

Fighting ambiguity of inverse problems in seismic imaging

Jörg Schleicher

University of Campinas & INCT-GP

Colóquio Brasileiro de Matemática
Rio de Janeiro, 30/07/2013

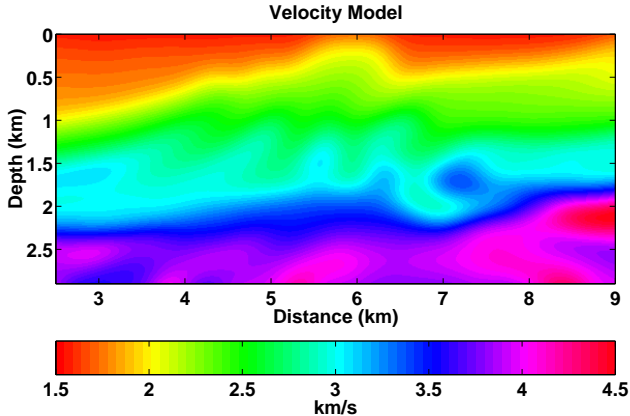
Contents

- 1 Introduction
- 2 Reflector-oriented regularization in slope tomography
- 3 Decomposition of sensitivity kernels in full-waveform inversion
- 4 Conclusions

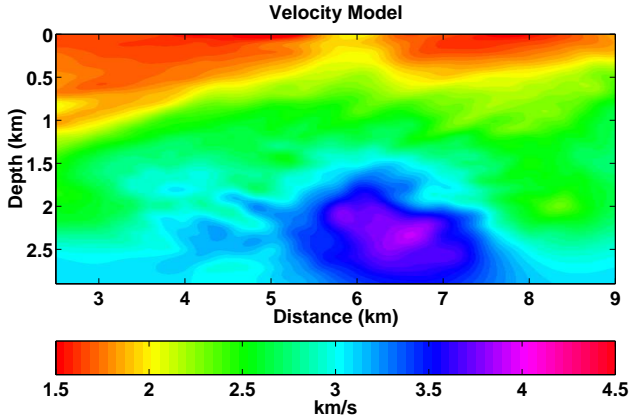
Contents

- 1 Introduction
- 2 Reflector-oriented regularization in slope tomography
- 3 Decomposition of sensitivity kernels in full-waveform inversion
- 4 Conclusions

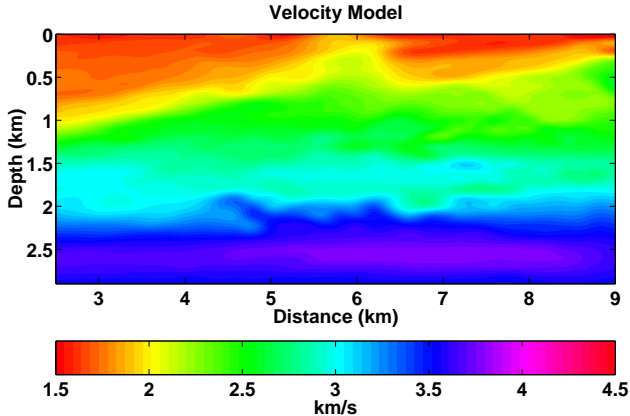
Motivation



Motivation



Motivation



Objectives

1 Slope tomography:

- Introduce geologically meaningful constraints
- Improve velocity model building for depth migration
- Better recover large scale structural features
- Improve convergence of layer and grid-based tomography

2 Full-waveform inversion:

- Decompose sensitivity kernels
- Understand contributions
- Invert only important ones

Objectives

- 1 Slope tomography:
 - Introduce geologically meaningful constraints
 - Improve velocity model building for depth migration
 - Better recover large scale structural features
 - Improve convergence of layer and grid-based tomography
- 2 **Full-waveform inversion:**
 - Decompose sensitivity kernels
 - Understand contributions
 - Invert only important ones

Inversion

Problem: invert

Nonlinear relationship between data and parameters

$$\mathbf{d} = \mathbf{F}(\mathbf{m})$$

$\mathbf{m} \equiv$ model parameters

$\mathbf{d} \equiv$ data parameters

$\mathbf{F} \equiv$ nonlinear functional (wave propagation)

Frechét derivatives

Solution:

Linear iterations: the Frechét derivative

$$\delta \mathbf{d} = \mathcal{D}\mathbf{F}(\mathbf{m}_0)\delta \mathbf{m}$$

$\mathbf{m}_0 \equiv$ reference model parameters

$\delta \mathbf{d} \equiv$ data perturbation

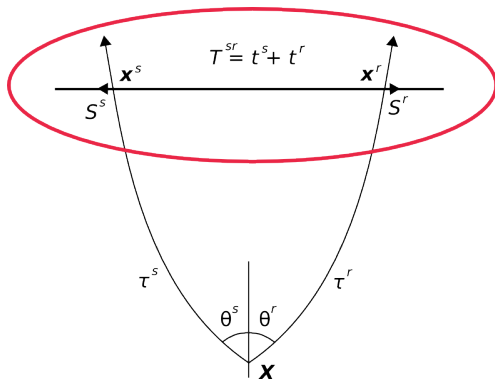
$\delta \mathbf{m} \equiv$ model parameters perturbations around \mathbf{m}_0

$\mathcal{D}\mathbf{F} \equiv$ Frechét derivative of \mathbf{F}

Contents

- 1 Introduction
- 2 Reflector-oriented regularization in slope tomography
- 3 Decomposition of sensitivity kernels in full-waveform inversion
- 4 Conclusions

What is slope tomography?

