Simultaneous Reconstruction of Coefficients and Source Parameters in Elliptic Systems Modelled with Many Boundary Values Problems

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The practical expression of linear elliptic partial differential equations found in most of the engineering application is represented by the following system, in which the fields may be a vector and coefficients can be represented by matrices and vectors according: To find $u(x)$ such that

$$\left\{ \begin{array}{ll} \nabla.(-c\nabla u - \alpha u + \gamma) + \beta.\nabla u + au = f & \text{if } x \in \Omega; \\ hu = g & \text{if } x \in \partial\Omega_D; \\ \nu.(c\nabla u + \alpha u - \gamma) + qu = g_\nu - h^* \mu & \text{if } x \in \partial\Omega_N; \end{array} \right. \quad (1)$$

where ν is the outward unit normal vector on $\partial\Omega := \partial\Omega_D \cup \Pi \cup \partial\Omega_N$,

- Boundary integral formulation for the inverse problem: Let G_ξ be the fundamental solution for the strongly elliptic system (1).
- Then the Calderón projector gap is:

$$\begin{bmatrix} \gamma u(\xi) \\ \mathcal{B}_\nu u(\xi) \end{bmatrix} = \begin{bmatrix} \int_\Omega \gamma_\xi [G_\xi](y) f(y,u) dy \\ \int_\Omega \mathcal{B}_{\nu_\xi} [G_\xi](y) f(y) dy \end{bmatrix} +$$

$$\begin{bmatrix} \frac{1}{2}(I_{x\to\xi} - T_{x\to\xi}) & S_{x\to\xi} \\ R_{x\to\xi} & \frac{1}{2}(I_{x\to\xi} + T_{x\to\xi}) \end{bmatrix} \begin{bmatrix} \gamma u(\xi) \\ \mathcal{B}_\nu u(\xi) \end{bmatrix}, \ (x,\xi) \in \Gamma \times \Gamma$$

- Boundary integral formulation for the inverse problem: Let v be the regular fundamental solution for the strongly elliptic system (1).
- Variational : for all $v \in (H_{\mathcal{L}^*}(\Omega)^m)^*$.

$$\int_\Omega v(x) f(x) dx = -\int_\Gamma \Big(\sum_{j=1}^{d} \nu_j \sum_{k=1}^{d} A_{jk} \partial_k v(x) + A_j v(x)\Big) u(x) d\sigma_x$$

$$+ \int_\Gamma v(x) \sum_{j=1}^{d} \nu_j \sum_{k=1}^{d} A_{jk} \partial_k u(x) d\sigma_x$$

- Direct problem (Closed operators): Variational formulation and stabilization via Babuska-Brezzi-Necas-Banach condition.
- Inverse problem (Compact operators): Variational or strong formulation and stabilization thorough Picard-Tikhonov-Landweber-Morozov-Banach regularization.
- Functional Analysis Framework: Closed Range Banach Theorem and Fredholm Operator Theory.
- Solution of Direct problems with minimization of Least square discrepancy between calculated and measured Neumann data.
- Many solutions of two equivalent direct problems and minimization of the discrepancy between these two solutions.(This work !)

- Given The Dirichlet to Newmann map $\Lambda_c : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$
- To find $u \in H^1(\Omega), c(x) \in L^\infty(\Omega)$ such that

$$\begin{cases} \nabla.(-c\nabla u) = 0 & \text{in } \Omega; \\ g_\nu = \Lambda_c[g] & \text{on } \partial\Omega; \end{cases} \tag{2}$$

- $Q_c = \int_\Omega c(x)||\nabla u(x)||^2 dx = \int_{\partial\Omega} \overline{g}(x)\Lambda_c[g](x)d\sigma(x)$

- In 1980, (Seminar on Numerical Analysis and its Applications to Continuum Physics, SBM, Rio de Janeiro), Calderón posed the following problem:
- Decide whether c is uniquely determined by Q_c, and, if so, calculate c in terms of Q_c.
- the uniqueness problem, for conductivity or any other coefficients, is an open problem that has been only partially solved.
- "Intrinsic non uniqueness in the inverse source problem";
- The Calderon Projector Gap is the same for all Cauchy datum used to estimate it.

- Some uniqueness class
  - The regular affine class "If the sources are restricted to the affine class of functions $\mathcal{C}(D;F) = \{f \in H^2(\Omega) : Df = F\}$ then we have uniqueness of the associated inverse source problem (Alves, Martins, Roberty, Colaço, Olander, 2007);
  - The characteristic class $f = F\chi_\omega$(P. Novikov, 1938, Isakov, 1990);
  - The mono and dipolar source class $\mathcal{A} = \{f := \sum_{j=1}^{m1} \lambda_j \delta_{x_j} + \sum_{j=1}^{m1} p_j \cdot \nabla \delta_{x_j}$(El Badia e Ha Duong, 2000).

