INVERSE HEAT TRANSFER PROBLEMS

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OUTLINE

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INTRODUCTION

Inverse heat transfer problems deal with the estimation of unknown quantities appearing in the mathematical formulation of physical processes in thermal sciences, by using measurements of temperature, heat flux, radiation intensities, etc.
INTRODUCTION

• Originally, inverse heat transfer problems have been associated with the estimation of an unknown boundary heat flux, by using temperature measurements taken below the boundary surface of a heat conducting medium.
• Recent technological advancements often require the use of involved experiments and indirect measurements, within the research paradigm of inverse problems.
• Nowadays, inverse analyses are encountered in single and multi-mode heat transfer problems, dealing with multi-scale phenomena.
• Applications range from the estimation of constant heat transfer parameters to the mapping of spatially and timely varying functions, such as heat sources, fluxes and thermophysical properties.
Consider the mathematical formulation of a heat transfer problem, which, for instance, can be linear or non-linear, one or multi-dimensional, involve one single or coupled heat transfer modes, etc.

We denote the vector of parameters appearing in such formulation as:

$$\mathbf{P}^T = [P_1, P_2, \ldots, P_N]$$

where *N is the number of parameters*

- These parameters can possibly be thermal conductivity components, heat transfer coefficients, heat sources, boundary heat fluxes, etc.
- They can represent constant values of such quantities, or the parameters of the representation of a function in terms of known basis functions.
Consider also that transient measurements are available within the medium, or at its surface, where the heat transfer processes are being mathematically formulated.

The vector containing the measurements is written as:

\[ \mathbf{Y}^T = \left( \tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_I \right) \]

\[ \tilde{Y}_i = \left( Y_{i1}, Y_{i2}, \ldots, Y_{iM} \right) \]

\[ M = \# \text{ of sensors} \]
\[ I = \# \text{ of transient measurements per sensor} \]

\[ D = MI = \# \text{ of measurements} \]

- The measured data are not limited to temperatures, but could also include heat fluxes, radiation intensities, etc.
The **statistical inversion approach** is based on the following principles (Jari P. Kaipio and Erkki Somersalo, *Computational and Statistical Methods for Inverse Problems*, Springer, 2004):

1. All variables included in the formulation are modeled as random variables.
2. The randomness describes the degree of information concerning their realizations.
3. The degree of information concerning these values is coded in the probability distributions.
4. The solution of the inverse problem is the posterior probability distribution.
• In many cases, the Posterior Probability Distribution does not allow an analytical treatment.
• Draw samples from the set $\Omega$ of all possible $\mathbf{P}$’s, each sample with probability $\pi(\mathbf{P}|\mathbf{Y})$.
• Get a set $\Theta = \{\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_M\}$ of samples distributed like the posterior distribution.
• Inference on $\pi(\mathbf{P}|\mathbf{Y})$ becomes inference on $\Theta = \{\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_M\}$, for example the mean of the samples in $\Theta$ give us an estimation of the average values of $\pi(\mathbf{P}|\mathbf{Y})$.
• We generally need the constant that normalizes the probability distribution:

**MARKOV CHAIN MONTE-CARLO METHODS**

(Metropolis-Hastings Algorithm)

• Very time consuming.
SOLUTION OF INVERSE PROBLEMS

Bayesian framework

Example: Estimation of Contact Failures in Layered Composites

- Metropolis-Hastings algorithm
- 2 layers
- Simulated Measurements
- TV prior
- 22 hours

Figure 5.a Exact temperature distribution at $Z = 1$ and $\tau = 0.065$ – two square failures of size 0.005 m

Figure 5.b Simulated measurements at $Z = 1$ and $\tau = 0.065$ – two square failures of size 0.005

Example: Estimation of Thermal Conductivity Components of Orthotropic Solids

\( k_1 \frac{\partial^2 T}{\partial x^2} + k_2 \frac{\partial^2 T}{\partial y^2} + k_3 \frac{\partial^2 T}{\partial z^2} = \frac{\partial T}{\partial t} \) in \( 0 < x < a, 0 < y < b, 0 < z < c \); \( t > 0 \)

\( T = 0 \) at \( x = 0 \) ; \( k_1 \frac{\partial T}{\partial x} = q_1(t) \) at \( x = a \), for \( t > 0 \)

