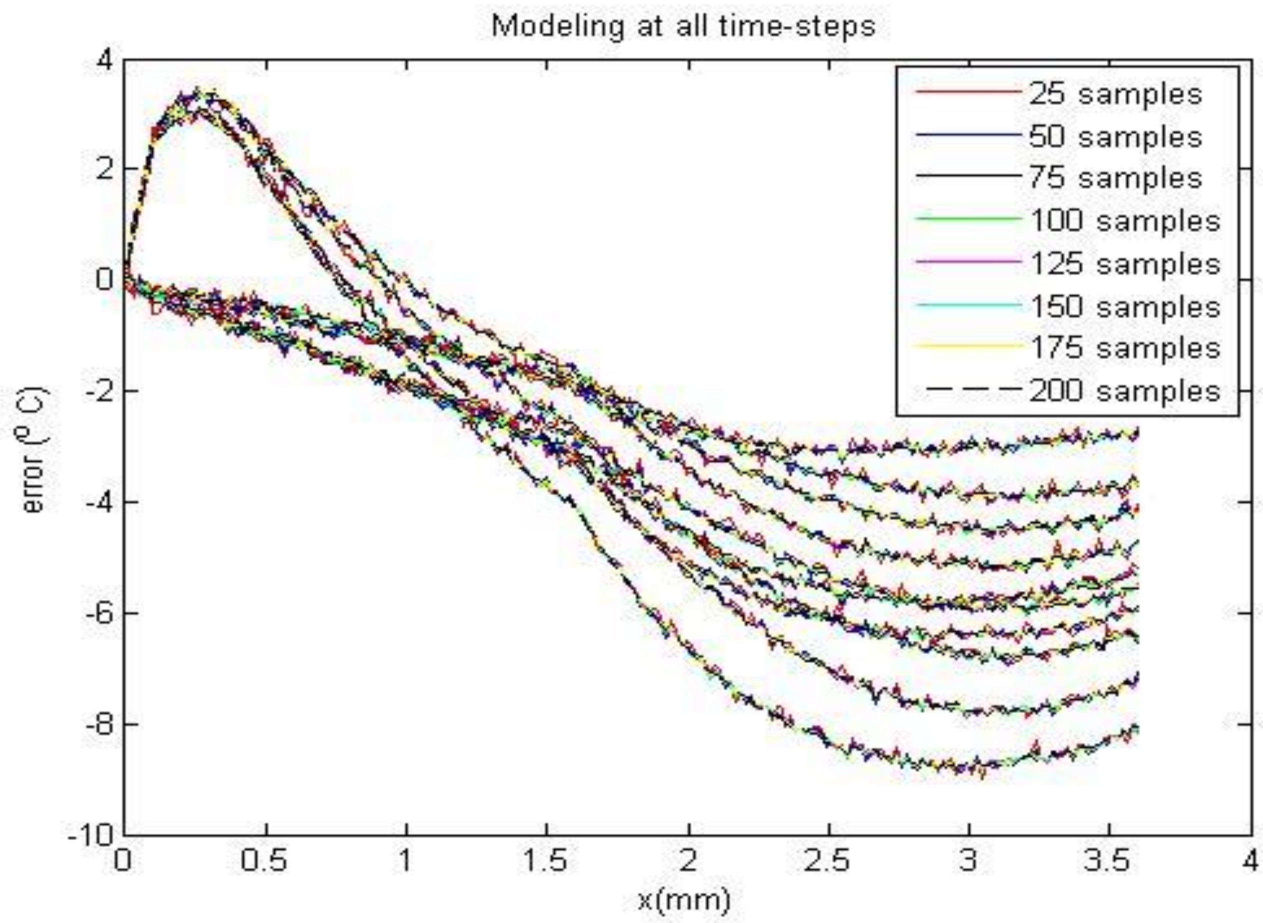
CONVERGENCE ANALYSIS OF THE MODELING ERROR