Lemma

Let V, W Banach spaces and V', W' its respective dual. Let $A \in \mathcal{L}(V, W)$ and $A^T \in \mathcal{L}(W', V')$ its transposed. Let $ker(A)$, $ker(A^T)$, $im(A)$ and $im(A^T)$ denotes its respective kernel and range. For $M \subset V$ and $N \subset W'$, let $M^\perp := \{v' \in V' | \forall m \in M, \langle v', m \rangle_{V' \times V} = 0\}$ and $N^\perp := \{v \in V | \forall n \in N, \langle n, v \rangle_{V' \times V} = 0\}$. Then, the following properties hold:

- $ker(A) = (im(A^T))^\perp$ *and* $ker(A^T) = (im(A))^\perp$;
- $\overline{im(A)} = (ker(A^T))^\perp$ *and* $\overline{im(A^T)} = (ker(A))^\perp$

We first apply this Fundamental Lemma to the operator

$$\mathcal{L}_0 : (H_0^2(\Omega))^m \to (L_2(\Omega))^m$$

where $(H_0^2(\Omega))^m := \{v \in (H^2(\Omega))^m | v|_{\partial\Omega} = 0; \mathcal{B}_\nu v|_{\partial\Omega} = 0\}$ has transpose

$$\mathcal{L}_0^* : (L_2(\Omega))^m \to (H^{-2}(\Omega))^m$$

Note that in this case, by (ii), $(H_{\mathcal{L}_0^*(\Omega)})^m = ker(\mathcal{L}_0^*) = (im(\mathcal{L}_0))^\perp = (\mathcal{L}_0((H_0^2(\Omega))^m))^\perp$ and, by (iii)

$$(L^2(\Omega))^m = (H_{\mathcal{L}_0^*}(\Omega))^m \oplus \overline{\mathcal{L}_0((H_0^2(\Omega)^m))}^{L^2(\Omega)} \tag{3}$$

Now we define the following space

$$(H^2_{D,N}(\Omega))^m := \{v \in (H^2(\Omega))^m | v|_{\Gamma_D} = 0; \mathcal{B}|_{\Gamma_N} v = 0\}$$

where $\Gamma_D \subset \partial\Omega$ and $\Gamma_N \subset \partial\Omega$ are arbitrary. Note that $H^2_0(\Omega))^m \subset (H^2_{D,N}(\Omega))^m$

The operator $\mathcal{L}_{0,D,N} : (H^2_{D,N}(\Omega))^m \to (L^2(\Omega))^m$ has transpose $\mathcal{L}^*_{0,D^c,N^c} : (L^2(\Omega))^m \to ((H^2_{D,N}(\Omega))^m)^* = (H^{-2}_{D^c,N^c}(\Omega))^m \subset (H^{-2}(\Omega))^m$ where $(H^{-2}_{D^c,N^c}(\Omega))^m$ is a set of distribution with trace support in $\partial\Omega\backslash\Gamma_D$ and conormal trace with support in $\partial\Omega\backslash\Gamma_N$ and whose kernel is

$$(H_{\mathcal{L}^*_{0,D^c,N^c}}(\Omega))^m = \{v \in (L^2(\Omega))^m | \mathcal{L}^*_{0,D^c,N^c} v = 0\}$$

Note that in this case, by (ii), $(H_{\mathcal{L}^*_{0,D^c,N^c}(\Omega)})^m = ker(\mathcal{L}^*_{0,D^c,N^c}) = (im(\mathcal{L}_{0,D,N}))^\perp = (\mathcal{L}_0((H^2_{D,N}(\Omega)))^m)^\perp$ and, by (iii)

$$(L^2(\Omega))^m = (H_{\mathcal{L}^*_{0,D^c,N^c}}(\Omega))^m \oplus \overline{\mathcal{L}_{0,D,N}((H^2_{D,N}(\Omega))^m)}^{L^2(\Omega)} \qquad (4)$$

Note that for $\Gamma_D = \Gamma_N = \Gamma$, decomposition (4) reduces to (3). Also that when Γ_D and Γ_N are a Lipschitz dissection of $\partial\Omega$, the fact that the unique solution v_χ of

$$P^{c*}_{\chi,0,0} \begin{cases} \mathcal{L}^*_{0,D^c,N^c} v = \chi & \text{if } x \in \Omega; \\ \gamma[v] = 0 & \text{if } x \in \partial\Omega_N = \partial\Omega \backslash \Gamma_D; \\ \tilde{\mathcal{B}_\nu} v = 0 & \text{if } x \in \partial\Omega_D = \partial\Omega \backslash \Gamma_D; \end{cases} \qquad (5)$$

is the trivial when $\chi(x) = 0$,and $(H_{\mathcal{L}^*_{0,D^c,N^c}}(\Omega))^m = (\{0\})^m$

- Characterizes materials parameters and source is a central question in the engineering project;
- it is important adequate existing engineering and multiphysics software to handle uncertainties in these parameters;
- be used as a tool for process experimental data;
- but respecting the actual engineering project project status of art.
- Applications when we have incomplete information about these coefficient and sources.