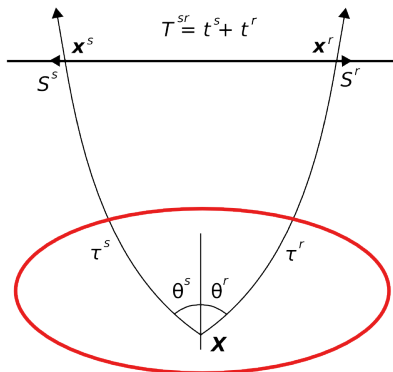


Data space:

$$\mathbf{d} = \{(\mathbf{x}^s, \mathbf{x}^r, T^{sr}, \mathbf{s}^s, \mathbf{s}^r)_n\}$$

for $n = 1, \dots, N$

What is slope tomography?



Model space:

$$\mathbf{m} = \{\mathbf{p}, (\mathbf{X}, \tau^s, \tau^r, \theta^s, \theta^r)_n\}$$

for $n = 1, \dots, N$

Frechét derivatives computation: dynamic ray tracing

The Hamiltonian (eikonal equation):

$$\mathcal{H}(\mathbf{x}, \mathbf{s}) = \frac{1}{2} (p(\mathbf{x})\mathbf{s} \cdot \mathbf{s} - 1) = 0$$

\mathbf{x} - position along the ray

\mathbf{s} - slowness vector along the ray

$p(\mathbf{x})$ - velocity square field

Frechét derivatives computation: dynamic ray tracing

$$\begin{aligned}
 \frac{d}{d\tau} \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} &= \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{H} \\ -\nabla_{\mathbf{s}} \mathcal{H} \end{bmatrix} \\
 \frac{d}{d\tau} \begin{bmatrix} \delta \mathbf{x} \\ \delta \mathbf{s} \end{bmatrix} &= \begin{bmatrix} \nabla_{\mathbf{s}} \nabla_{\mathbf{x}}^T \mathcal{H} & \nabla_{\mathbf{s}} \nabla_{\mathbf{s}}^T \mathcal{H} \\ -\nabla_{\mathbf{x}} \nabla_{\mathbf{x}}^T \mathcal{H} & -\nabla_{\mathbf{x}} \nabla_{\mathbf{s}}^T \mathcal{H} \end{bmatrix} \begin{bmatrix} \delta \mathbf{x} \\ \delta \mathbf{s} \end{bmatrix} \\
 &\quad + \begin{bmatrix} \nabla_{\mathbf{s}} \nabla_{\mathbf{p}}^T \mathcal{H} \delta \mathbf{p} \\ -\nabla_{\mathbf{x}} (\nabla_{\mathbf{p}}^T \mathcal{H} \delta \mathbf{p}) \end{bmatrix}.
 \end{aligned}$$

Reference ray

Frechét derivatives computation: dynamic ray tracing

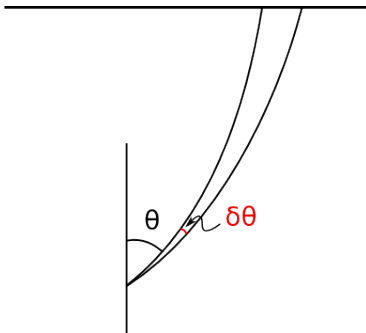
$$\begin{aligned}
 \frac{d}{d\tau} \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} &= \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{H} \\ -\nabla_{\mathbf{s}} \mathcal{H} \end{bmatrix} \\
 \frac{d}{d\tau} \begin{bmatrix} \delta \mathbf{x} \\ \delta \mathbf{s} \end{bmatrix} &= \begin{bmatrix} \nabla_{\mathbf{s}} \nabla_{\mathbf{x}}^T \mathcal{H} & \nabla_{\mathbf{s}} \nabla_{\mathbf{s}}^T \mathcal{H} \\ -\nabla_{\mathbf{x}} \nabla_{\mathbf{x}}^T \mathcal{H} & -\nabla_{\mathbf{x}} \nabla_{\mathbf{s}}^T \mathcal{H} \end{bmatrix} \begin{bmatrix} \delta \mathbf{x} \\ \delta \mathbf{s} \end{bmatrix} \\
 &\quad + \begin{bmatrix} \nabla_{\mathbf{s}} \nabla_{\mathbf{p}}^T \mathcal{H} \delta \mathbf{p} \\ -\nabla_{\mathbf{x}} (\nabla_{\mathbf{p}}^T \mathcal{H} \delta \mathbf{p}) \end{bmatrix}.
 \end{aligned}$$

Paraxial rays

Initial Conditions

Slowness Direction:

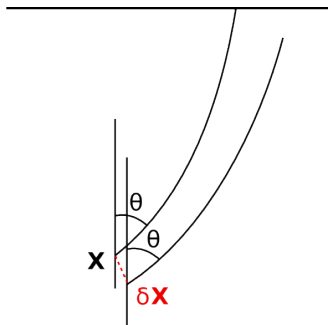
$$\delta \mathbf{x} = \mathbf{0} \quad \text{and} \quad \delta \mathbf{s} = s \left(\mathbf{I} - \frac{\mathbf{n} \nabla_{\mathbf{s}}^T \mathcal{H}}{\nabla_{\mathbf{s}}^T \mathcal{H} \mathbf{n}} \right) \frac{d\mathbf{n}}{d\theta} \delta \theta$$



Initial Conditions

Scattering point position:

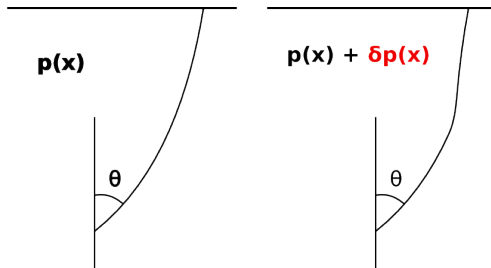
$$\delta \mathbf{x} = \delta \mathbf{X} \quad \text{and} \quad \delta \mathbf{s} = - \frac{\nabla_{\mathbf{s}} \mathcal{H}}{\|\nabla_{\mathbf{s}} \mathcal{H}\|} \frac{\nabla_{\mathbf{x}}^T \mathcal{H} \delta \mathbf{X}}{\|\nabla_{\mathbf{s}} \mathcal{H}\|}$$