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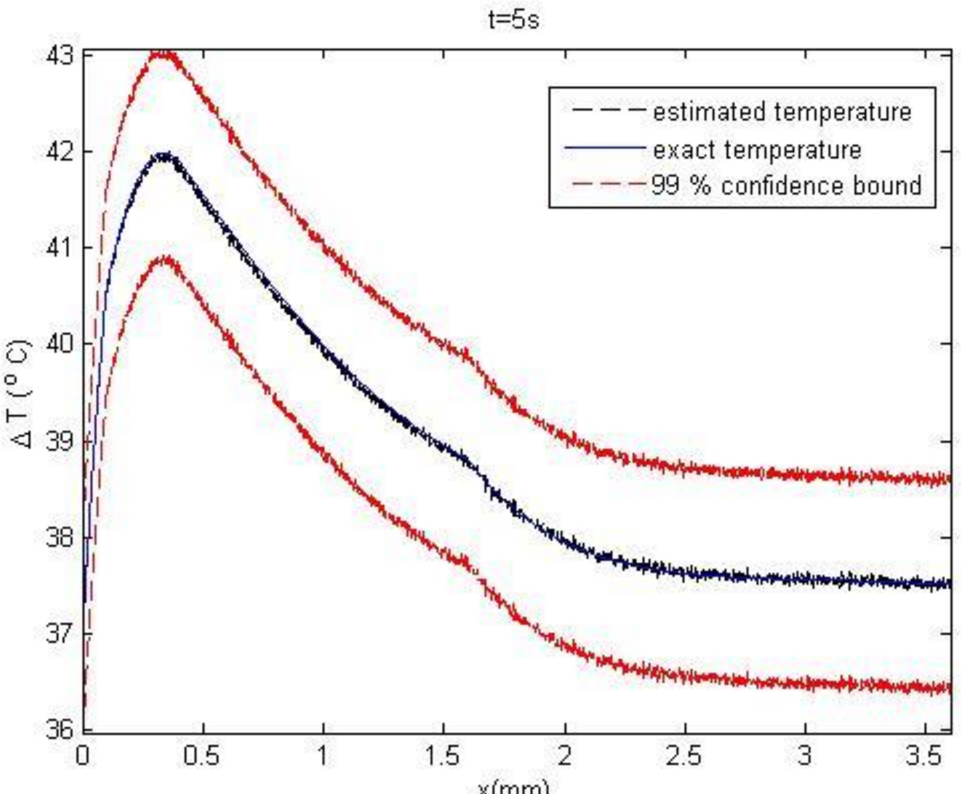
ESTIMATED TEMPERATURE – PARTICLE FILTER
ASIR+AEM (sensor at x=0.5 mm)