\( T = 0 \) at \( y = 0 \) ; \( k_2 \frac{\partial T}{\partial y} = q_2(t) \) at \( y = b \), for \( t > 0 \)

\( T = 0 \) at \( z = 0 \) ; \( k_3 \frac{\partial T}{\partial z} = q_3(t) \) at \( z = c \), for \( t > 0 \)

\( T = 0 \) for \( t = 0 \); in \( 0 < x < a, 0 < y < b, 0 < z < c \)
Example: Estimation of Thermal Conductivity Components of Orthotropic Solids – Interpolation of the Likelihood with RBF’s

Exact Likelihood (48 seconds)  Interpolated Likelihood (1.8 seconds)

Example: Characterization of Heterogeneous Media
YES! The likelihood is Gaussian!
Example: Characterization of Heterogeneous Media

Figure 7 – (a) Infrared camera image acquired at the moment the DC source is switched on. (b) Infrared camera image acquired after some elapsed time during heating period.

The number of pixels in the vertical direction for the configuration that has been tested provides the total number of **328 spatial measurements** along the 8 cm of the plate.
Another advancement of the present study was the solution of the inverse problem in the transformed field, from the integral transformation of the experimental temperature data, thus compressing the experimental measurements in the space variables into a few transformed fields. Once the experimental temperature readings have been obtained, one proceeds to the integral transformation of the temperature field at each time through the integral transform pair below:

\[
\text{Transform } \quad T_{\text{exp}, i}(t) = \int_0^{L_x} w(x)\tilde{\psi}_i(x)[T_{\text{exp}}(x,t) - T_\infty]dx
\]

\[
\text{Inverse } \quad T_{\text{exp}}(x,t) = T_\infty + \sum_{i=0}^{N_i} \tilde{\psi}_i(x)T_{\text{exp}, i}(t)
\]

Example: Characterization of Heterogeneous Media
Data Compression

These are in fact the quantities that are employed in the inverse problem analysis. Therefore, a significant data reduction of more than 95% is achieved, as one chooses to solve the inverse problem in the transformed temperature domain.
SOLUTION OF INVERSE PROBLEMS
Bayesian framework

Transformed Potentials

Temperature (°C)

SOLUTION OF INVERSE PROBLEMS

Bayesian framework

Example: Characterization of Heterogeneous Media – Nodal Approach

Thin plate: Lumped model in z

\[
C(x, y) \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left[ k(x, y) \frac{\partial T}{\partial x} \right] + \frac{\partial}{\partial y} \left[ k(x, y) \frac{\partial T}{\partial y} \right] - h(x, y)(T - T_\infty) + g(x, y)
\]

By writing the equation above in non-conservative form:

\[
\frac{\partial T}{\partial t} = a(x, y) \nabla^2 T + \frac{1}{C(x, y)} \left[ \frac{\partial k(x, y)}{\partial x} \frac{\partial T}{\partial x} + \frac{\partial k(x, y)}{\partial y} \frac{\partial T}{\partial y} \right] - H(x, y)(T - T_\infty) + G(x, y)
\]

\[
a(x, y) = \frac{k(x, y)}{C(x, y)} \quad H(x, y) = \frac{h(x, y)}{C(x, y)} \quad G(x, y) = \frac{g(x, y)}{C(x, y)}
\]

SOLUTION OF INVERSE PROBLEMS

Bayesian framework

Example: Characterization of Heterogeneous Media – Nodal Approach

\[ Y_{ij} = J_{ij} P_{ij} \]

\[ J_{ij} = \begin{bmatrix} L_{1,i,j} & Dx_{1,i,j} & Dy_{1,i,j} & -\Delta t(T_{1,i,j}^{t} - T_{\infty}) & \Delta t \\ L_{2,i,j} & Dx_{2,i,j} & Dy_{2,i,j} & -\Delta t(T_{2,i,j}^{t} - T_{\infty}) & \Delta t \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{n,i,j} & Dx_{n,i,j} & Dy_{n,i,j} & -\Delta t(T_{n,i,j}^{t} - T_{\infty}) & \Delta t \end{bmatrix} \]

\[ Y_{ij} = \begin{bmatrix} Y_{i,j}^{1} \\ Y_{i,j}^{2} \\ \vdots \\ Y_{i,j}^{n} \end{bmatrix} \]

\[ P_{ij} = \begin{bmatrix} a_{i,j} \\ \delta_{i,j}^{x} \\ \delta_{i,j}^{y} \\ H_{i,j} \\ G_{i,j} \end{bmatrix} \]