- This work is addressed to investigate the class of problems in which we want determine unknown parameters in the functions that characterize these coefficients and sources.
- To compensate this incomplete information that ill-posed the problem, we suppose that both, Neumann and Dirichlet data, are prescribed for *many boundary value problems.*
- *These problems are formulated for the same physical coefficients and source which depend on the same set of unknown parameters.*

- $\mathcal{L}u = -\sum_{j=1}^{d}(\sum_{k=1}^{d} \partial_j(A_{jk}\partial_k)u + A_j\partial_j u) + Au$
- $(A_{jk}, A_j, A) : \Omega \to \mathbb{R}^{m\times m}$.
- u is a column vector with m scalar fields and $\mathcal{L}u : \Omega \to \mathbb{R}^m$
- strongly elliptic system.

$$\mathcal{L}_0 u = -\sum_{j=1}^{d} \partial_j \mathcal{B}_j u \text{ where } \mathcal{B}_j = \sum_{k=1}^{d} A_{jk}\partial_k \tag{6}$$

- Ω is a Lipschitz domain and γ is the trace operator
- the conormal derivative is

$$\mathcal{B}_\nu u = \sum_{j=1}^{d} \nu_j \gamma[\mathcal{B}_j u] \tag{7}$$

Let Ω a domain with Lipschitz dissection boundary $\partial\Omega = \partial\Omega_N \cup \Pi \cup \partial\Omega_N$. The mixed boundary value problem for the physical model given by (1) is given by the well posed problem P_{f,g_D,g_N} : To find $u \in H^1(\Omega)^m$ such that

$$P_{f,g_D,g_N} \begin{cases} \mathcal{L}u = f & \text{if } x \in \Omega; \\ \gamma[u] = g_D & \text{if } x \in \partial\Omega_D; \\ \mathcal{B}_\nu u = g_N & \text{if } x \in \partial\Omega_N; \end{cases} \tag{8}$$

we can show that (8) has the following weak formulation W_{f,g_D,g_N}

$$\begin{cases} (\mathcal{L}u, v)_\Omega + (\mathcal{B}_\nu u, \gamma[v])_{\partial\Omega} = \Phi(u, v) = & \\ = (f, v)_\Omega + (g_N, \gamma[v])_{\partial\Omega_N} & \text{if } v \in H^1_D(\Omega)^m; \\ \gamma[u] = g_D & \text{if } x \in \partial\Omega_D; \end{cases} \tag{9}$$

Definition

When $u = u^+ + u^- \in L^2(\mathbb{R}^d)^m$, with $u^\pm \in H^1(\Omega^\pm)^m$, has compact support in $\mathbb{R}^d$ and $f = f^+ + f^- \in H^{-1}(\mathbb{R}^d)^m$, we can enunciate the *Third Green Identity*

$$u = \mathcal{G}f + DL[u]_\Gamma - SL[\mathcal{B}_\nu u]_\Gamma \text{ on } \mathbb{R}^d. \tag{10}$$

Definition

When the mixed boundary value problem is posed with a non null source, P_{f,g^D,g_ν^N}, we have a gap in the Calderón projector:

$$\begin{bmatrix} \gamma u(\xi) \\ \mathcal{B}_\nu u(\xi) \end{bmatrix} = \begin{bmatrix} \int_\Omega \gamma_\xi[G_\xi](y)f(y)dy \; \int_\Omega \mathcal{B}_{\nu_\xi}[G_\xi](y)f(y)dy \end{bmatrix} +$$

$$\begin{bmatrix} \frac{1}{2}(I_{x\to\xi} - T_{x\to\xi}) & S_{x\to\xi} \\ R_{x\to\xi} & \frac{1}{2}(I_{x\to\xi} + T_{x\to\xi}) \end{bmatrix} \begin{bmatrix} \gamma u(\xi) \\ \mathcal{B}_\nu u(\xi) \end{bmatrix}, \; (x,\xi) \in \Gamma \times \Gamma$$

Matrix equation for Calderón Projector Gap Lipschtiz Boundary Dissection:

$$\begin{bmatrix} \gamma u(\xi)|_{\Gamma_D} \\ \gamma u(\xi)|_{\Gamma_N} \\ \mathcal{B}_\nu u(\xi)|_{\Gamma_D} \\ \mathcal{B}_\nu u(\xi)|_{\Gamma_N} \end{bmatrix} = \begin{bmatrix} \int_\Omega \gamma_\xi G_\xi|_{\Gamma_D}(y)f(y)dy \\ \int_\Omega \gamma_x i G_\xi|_{\Gamma_N}(y)f(y)dy \\ \int_\Omega \mathcal{B}_{\nu_\xi} G_\xi|_{\Gamma_D}(y)f(y)dy \\ \int_\Omega \mathcal{B}_{\nu_\xi} G_\xi|_{\Gamma_N}(y)f(y)dy \end{bmatrix} +$$