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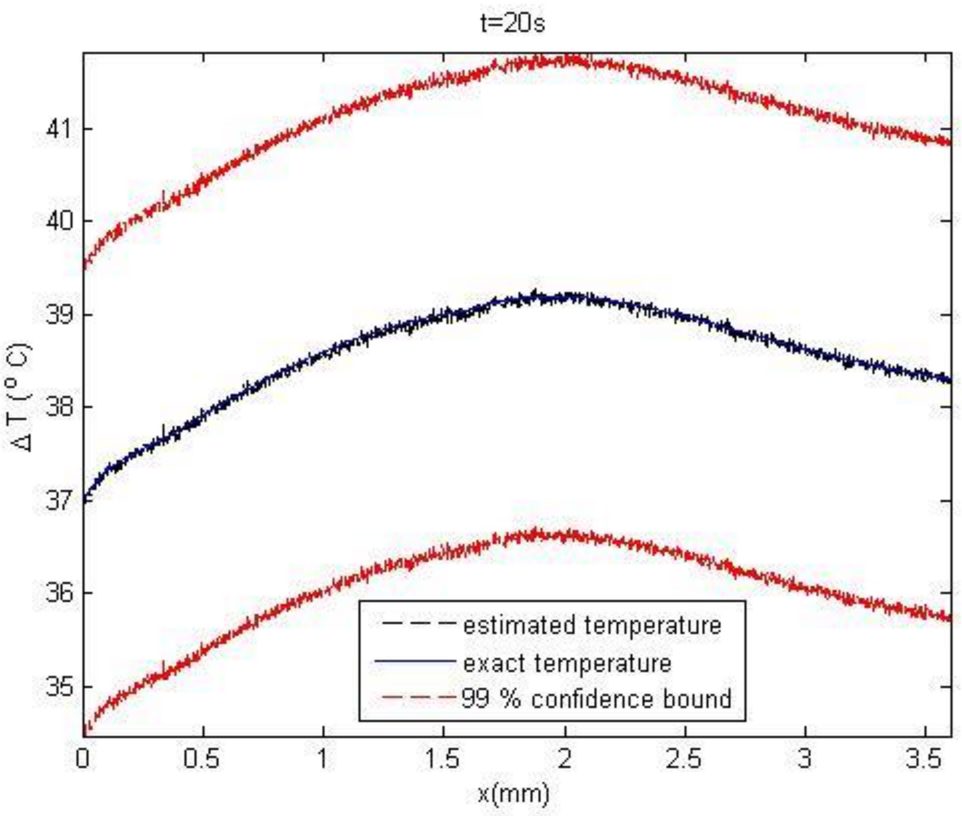
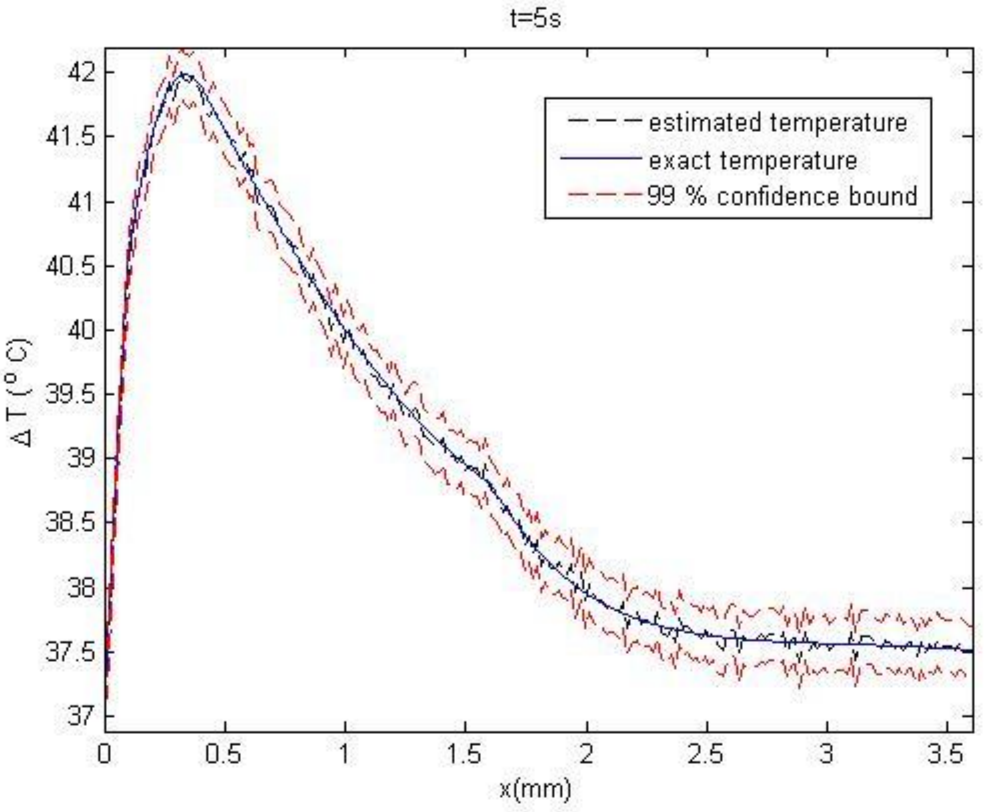


SOLUTION OF INVERSE PROBLEMS
Improvement of solutions with reduced models

ESTIMATED TEMPERATURE – PARTICLE FILTER
ASIR+AEM (sensor at x=1.5 mm)