Initial Conditions

Velocity model parameters:

$$\delta \mathbf{x} = \mathbf{0} \quad \text{and} \quad \delta \mathbf{s} = - \frac{\nabla_{\mathbf{s}} \mathcal{H}}{\|\nabla_{\mathbf{s}} \mathcal{H}\|} \frac{\nabla_{\mathbf{p}}^T \mathcal{H} \delta \mathbf{p}}{\|\nabla_{\mathbf{s}} \mathcal{H}\|}$$



Linear iterations

- 1- reference model \mathbf{m}_0
- 2- ray tracing is performed to calculate synthetic data (\mathbf{d}^c),
 $\delta\mathbf{d} = \mathbf{d}^{OBS} - \mathbf{d}^c$
- 3- compute Frechét derivatives $\mathcal{D}\mathbf{F}(\mathbf{m}_0)$
- 4- solve for model perturbations $\delta\mathbf{m}$
- 5- Update reference model $\mathbf{m}_0 \leftarrow \mathbf{m}_0 + \delta\mathbf{m}$
- 6- If updated model fits the data within a specified tolerance
stop; otherwise, iterate

Linearized inversion

Estimation of a model consistent with the data:

$$\min_{\mathbf{m}} \|\delta \mathbf{d} - \mathcal{D}\mathbf{F}(\mathbf{m}_0)\delta \mathbf{m}\|_2$$

Linearized inversion

Estimation of a model consistent with the data:

$$\min_{\mathbf{m}} \|\delta \mathbf{d} - \mathcal{D}\mathbf{F}(\mathbf{m}_0)\delta \mathbf{m}\|_2$$

Problem:

There is no unique solution!

Linearized inversion

Estimation of a model consistent with the data:

$$\min_{\mathbf{m}} \|\delta \mathbf{d} - \mathcal{D}\mathbf{F}(\mathbf{m}_0)\delta \mathbf{m}\|_2$$

Problem:

There is no unique solution!

Remedy:

Constrain the solution with additional properties.
Regularization: *smoothness of the velocity field*.

Smoothness constraints

- Minimum curvature constraints
 - Minimize Laplacian
 - Minimize second derivatives independently
- Minimum inhomogeneity constraints
 - Minimize first derivatives independently
 - Minimize directional derivatives along potential reflectors

Slope tomography objective function

$$\begin{aligned}
 \Phi(\mathbf{m}; \lambda_i) = & \|\mathbf{d} - \mathbf{F}(\mathbf{m})\|_2^2 + \lambda_0^2 \|\mathbf{m} - \mathbf{m}_0\|_2^2 \\
 & + \lambda_1^2 \|\mathbf{D}_1 \mathbf{p}\|_2^2 + \lambda_2^2 \|\mathbf{D}_3 \mathbf{p}\|_2^2 \\
 & + \lambda_3^2 \|\mathbf{D}_1^2 \mathbf{p}\|_2^2 + \lambda_4^2 \|\mathbf{D}_3^2 \mathbf{p}\|_2^2 \\
 & + \lambda_5^2 \|(\mathbf{D}_1^2 + \mathbf{D}_3^2) \mathbf{p}\|_2^2 \\
 & + \lambda_6^2 \|\mathbf{D}_r \mathbf{p}\|_2^2
 \end{aligned}$$

Do not get too far from a prior (previous or initial) model

Slope tomography objective function

$$\begin{aligned}\Phi(\mathbf{m}; \lambda_i) = & \|\mathbf{d} - \mathbf{F}(\mathbf{m})\|_2^2 + \lambda_0^2 \|\mathbf{m} - \mathbf{m}_0\|_2^2 \\ & + \lambda_1^2 \|\mathbf{D}_1 \mathbf{p}\|_2^2 + \lambda_2^2 \|\mathbf{D}_3 \mathbf{p}\|_2^2 \\ & + \lambda_3^2 \|\mathbf{D}_1^2 \mathbf{p}\|_2^2 + \lambda_4^2 \|\mathbf{D}_3^2 \mathbf{p}\|_2^2 \\ & + \lambda_5^2 \|(\mathbf{D}_1^2 + \mathbf{D}_3^2) \mathbf{p}\|_2^2 \\ & + \lambda_6^2 \|\mathbf{D}_r \mathbf{p}\|_2^2\end{aligned}$$

Gradient smoothness

Slope tomography objective function

$$\begin{aligned}
 \Phi(\mathbf{m}; \lambda_i) = & \|\mathbf{d} - \mathbf{F}(\mathbf{m})\|_2^2 + \lambda_0^2 \|\mathbf{m} - \mathbf{m}_0\|_2^2 \\
 & + \lambda_1^2 \|\mathbf{D}_1 \mathbf{p}\|_2^2 + \lambda_2^2 \|\mathbf{D}_3 \mathbf{p}\|_2^2 \\
 & + \lambda_3^2 \|\mathbf{D}_1^2 \mathbf{p}\|_2^2 + \lambda_4^2 \|\mathbf{D}_3^2 \mathbf{p}\|_2^2 \\
 & + \lambda_5^2 \|(\mathbf{D}_1^2 + \mathbf{D}_3^2) \mathbf{p}\|_2^2 \\
 & + \lambda_6^2 \|\mathbf{D}_r \mathbf{p}\|_2^2
 \end{aligned}$$