$$\left[\begin{matrix} \frac{1}{2}(I^{DD}_{x\to\xi} - T^{DD}_{x\to\xi}) & -T^{ND}_{x\to\xi} & S^{DD}_{x\to\xi} & S^{ND}_{x\to\xi} \\ -T^{DN}_{x\to\xi} & \frac{1}{2}(I^{NN}_{x\to\xi} - T^{NxN}_{x\to\xi}) & S^{DN}_{x\to\xi} & S^{NN}_{x\to\xi} \\ R^{DD}_{x\to\xi} & R^{ND}_{x\to\xi} & \frac{1}{2}(I^{DD}_{x\to\xi} + \tilde{T}^{*DD}_{x\to\xi}) & \tilde{T}^{*ND}_{x\to\xi} \\ R^{DN}_{x\to\xi} & R^{NN}_{x\to\xi} & \tilde{T}^{*DN}_{x\to\xi} & \frac{1}{2}(I^{NN}_{x\to\xi} + \tilde{T}^{*NN}_{x\to\xi} \end{matrix} \right.$$

Lemma

For a given association of a Lipschitz domain with a source distribution, the Calderón projector gap is as a restriction which the Cauchy data must satisfy in order to be a consistent data with boundary value problems.

Definition

Let us consider two mixed boundary value problems P_{f_I,g^I,g^I_ν} and $P_{f_{II},g^{II},g^{II}_\nu}$ defined on the same Lipschitz domain Ω. We say that these problems are complementary if $f_I = f_{II}$, $\Gamma^I_D = \Gamma^{II}_N$, $\Gamma^{II}_D = \Gamma^I_N$ and there exist a Cauchy data (g, g_ν) such that

$$g^I = g\chi_{\Gamma^I_D} \text{ and } g^{II} = g\chi_{\Gamma^{II}_D}.$$

$$g^I_\nu = g_\nu\chi_{\Gamma^I_D} \text{ and } g^{II}_\nu = g_\nu\chi_{\Gamma^{II}_D}.$$

Theorem

Suppose that two mixed boundary value problems P_{f_I, g^I, g_ν^I} and $P_{f_{II}, g^{II}, g_\nu^{II}}$ has solutions u_I and u_{II}, respectively. If they are complementary, then

$$u_I = u_{II}.$$

Proof:

$$g(x) = g^I(x)\chi_{\Gamma_D^I}(x) + g^{II}(x)\chi_{\Gamma_N^I}(x) = g^I(x)\chi_{\Gamma_N^{II}}(x) + g^{II}(x)\chi_{\Gamma_D^{II}}(x)$$

and

$$g_\nu(x) = g_\nu^I(x)\chi_{\Gamma_D^I}(x) + g_\nu^{II}(x)\chi_{\Gamma_N^I}(x) = g_\nu^I(x)\chi_{\Gamma_N^{II}}(x) + g_\nu^{II}(x)\chi_{\Gamma_D^{II}}(x).$$

- Denoting $f = fI = f_{II}$,
- the solution will be, via boundary integral equation method,

$$u(x) = \int_{\Omega} G_{\xi}(x) f(x) dx - DL[g](x) + SL[g_{\nu}](x).$$

- By taking the trace and the conormal trace, we see that it satisfies the Calderón gap projection dissection equation.
- So, Cauchy data obtained by the extension formulates a unique problem with integral representation.

- Unknowns parameters related with the support of inclusions inside Ω where the coefficient has some different functional description, or even with the functional description itself.
- We consider these parameters collected in a parameter vector $\alpha = [\alpha_1, \alpha_2, ..., \alpha_{NA}]^T$ and that the coefficients and source are represented as

$$\{A_{jk} := [a^{jk}_{pq}(\alpha, x) \ , \ A_j := a^j_{pq}(\alpha, x) \ , \ A := [a(\alpha, x)] \ , \ f(\alpha, x)\},$$

- which for $p, q = 1, ..., m$ are functions from $\Omega \times [\beta_1, \beta_2]$ into $\mathbb{R}^{m\times m}$.

- $\alpha \in [\beta_1, \beta_2] \in \mathbb{R}^{NA}$.
- the strongly elliptic operator model with parameter dependence are formally written as:

$$\mathcal{L}_\alpha u = f_\alpha \ , \ x \in \Omega \text{ and } \alpha \in [\alpha_1, \alpha_2].$$

- The indeterminacy of α is compensate with the over prescription of boundary conditions: For $p = 1, ..., NP$,

$$\gamma u^{(p)} = g^{(p)} \text{ and } \mathcal{B}_\nu u^{(p)} = g_\nu^{(p)}.$$

- Steklov-Poincaré operator, which is an extended definition of the Dirichlet to Neumann map for this kind of system, at some points in the trace space

$$(\gamma u^{(p)}, \mathcal{B}_\nu u^{(p)}) = (g^{(p)}, g_\nu^{(p)}).$$

- This set of (NP) Cauchy data fully prescribed at the boundary can be used to formulated a non unique set with $2(NP)$ well posed direct problems by using some Lipschitz of the Boundary Γ.