In the nodal strategy, the sensitivity matrix is approximately computed with the measurements:

\[ \pi(P,J|Y) \propto \pi(Y|P,J)\pi(P)\pi(J) \]

SOLUTION OF INVERSE PROBLEMS
Bayesian framework

Example: Characterization of Heterogeneous Media – Nodal Approach
Example: Characterization of Heterogeneous Media – Nodal Approach

a (m²/s) mapping with MH. $\mu = 2.529 \times 10^{-7}$ and $\sigma = 4.8277 \times 10^{-9}$

G (K/s) mapping with MH
SOLUTION OF INVERSE PROBLEMS

Bayesian framework

Example: Characterization of Heterogeneous Media – Nodal Approach

Residuals at i = 38 and j = 40 for all time step

Residuals at i = 64 and j = 60 for all time step

Residuals at i = 30 and j = 90 for all time step

Residuals at i = 58 and j = 59 for all time step
SOLUTION OF INVERSE PROBLEMS

Example: Non linear 3D heat conduction
Estimation of \( q(x,y) \) with measurements of \( T(x,y,0,t) \)

Helcio R. B. Orlande, George S. Dulikravich, Markus Neumayer, Daniel Watzenig, Marcelo J. Colaço,
SOLUTION OF INVERSE PROBLEMS

Complete model

\[ C(T_c) \frac{\partial T_c(x, y, z, t)}{\partial t} = \frac{\partial}{\partial x} \left[ k(T_c) \frac{\partial T_c}{\partial x} \right] + \frac{\partial}{\partial y} \left[ k(T_c) \frac{\partial T_c}{\partial y} \right] + \frac{\partial}{\partial z} \left[ k(T_c) \frac{\partial T_c}{\partial z} \right] \]

in \(0 < x < a, 0 < y < b, 0 < z < c\), for \(t > 0\)

\[ \frac{\partial T_c}{\partial x} = 0 \quad \text{at } x = 0 \text{ and } x = a, 0 < y < b, 0 < z < c, \text{ for } t > 0 \]

\[ \frac{\partial T_c}{\partial y} = 0 \quad \text{at } y = 0 \text{ and } y = b, 0 < x < a, 0 < z < c, \text{ for } t > 0 \]

\[ \frac{\partial T_c}{\partial z} = 0 \quad \text{at } z = 0, 0 < x < a, 0 < y < b, \text{ for } t > 0 \]

\[ k(T_c) \frac{\partial T_c}{\partial z} = q(x, y) \quad \text{at } z = c, 0 < x < a, 0 < y < b, \text{ for } t > 0 \]

\[ T_c = T_0 \quad \text{for } t = 0, \text{ in } 0 < x < a, 0 < y < b, 0 < z < c \]
Reduced models: Linear problem with properties at $T^*$

\[
C^* \frac{\partial \bar{T}(x, y, t)}{\partial t} = \frac{\partial}{\partial x} \left[ k^* \frac{\partial \bar{T}}{\partial x} \right] + \frac{\partial}{\partial y} \left[ k^* \frac{\partial \bar{T}}{\partial y} \right] + \frac{q(x, y)}{c}
\]

in $0 < x < a$, $0 < y < b$, for $t > 0$

\[
\frac{\partial \bar{T}}{\partial x} = 0 \quad \text{at} \ x = 0 \text{ and } x = a, \ 0 < y < b, \ \text{for} \ t > 0
\]

\[
\frac{\partial \bar{T}}{\partial y} = 0 \quad \text{at} \ y = 0 \text{ and } y = b, \ 0 < x < a, \ \text{for} \ t > 0
\]

\[
\bar{T} = T_0 \quad \text{for} \ t = 0, \ \text{in} \ 0 < x < a, \ 0 < y < b
\]

where

\[
\bar{T}(x, y, t) = \frac{1}{c} \int_{z=0}^c T(x, y, z, t) \, dz
\]
Reduced models: Linear problem with properties at $T^*$

Classical Lumped Formulation:
Temperature gradients across the thickness of the plate are fully neglected.

$$T(x, y, 0, t) = T(x, y, c, t) = \bar{T}(x, y, t)$$
In general, the direct problem solution with the complete model took around 7.2 s, while the solution with the reduced model took around 0.09 s of CPU time.