Curvature smoothness

Slope tomography objective function

$$\begin{aligned}
 \Phi(\mathbf{m}; \lambda_i) = & \|\mathbf{d} - \mathbf{F}(\mathbf{m})\|_2^2 + \lambda_0^2 \|\mathbf{m} - \mathbf{m}_0\|_2^2 \\
 & + \lambda_1^2 \|\mathbf{D}_1 \mathbf{p}\|_2^2 + \lambda_2^2 \|\mathbf{D}_3 \mathbf{p}\|_2^2 \\
 & + \lambda_3^2 \|\mathbf{D}_1^2 \mathbf{p}\|_2^2 + \lambda_4^2 \|\mathbf{D}_3^2 \mathbf{p}\|_2^2 \\
 & + \lambda_5^2 \|(\mathbf{D}_1^2 + \mathbf{D}_3^2) \mathbf{p}\|_2^2 \\
 & + \lambda_6^2 \|\mathbf{D}_r \mathbf{p}\|_2^2
 \end{aligned}$$

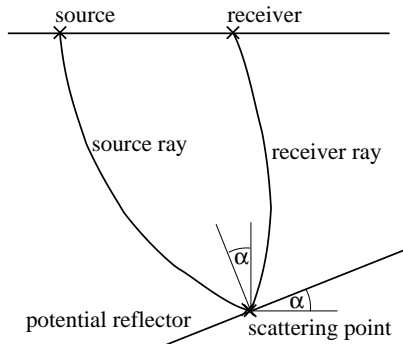
Laplacian isotropic smoothness

Slope tomography objective function

$$\begin{aligned}
 \Phi(\mathbf{m}; \lambda_i) = & \|\mathbf{d} - \mathbf{F}(\mathbf{m})\|_2^2 + \lambda_0^2 \|\mathbf{m} - \mathbf{m}_0\|_2^2 \\
 & + \lambda_1^2 \|\mathbf{D}_1 \mathbf{p}\|_2^2 + \lambda_2^2 \|\mathbf{D}_3 \mathbf{p}\|_2^2 \\
 & + \lambda_3^2 \|\mathbf{D}_1^2 \mathbf{p}\|_2^2 + \lambda_4^2 \|\mathbf{D}_3^2 \mathbf{p}\|_2^2 \\
 & + \lambda_5^2 \|(\mathbf{D}_1^2 + \mathbf{D}_3^2) \mathbf{p}\|_2^2 \\
 & + \lambda_6^2 \|\mathbf{D}_r \mathbf{p}\|_2^2
 \end{aligned}$$

Smoothness along reflectors

Regularization along the reflectors



$D_r p$ operator

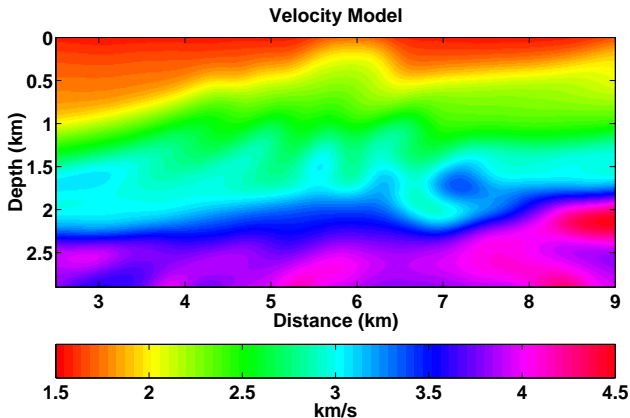
$$\alpha = \frac{\theta_s + \theta_r}{2}$$

$$\mathbf{n}(\alpha; \mathbf{X}) \times \nabla p(\mathbf{X}) = \mathbf{0}$$

Slope tomography linear iterations

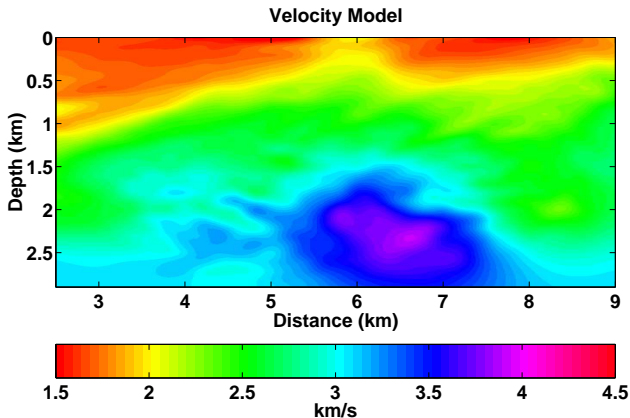
$$\begin{bmatrix} \mathcal{D}\mathbf{F}(\mathbf{m}_0) \\ \lambda_0 \mathbf{I} \\ \lambda_1 \mathbf{D}_1 \\ \lambda_2 \mathbf{D}_3 \\ \lambda_3 \mathbf{D}_1^2 \\ \lambda_4 \mathbf{D}_3^2 \\ \lambda_5 (\mathbf{D}_1^2 + \mathbf{D}_3^2) \\ \lambda_6 \mathbf{D}_r \end{bmatrix} \delta \mathbf{m} = \begin{bmatrix} \delta \mathbf{d} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