- Choose some Lipschitz dissection of Γ associated with problem (p) and given by

$$\Gamma = \Gamma_I^{(p)} \cup \Pi^{(p)} \cup \Gamma_{II}^{(p)},$$

- $\Gamma_I^{(p)}$ and $\Gamma_{II}^{(p)}$ are disjoint, eventually-empty or non connected, relatively open subsets of Γ, having $\Pi^{(p)}$ as their common boundary.
- Consider also the restriction for Cauchy data for problem (p) associated with this partition:

$$\begin{cases} (g^{I(p)}, g_\nu^{I(p)}) = (\gamma u^{(p)}, \mathcal{B}_\nu u^{(p)})|_{\Gamma_I} = (g^{(p)}|_{\Gamma_I}, g_\nu^{(p)}|_{\Gamma_I}) \\ (g^{II(p)}, g_\nu^{II(p)}) = (\gamma u^{(p)}, \mathcal{B}_\nu u^{(p)})|_{\Gamma_{II}} = (g^{(p)}|_{\Gamma_{II}}, g_\nu^{(p)}|_{\Gamma_{II}}). \end{cases}$$

- For each one of these Cauchy data of the Lipschitz dissection,
- we can formulate two complementaries well posed sets of mixed boundary values problems, respectively,

$$P^{\alpha}_{f_\alpha, g_I^{(p)}, g_{II\nu}^{(p)}} \text{ and } P^{\alpha}_{f_\alpha, g_{II}^{(p)}, g_{I\nu}^{(p)}}$$

- Given a guess set of parameter α, for problems $p = 1, ..., NP$, to find complementary solutions $u_I^{(p)}$ and $u_{II}^{(p)}$ such that

$$\begin{cases} u_I^{(p)} \text{ solution of } P^{\alpha}_{f_\alpha, g_I^{(p)}, g_{II\nu}^{(p)}} \\ u_{II}^{(p)} \text{ solution of } P^{\alpha}_{f_\alpha, g_{II}^{(p)}, g_{I\nu}^{(p)}}. \end{cases} \tag{11}$$

- The partition done with the Lipschitz dissection is arbitrary.
- It can also be different for different problems in the many boundary values problems set.
- If necessary, we can do two or more dissection for the same problem.
- The correctness of this procedure will depends on the information about the parameters that it produces.
- One basic rule of thumb is that partitions must be chosen in a way to avoid the guess direct problems to be non well posed.

Lemma

Suppose that in the model given by operator $\mathcal{L}_\alpha$ and source f_α, characterized by the parameter set α, the associated Cauchy boundary data are given by

$$\gamma u^{(p)} = g^{(p)} \text{ and } \mathcal{B}_\nu u^{(p)} = g_\nu^{(p)}, \; forp = 1, ..., NP.$$

If for some p and for some Lipschitz dissection we have $u_I^{(p)}$ and $u_{II}^{(p)}$ solutions of problem $P^\alpha_{f_\alpha, g_I^{(p)}, g_{II\nu}^{(p)}}$ and $P^\alpha_{f_\alpha, g_{II}^{(p)}, g_{I\nu}^{(p)}}$, then

$$u_I^{(p)} = u_{II}^{(p)}.$$

- The idea now is explore the fact that these two set of solutions indexed by I and II must be, under ideal conditions, equal for each problem (p), as has been stated in Theorem of Complementary solutions and Lemma on Solution with consistent Cauchy data
- and create some discrepancy function that measures observed differences for guess value of the parameters.
- The sup norm in the solution space for the direct problems can be adopted as measures

$$d(\alpha, u_I, u_{II}) = \max_{p=1}^{NP} \sup_{x \in \Omega)} |u_I^{(p)}(\alpha, x) - u_{II}^{(p)}(\alpha, x)|,$$

- Optimization problem: In the guess set of parameters $\alpha \in \{[\alpha_1, \alpha_2] \subset \mathbb{R}^{NA}\}$, to find $\overline{\alpha}$ that minimizes the discrepancy between Lipschitz dissected solutions.
- Use of solvers based on finite elements method,...etc.
- Of course, the boundary integral methodology or the Green's function methodologies can also be used, but this is not the more conventional procedure.
- From computational point of view, minimization of the discrepancy functional can be easily implemented if the algorithm does not require the computations of gradients of the solution with respect to the parameters.
- Nelder-Mead Simplex method in low dimensions.