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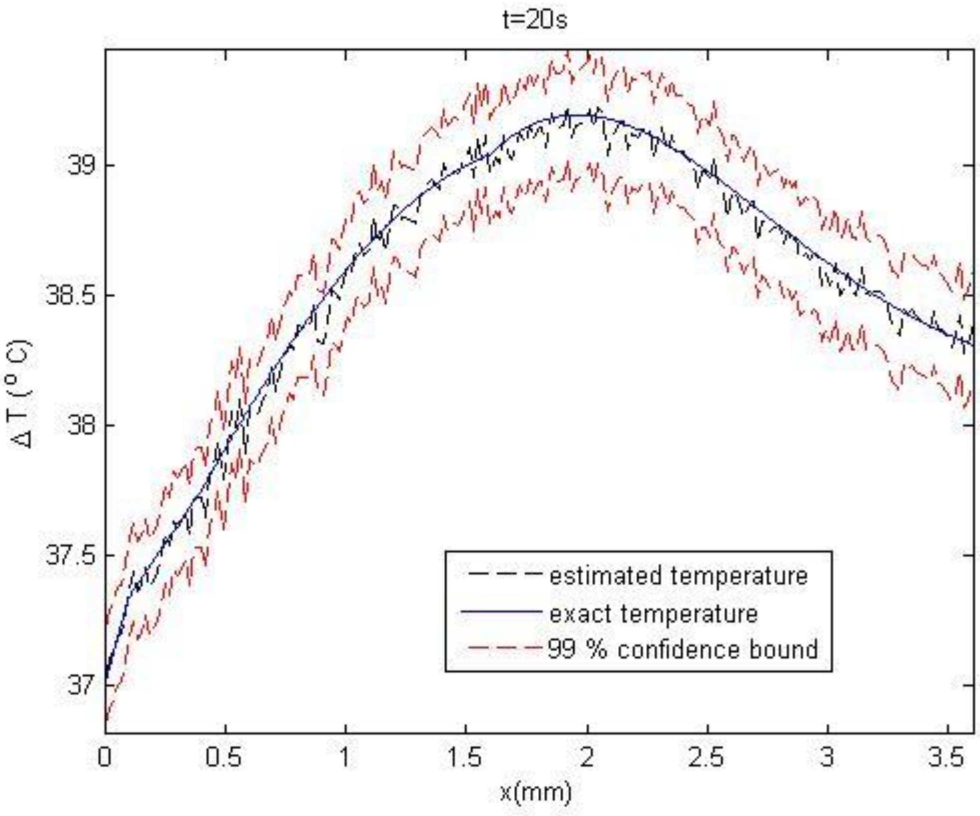
ESTIMATED TEMPERATURE – PARTICLE FILTER ASIR+AEM (sensor at x=1.5 mm)



CONCLUSIONS

• With the recent advancement of fast and affordable computational resources, sampling methods have become more popular within the community dealing with the solution of inverse problems. These methods are backed up by the statistical theory within the Bayesian framework, being quite simple in terms of application and not restricted to any prior distribution for the unknowns or models for the measurement errors.

CONCLUSIONS

• If the number of unknowns is too large, thus requiring a large number of samples to represent the posterior distribution, or the solution of the direct problem is too expensive in terms of computational time, the application of sampling methods may still be prohibitive nowadays.

• The use of surrogate models or response surfaces for the solution of the direct problem are useful for the reduction of the computational time, specially if used with the Approximation Error Approach.

• More efficient sampling algorithms are under development, e.g., the Delayed Acceptance Metropolis-Hastings algorithm.

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ACKNOWLEDGEMENTS

- **Prof. Wagner Muniz**
- **Prof. Antônio Leitão**
- **Prof. Andreas Rieder**
- Conselho Nacional de Desenvolvimento Científico e Tecnológico (**CNPq**).
- Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (**CAPES**).
- Fundação Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro (**FAPERJ**).