Regularization



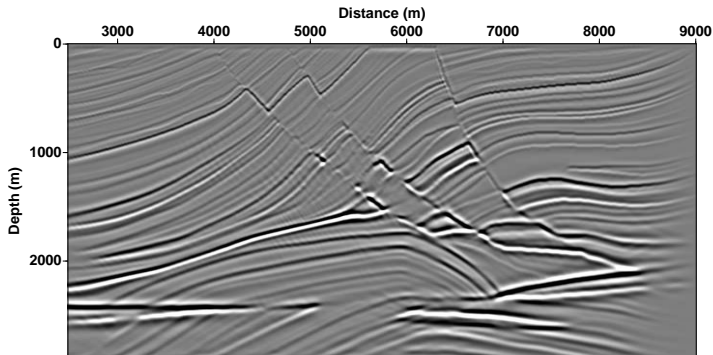
Exact model

Regularization



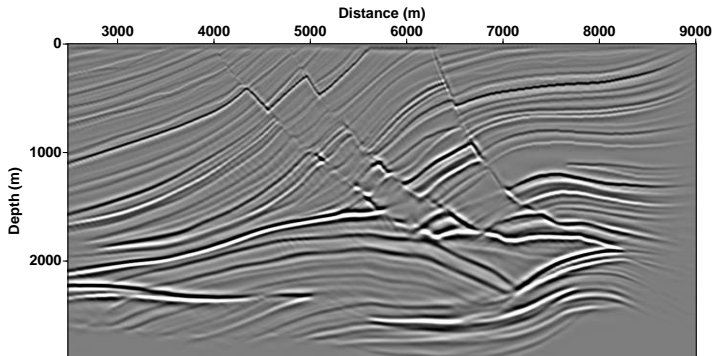
Minimization of Laplacian

Pre-Stack depth migration



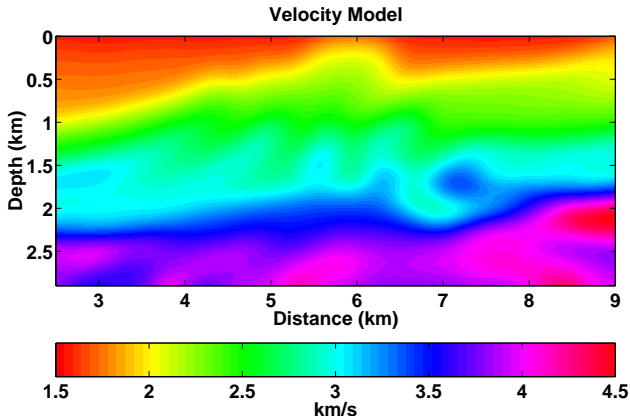
Exact model

Pre-Stack depth migration



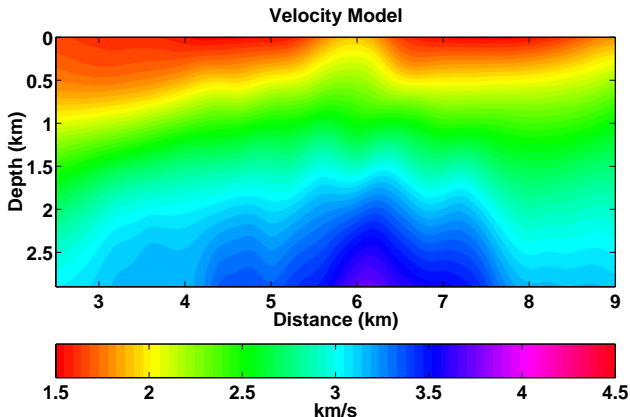
Minimization of Laplacian

Regularization



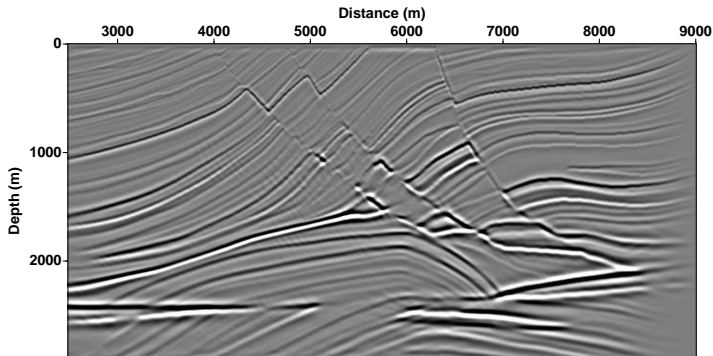
Exact model

Regularization



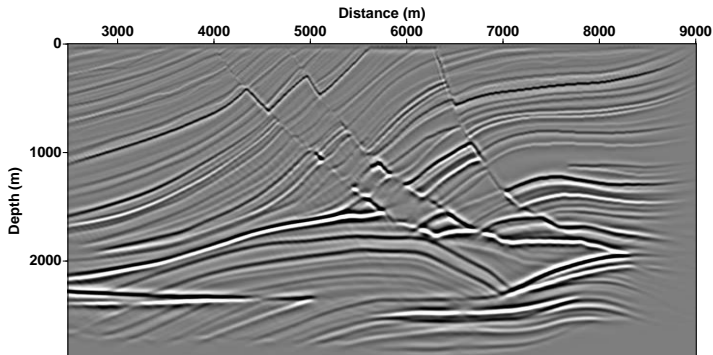
Minimization of curvature

Pre-Stack depth migration



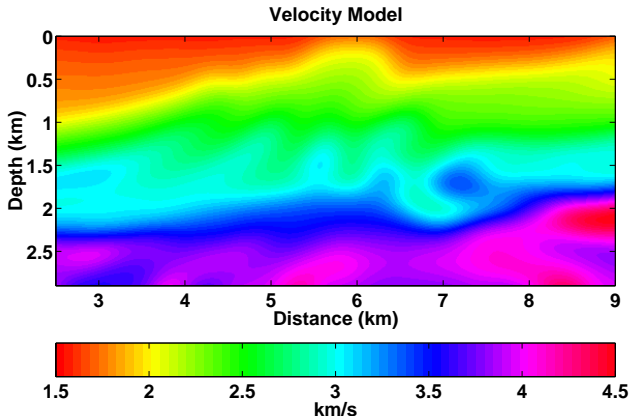
Exact model

Pre-Stack depth migration