Lemma

Suppose a model given by $\mathcal{L}_\alpha$*,* f_α *and* $(\gamma[u], \mathcal{B}_\nu[u]) = (g, g_\nu)$ *dissected according the Lipschitz dissection. Then, if for* $\alpha = \alpha^{(0)}$ *presents a discrepancy (* $u^I_{\alpha^{(0)}} \neq u^{II}_{\alpha^{(0)}}$*), then for any* α *near* $\alpha^{(0)}$*,*

$$\int \chi(x)(u^{II}_\alpha(x) - u^I_\alpha(x))dx = \mathcal{H} + \int_\Omega (v^{II}_0(x) - v^I_0(x)) f_0(x, \alpha^{(0)})dx +$$

$$\sum_{i=1}^{NA}(\alpha_i - \alpha_i^{(0)}) \int_\Omega (v^{II}_0(x)\mathcal{L}_\alpha \frac{\partial u^{II}_0}{\partial \alpha_i} - v^I_0(x)\mathcal{L}_\alpha \frac{\partial u^I_0}{\partial \alpha_i})dx + O(\|\alpha_i - \alpha_i^{(0)}\|^2)$$

where v^I_0*,* v^{II}_0 *are solutions of boundary complementary adjoint homogeneous auxiliary problems in th sence of Lipschtz disection with mass one source* $\chi \in L^2(\Omega)$ *and* $\mathcal{H}$ *depends on Cauchy data.*

- The numerical experiment that illustrate this work is a model in which the square $(-1, +1) \times (-1, +1)$
- has in its interior a small rectangle with has unknown center, unknown edges a and b,
- which supports unknown parameters related with the conductivity, c, the potential, a, and the source intensity, f.
- Cauchy data are synthetically generated with a problem in which parameters value are known equal to 1 in the exterior of the small rectangle, and all equal 2 in the interior.
- Also the unknown information about the rectangle used are center at the origin and side $a = b = .2$.

The parameters in operator $\mathcal{L}$ are:

$$m = 1 \; ;$$

$$A_{jk} = 1 + (c-1)\chi(x_0 - \frac{1}{2}x_1, x_0 + \frac{1}{2}x_1)\chi(y_0 - \frac{1}{2}y_1, y_0 + \frac{1}{2}y_1)\delta_{jk} \; ;$$

$$A_j = 0 \; ;$$

$$A = 1 + (a-1)\chi(x_0 - \frac{1}{2}x_1, x_0 + \frac{1}{2}x_1)\chi(y_0 - \frac{1}{2}y_1, y_0 + \frac{1}{2}y_1)\delta_{jk} \; ;$$

$$f(x) = 1 + (f-1)\chi(x_0 - \frac{1}{2}x_1, x_0 + \frac{1}{2}x_1)\chi(y_0 - \frac{1}{2}y_1, y_0 + \frac{1}{2}y_1)\delta_{jk} \; ;$$

and the set of parameters are $\alpha = (x_0, y_0, x_1, y_1, c, a, f) \in \mathbb{R}^7$.

-

  $$\Gamma = \Gamma_{y=-1} \cup \Pi_{(-1,+1)} \cup \Gamma_{x=+1} \cup \Pi_{(+1,+1)} \cup \Gamma_{y=+1} \cup \Pi_{(-1,+1)} \cup \Gamma_{x=-1} \cup \Pi_{(}$$

  and is counterclockwise oriented.
- Cauchy data are synthetically produced by solving a set of 5 Dirichlet direct problems
- with parameters $\alpha = (0, 0, 1, 1, 2, 2, 2) \in \mathbb{R}^7$ with quadratic Lagrange finite elements method.

(0) $g|_{\Gamma_{y=-1}} = 0$; $g|_{\Gamma_{x=+1}} = 0$; $g|_{\Gamma_{y=+1}} = 0$; $g|_{\Gamma_{x=-1}} = 0$;

(1) $g|_{\Gamma_{y=-1}} = (1-x)(1+x)$; $g|_{\Gamma_{x=+1}} = 0$; $g|_{\Gamma_{y=+1}} = 0$; $g|_{\Gamma_{x=-1}} = 0$;

(2) $g|_{\Gamma_{y=-1}} = 0$; $g|_{\Gamma_{x=+1}} = (1-y)(1+y)$; $g|_{\Gamma_{y=+1}} = 0$; $g|_{\Gamma_{x=-1}} = 0$;

(3) $g|_{\Gamma_{y=-1}} = 0$; $g|_{\Gamma_{x=+1}} = 0$; $g|_{\Gamma_{y=+1}} = (1-x)(1+x)$; $g|_{\Gamma_{x=-1}} = 0$;

(4) $g|_{\Gamma_{y=-1}} = 0$; $g|_{\Gamma_{x=+1}} = 0$; $g|_{\Gamma_{y=+1}} = 0$; $g|_{\Gamma_{x=-1}} = (1-y)(1+y)$.

generating Neumann data
$\{g_\nu^{(p)}|_{\Gamma_{y=-1}}$; $g_\nu^{(p)}|_{\Gamma_{x=+1}}$; $g_\nu^{(p)}|_{\Gamma_{y=+1}}$; $g_\nu^{(p)}|_{\Gamma_{x=-1}}$; $p = 0, 1, 2, 3, 4\}$.