**Improved Lumped Formulation:**

\[
H_{1,1} \text{ formula (correct trapezoidal rule): } \bar{T}(x, y, t) \approx \frac{1}{2} [T(x, y, 0, t) + T(x, y, c, t)] + \frac{c}{12} \left[ \frac{\partial T}{\partial z} \bigg|_{z=0} - \frac{\partial T}{\partial z} \bigg|_{z=c} \right]
\]

\[
H_{0,0} \text{ formula (trapezoidal rule): } \int_{z=0}^{c} \frac{\partial T(x, y, z, t)}{\partial z} dz = T(x, y, c, t) - T(x, y, 0, t) \approx \frac{c}{2} \left[ \frac{\partial T}{\partial z} \bigg|_{z=0} + \frac{\partial T}{\partial z} \bigg|_{z=c} \right]
\]

\[
T(x, y, 0, t) = \bar{T}(x, y, t) - \frac{c}{6k} \ast q(x, y)
\]

\[
T(x, y, c, t) = \bar{T}(x, y, t) + \frac{c}{3k} \ast q(x, y)
\]
Error of the Direct Problem Solution at the final time:

Classical Lumped Model

Improved Lumped Model

\[ q(x_i, y_j) = \begin{cases} 
10^7 \text{Wm}^{-2}, & \text{for } 8 \leq i \leq 10 \text{ and } 8 \leq j \leq 10 \\
10^7 \text{Wm}^{-2}, & \text{for } 18 \leq i \leq 20 \text{ and } 18 \leq j \leq 20 \\
0, & \text{elsewhere}
\end{cases} \]

Orlande, H.R.B., Dulikravich, G., Inverse Heat Transfer Problems and their Solutions within the Bayesian Framework, ECCOMAS Special Interest Conference, Numerical Heat Transfer 2012, 4-6 September 2012, Gliwice-Wrocław, Poland
SOLUTION OF INVERSE PROBLEMS
Improvement of solutions with reduced models

DELAYED ACCEPTANCE METROPOLIS-HASTINGS ALGORITHM

1. Sample a Candidate Point $P^*$ from a proposal distribution $p(P^*,P^{(t-1)})$.
2. Calculate the acceptance factor with the surrogate model:

$$
\alpha = \min \left[ 1, \frac{\pi(P^*|Y) p(P^{(t-1)}, P^*)}{\pi(P^{(t-1)}|Y) p(P^*, P^{(t-1)})} \right]
$$

3. Generate a random value $U$ that is uniformly distributed on $(0,1)$.
4. If $U \leq \alpha$, proceed to step 5. Otherwise, return to step 1.
5. Calculate a new acceptance factor with the complete model:

$$
\alpha_c = \min \left[ 1, \frac{\pi_c(P^*|Y) p(P^{(t-1)}, P^*)}{\pi_c(P^{(t-1)}|Y) p(P^*, P^{(t-1)})} \right]
$$

6. Generate a new random value $U_c$ which is uniformly distributed on $(0,1)$.
7. If $U_c \leq \alpha_c$, set $P^{(t)} = P^*$. Otherwise, set $P^{(t)} = P^{(t-1)}$.
8. Return to step 1.

where $\pi(P|Y)$ and $\pi_c(P|Y)$ are the posterior distributions with the likelihoods computed with the surrogate model and with the complete model, respectively.
SOLUTION OF INVERSE PROBLEMS

PRIOR DISTRIBUTIONS
Total variation non-informative prior

\[ \pi(P) \propto \exp[-\alpha TV(P)] \]

\[ TV(P) = \sum_{i=2}^{I-1} \sum_{j=2}^{J-1} \Delta y \left[ |q(x_i, y_j) - q(x_{i+1}, y_j)| + |q(x_i, y_j) - q(x_{i-1}, y_j)| \right] + \]
\[ + \Delta x \left[ |q(x_i, y_j) - q(x_i, y_{j+1})| + |q(x_i, y_j) - q(x_i, y_{j-1})| \right] \]

In the approximation error model (AEM) approach, the statistical model of the approximation error is constructed and then represented as additional noise in the measurement model [1,19-23]. With the hypotheses that the measurement errors are additive and independent of the parameters \( \mathbf{P} \), one can write

\[
\mathbf{Y} = \mathbf{T}_c(\mathbf{P}) + \mathbf{e}
\]  