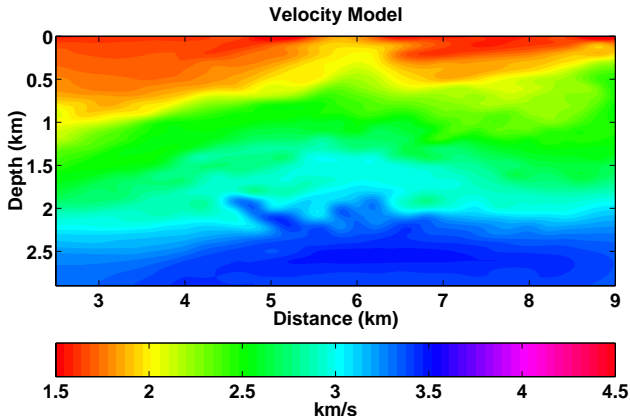
Minimization of curvature

Regularization



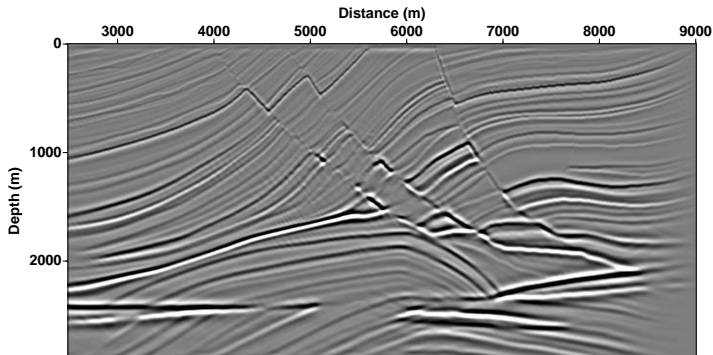
Exact model

Regularization



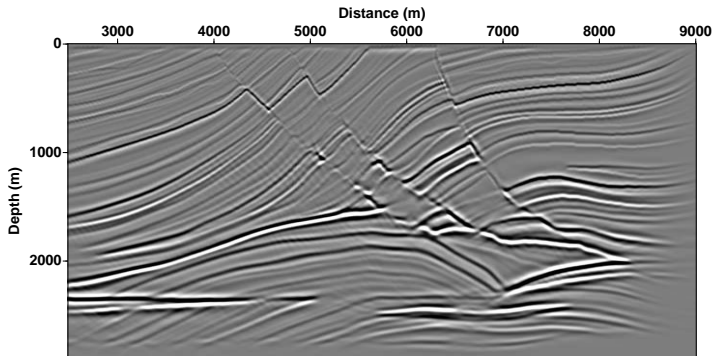
Minimization of gradient

Pre-Stack depth migration



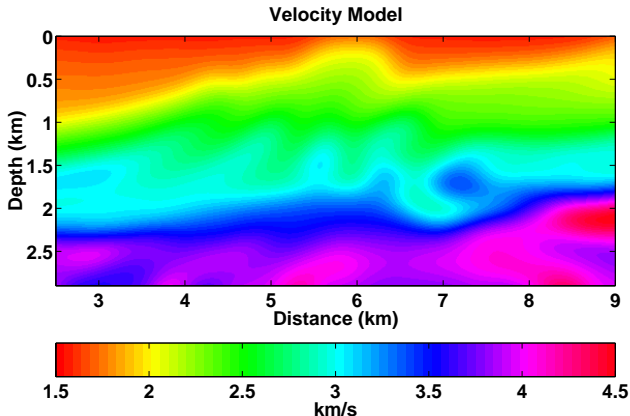
Exact model

Pre-Stack depth migration



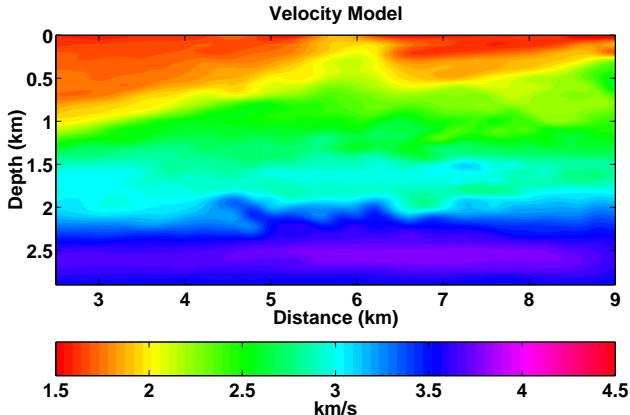
Minimization of gradient

Regularization



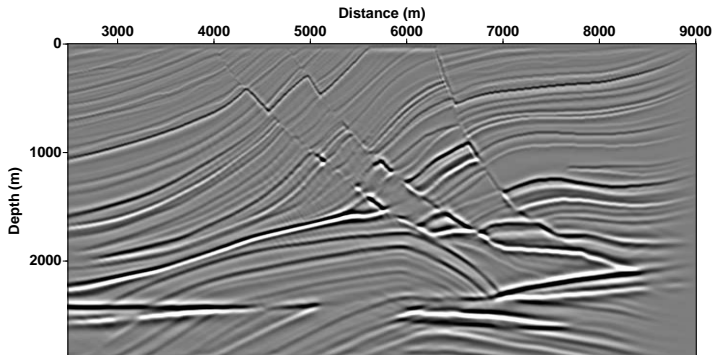
Exact model

Regularization



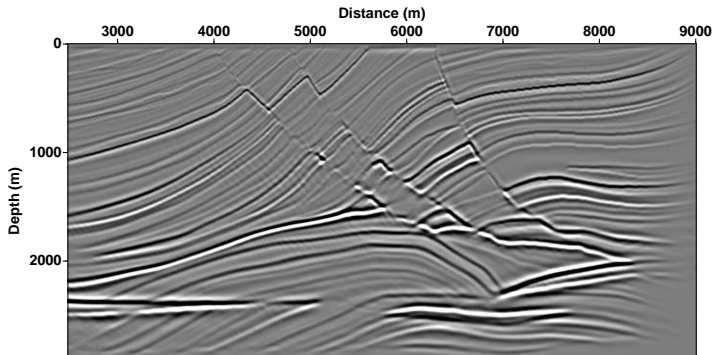
Minimization of derivative along reflectors