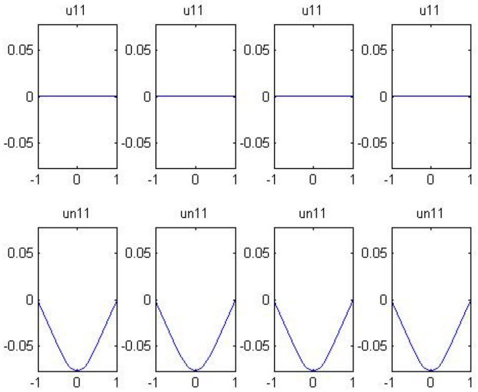


Figure : Cauchy Data Dirichelet problems (0)

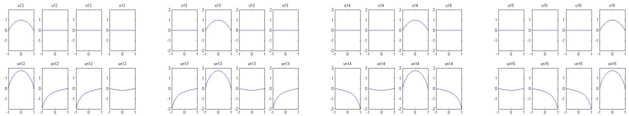


Figure : Cauchy Data Dirichelet problems (1),(2),(3),(4)

They has been interpolated with piecewise cubic splines and used in the inverse algorithm. The Boundary of the square has been dissected in two non connected parts composed by

$$\Gamma_I = \Gamma_{y=-1} \cup \Gamma_{y=+1} \text{ and } \Gamma_{II} = \Gamma_{x=-1} \cup \Gamma_{x=+1}.$$

- Ten problems formulated with the dissection of these Cauchy data can now be used to evaluate
- the discrepancy functional based on the following sup norm:

$$d(\alpha, U_I, U_{II}) = max_{p=1}^{NP}(\sup_{x \in \Omega} |u_I^{(p)}(\alpha, x) - u_{II}^{(p)}(\alpha, x)|).$$

- The search starts with random generated initial data in the intervals $[0,2]^7$ for the 7 unknown parameters

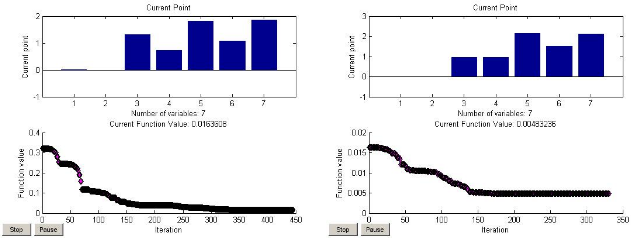


Figure : Iterative simultaneous reconstruction of rectangle shape, conductivity, absorption and source

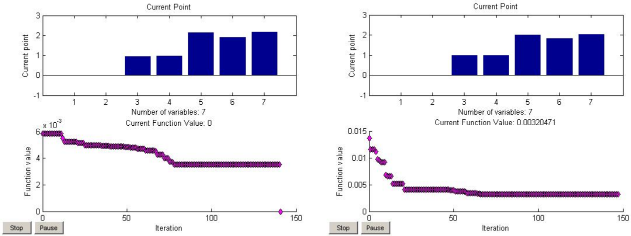


Figure : Iterative simultaneous reconstruction of rectangle shape, conductivity, absorption and source

- We proposed a methodology for reconstruction of unknown parameters associated with coefficients and source in strongly elliptic system.
- To make it clear, we also introduce the most important mathematical concepts involved in the solution of the strongly elliptic problem with integral equations at the boundary of a Lipschitz domain.
- In the inverse problem, the existence of the unknown parameters is compensate with the prescription of many Cauchy data related experimentally with the same set of parameters.

- We demonstrate that a discrepancy functional depending on the the parameters must be minimized in order to be consistent with the given Cauchy data.
- The main ideas used to develop this formulation are Lipschitz Dissection and Calderón Projector Gap.
- In this first work, the optimization methodology is numerically investigate with non differentiable Nelder-Mead search algorithm.
- Numerical results are presented to illustrate the ideas. Further research involving differentiability and the use of differentiable algorithms are currently been investigated.