(16)

where \( \mathbf{T}_c(\mathbf{P}) \) is the sufficiently accurate solution of the complete model given by equations (1.a-h). The vector of measurement errors, \( \mathbf{e} \) are assumed here to be Gaussian, with zero mean and known covariance matrix \( \mathbf{W} \).
SOLUTION OF INVERSE PROBLEMS
Improvement of solutions with reduced models

APPROXIMATION ERROR MODEL

\[ Y = T(P) + [T_c(P) - T(P)] + \varepsilon \]

By defining the error between the complete and the surrogate model solutions as

\[ \varepsilon = [T_c(P) - T(P)] \]

equation (17) can be written as

\[ Y = T(P) + \eta \]

where

\[ \eta = \varepsilon + e \]
SOLUTION OF INVERSE PROBLEMS
Improvement of solutions with reduced models

APPROXIMATION ERROR MODEL

\[ \eta \text{ is modeled as a Gaussian variable} \]

\[ \widetilde{\pi}(\gamma, P | Y) \propto \gamma^{(M+2)/2} \exp \left\{ -\frac{1}{2} [Y - T(P) - \overline{\eta}]^T \tilde{W}^{-1} [Y - T(P) - \overline{\eta}] - \frac{1}{2} \gamma (P - \mu)^T \Gamma^{-1} (P - \mu) - \frac{1}{2} \left( \frac{\gamma}{\gamma_0} \right)^2 \right\} \]

Enhanced error model:

\[ \overline{\eta} \approx \overline{\varepsilon} \]

\[ \tilde{W} \approx W_\varepsilon + W \]

Gaussian prior

Energy Balance: \( q(x_i, y_j) = C^* c \frac{dT(x_i, y_j)}{dt} \)

In order to generate this physically motivated Gaussian prior, and at the same time not violate the Bayesian principle that the prior is the information for the unknowns (coded in the form of probability distribution functions) that is available before the measurements are taken, we assume here that another kind of measurements is also available. Such other kind of measurements is only used to generate the prior, and is considered independent of the temperature measurements used in the inverse analysis, that is, for the computation of the likelihood.
<table>
<thead>
<tr>
<th>Test case</th>
<th>Flux</th>
<th>Prior</th>
<th>Approach</th>
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</thead>
<tbody>
<tr>
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<td>TV</td>
<td>-</td>
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<tr>
<td>2</td>
<td>B</td>
<td>TV</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
<td>TV</td>
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<td>12</td>
<td>C</td>
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\[ \sigma = 0.02 \text{ K} \]

<table>
<thead>
<tr>
<th>Test case</th>
<th>CPU Time (h)</th>
<th>Acceptance ratio (%)</th>
<th>RMS Error (W/m(^2))</th>
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</tr>
</tbody>
</table>

\[ \sigma = 1.25 \text{ K} \]
Gaussian prior + AEM - $\sigma = 0.02$ K
TV prior + DAMH - $\sigma = 1.25$ K
$\sigma = 1.25 \text{ K}$

Gaussian prior

Gaussian prior + AEM
CONVECTIVE EFFECTS IN LIQUIDS CHARACTERIZED BY THE LINE HEAT SOURCE PROBE

CONVECTIVE EFFECTS IN LIQUIDS CHARACTERIZED BY THE LINE HEAT SOURCE PROBE

CONVECTIVE EFFECTS IN LIQUIDS CHARACTERIZED BY THE LINE HEAT SOURCE PROBE

SOLUTION OF INVERSE PROBLEMS
Improvement of solutions with reduced models

HYPERTHERMIA TREATMENT OF CANCER - NANOPARTICLES

SOLUTION OF INVERSE PROBLEMS
Improvement of solutions with reduced models

COMPLETE MODEL FOR THE FLUENCE RATE

\[
\nabla \cdot \left[ -D^*(\mathbf{r}) \nabla \Phi_d(\mathbf{r}) + \frac{\sigma_s^*(\mathbf{r}) g^*(\mathbf{r})}{\mu_{tr}^*(\mathbf{r})} \Phi_c(\mathbf{r}) i \right] + \sigma_a(\mathbf{r}) \Phi_d(\mathbf{r}) = \sigma_s^*(\mathbf{r}) \Phi_c(\mathbf{r})
\]

\[
\mathbf{n} \cdot \left( -D(\mathbf{r}) \nabla \Phi_d(\mathbf{r}) + \frac{\sigma_s^*(\mathbf{r}) g^*(\mathbf{r})}{\mu_{tr}^*(\mathbf{r})} \Phi_c(\mathbf{r}) i \right) + \frac{1}{2A} \Phi_d(\mathbf{r}) = 0
\]