Pre-Stack depth migration



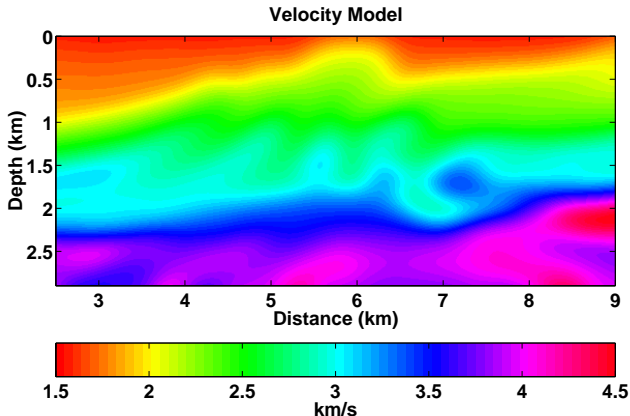
Exact model

Pre-Stack depth migration



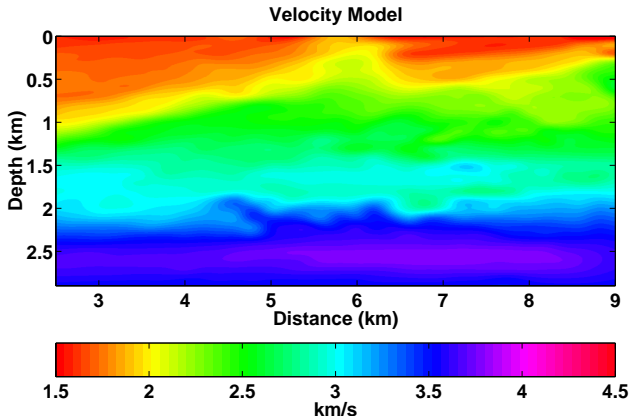
Minimization of derivative along reflectors

Regularization



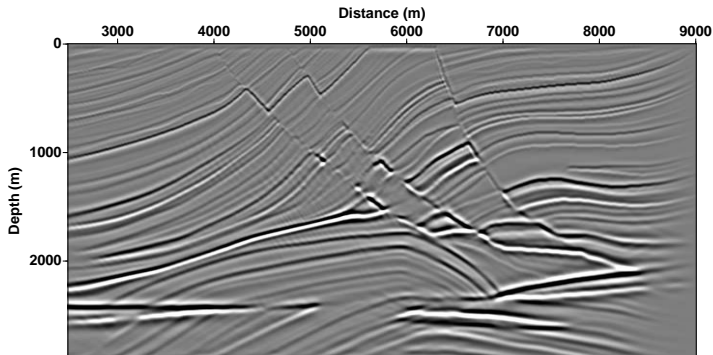
Exact model

Regularization



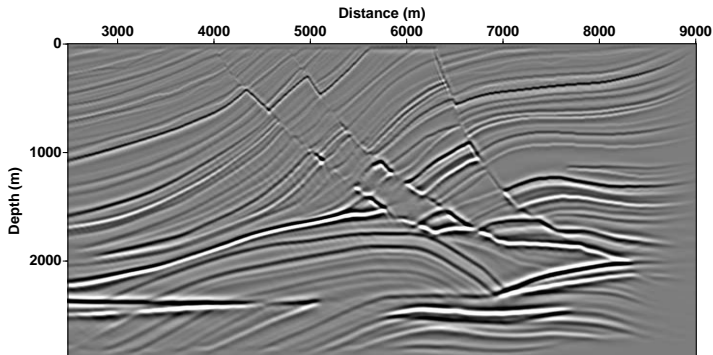
Minimization of gradient and derivative along reflectors