- square $(-1,+1)\times(-1,+1)$ has in its interior sources with intensity equal 1
- and supported on two circles with unknown center and radius (x_{c1}, y_{c1}, R_{c1}) and (x_{c2}, y_{c2}, R_{c2}), respectively.
- Laplace. Even for this very simple problem there is no mathematical proof of uniqueness of reconstruction from boundary data.
- finite elements method can avoid Green's function.
- Nelder-Mead Simplex Method with random generated initial data

The Finite elements solution used for produce Cauchy data

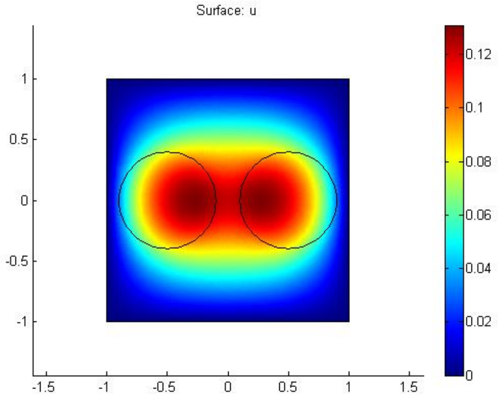


Figure : Two circle inside an square model

	x_{c1}	y_{c1}	R_{c1}	x_{c2}	y_{c2}	R_{c2}	AREA
exact	-.5	0	.2	0.5	0	.2	.253
initial	-.4	0	.3	.4	0	.3	—
final	-.5127	-.0006	.1982	.5384	-.0000	.2103	.2644

Table : Two Circles Inside a square reconstructed as it is.

Reconstructed values after 140 iter info existence two sources

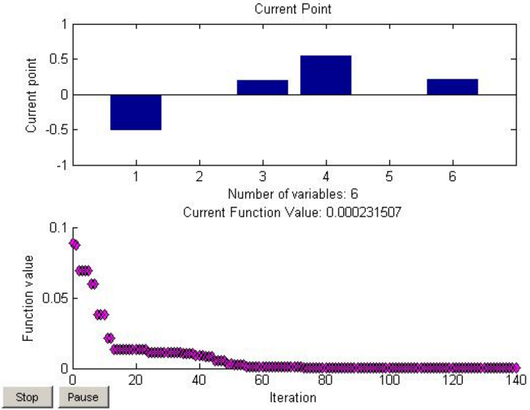


Figure : Convergence results for the two circles inside a square model.

	x_c	y_c	R_c	AREA
initial	.00005	.0005	.3	.2513
final	0.0001	.0005	.2840	.2534
initial	.5	.5	.3	.2513
final	-.2847	.0065	.2517	.1991
initial	-.1	.1	.3	.3
final	-.2774	-..0019	.2517	.1991
initial	-.05	.05	.3	.3
final	-..3109	.00500	.2457	.1897

Table : Two circle inside a square reconstructed as one circle inside a square.

Two squares inside a square

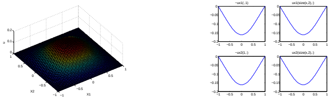
The Fourier series has been truncated with 100 and the number of collocations points is 40.

Table : Characteristic source dimensions

Type of source	a_1	b_1	a_2	b_2
Exact source	-0.5000	+0.5000	-0.5000	+0.5000
Random source	-0.4074	+0.0635	-0.4529	+0.4576
Recosntructed 80 iter	-0.4975	+0.0962	-0.7446	+0.9357
Reconstructed 286 iter	-0.4970	+0.4970	-0.4968	+0.4968

- The solution of the Dirichelet problem and its Neumann data.
- The discrepancy functional with exact source parameters, is $4.2047e-006$. Convergence is quite satisfactory when the Discrepancy become close to this value for 286 iterations.

Figure : Homogeneous Dirichlet model solution with boundary data.



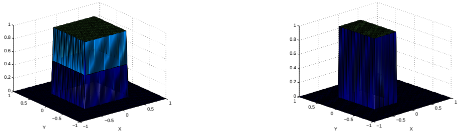


Figure : Exact and Random generate source support.

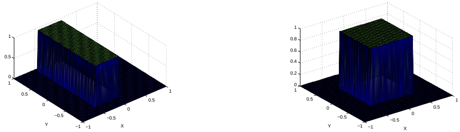


Figure : Reconstructed source for 80 and 286 iterations.

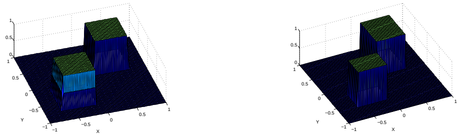


Figure : Exact and Random generate source biconnected support.

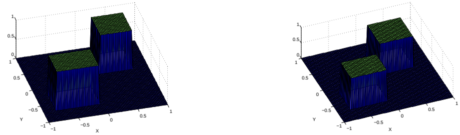


Figure : Reconstructed biconnected source for 82 and 675 iterations.

- The non injective behaviour of the inverse problem of source reconstruction by using only boundary data is given by the non one to one behaviour of the normal trace induced by the elliptic operator L on the domain boundary;
- In the moment, the uniqueness can be proved for some special class, such as, the regular affine, the characteristic and the distributional monopole or dipole sources;
- One experiment shown that at least for one kind of source with non connected support the reconstruction can be done successfully;
- If the support are not supposed has at least two connected components, the reconstruction fails;
- The introduced method of split solutions seen to be very promising.

We recognize that more mathematical analysis and computational experiments can be done. Also, some discussion about the statistical meaning of the concepts here introduced are not presented. These topics will postpone to be presented in future works.

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