Where \( \Phi_c(\mathbf{r}) \) is given by Beer Law

\[
\sigma_s^* = \sigma_s (1 - g^2) \quad g^* = \frac{g}{1 + g}
\]

\[
\mu_{tr} = \sigma_a + \sigma_s (1 - g) \quad D = 1 / 3 \mu_{tr} \quad A = \frac{1 - r_{id}}{1 + r_{id}}
\]
SOLUTION OF INVERSE PROBLEMS
Improvement of solutions with reduced models

REduced model for the Fluence rate
Semi-infinite medium irradiated by a wide collimated beam with refractive index mismatched boundaries - Welch, (2011)

\[ F_+ (z) = \frac{S_+ - r_{id} S_-}{(\mu_t^2 - \mu_{eff}^2)(1 - r_{id})} \exp(-\mu_{eff} z) - \frac{S_+}{(\mu_t^2 - \mu_{eff}^2)} \exp(-\mu_t z) \]

\[ F_- (z) = \frac{q(S_+ - r_{id} S_-)}{(\mu_t^2 - \mu_{eff}^2)(1 - r_{id})} \exp(-\mu_{eff} z) - \frac{S_-}{(\mu_t^2 - \mu_{eff}^2)} \exp(-\mu_t z) \]

Where \( S_+ = (\mu_s / 4) \left[ (5 + 9g) \mu_a + 5\mu_s \right] \)

\( S_- = (\mu_s / 4) \left[ (1 - 3g) \mu_a + \mu_s \right] \)

\( q = (\mu_{eff} - 2\mu_a) / (\mu_{eff} + 2\mu_a) \)

The total fluence rate is given as :

\( \Phi_t (z) = 2 \left[ F_+ (z) + F_- (z) \right] + \Phi_c (z) \)
**SOLUTION OF INVERSE PROBLEMS**

**Improvement of solutions with reduced models**

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**BIOHEAT TRANSFER EQUATION**

\[
\rho(r,t)c(r,t) \frac{\partial T(r,t)}{\partial t} = k_c(r,t) \nabla T(r,t) + Q_{bio}(r,t) + Q_{laser}(r,t)
\]

\[T(r,t) = T_b\]

\[k_c(r,t) \frac{\partial T(r,t)}{\partial r} + hT(r,t) = hT_b\]

\[T(r,t) = T_b, \quad t > 0\]

\[Q_{laser}(r,t) = \sigma_a(r) \Phi_t(r,t)\]

\[Q_{bio}(r,t) = \rho_b(r,t)c_b(r,t)v_b(r,t)[T(r,t) - T_b(r,t)] + Q_m(r,t)\]
SOLUTION OF INVERSE PROBLEMS
Improvement of solutions with reduced models

CONVERGENCE ANALYSIS OF THE MODELING ERROR
SOLUTION OF INVERSE PROBLEMS
Improvement of solutions with reduced models

ESTIMATED TEMPERATURE – PARTICLE FILTER
ASIR+AEM (sensor at x=0.5 mm)
SOLUTION OF INVERSE PROBLEMS
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CONCLUSIONS

• With the recent advancement of fast and affordable computational resources, sampling methods have become more popular within the community dealing with the solution of inverse problems. These methods are backed up by the statistical theory within the Bayesian framework, being quite simple in terms of application and not restricted to any prior distribution for the unknowns or models for the measurement errors.
CONCLUSIONS

• If the number of unknowns is too large, thus requiring a large number of samples to represent the posterior distribution, or the solution of the direct problem is too expensive in terms of computational time, the application of sampling methods may still be prohibitive nowadays.
• The use of surrogate models or response surfaces for the solution of the direct problem are useful for the reduction of the computational time, specially if used with the Approximation Error Approach.
• More efficient sampling algorithms are under development, e.g., the Delayed Acceptance Metropolis-Hastings algorithm.
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