Pre-Stack depth migration



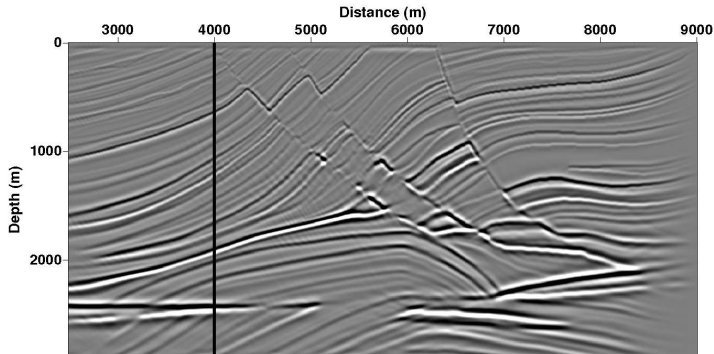
Exact model

Pre-Stack depth migration

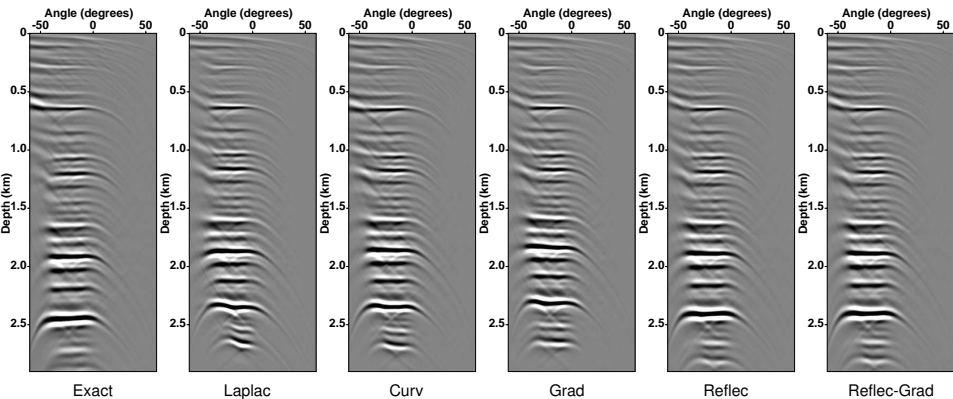


Minimization of gradient and derivative along reflectors

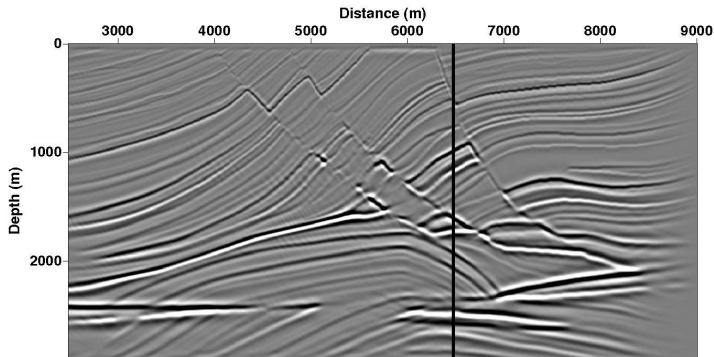
Angle domain image gathers: $x=4.0$ km



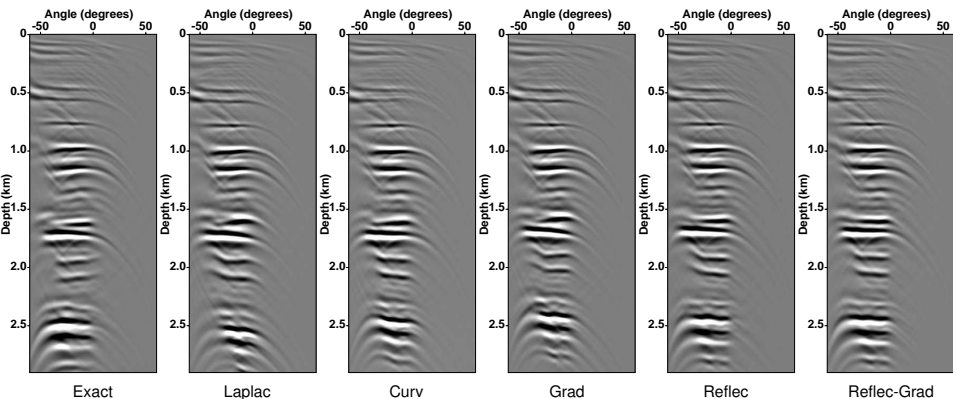
Angle domain image gathers: $x=4.0$ km



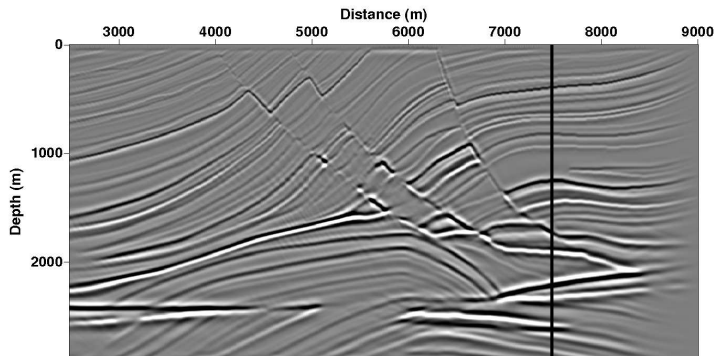
Angle domain image gathers: $x=6.5$ km



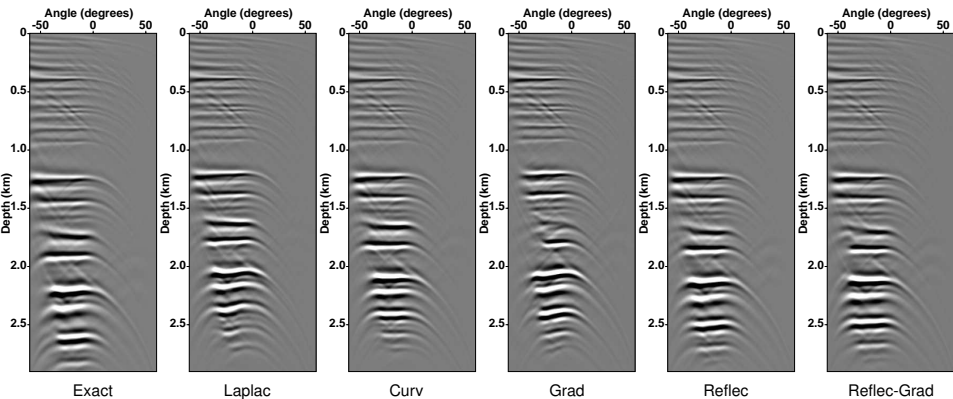
Angle domain image gathers: $x=6.5$ km



Angle domain image gathers: $x=7.5$ km



Angle domain image gathers: $x=7.5$ km



Discussion

- **Inverted velocity models depend strongly on regularization**
 - Pure curvature constraints produced worst results
 - Migrations results are less sensitive to regularization than velocity models
- Regularization along the dip of possible reflectors
 - Implements in a natural way in slope tomography
 - Reduces the differences between layer based and grid based velocity model parameterizations
 - Highlights structural features in the velocity model
 - Improves the velocity model in areas of poor ray coverage in a geologically plausible way

Discussion

- Inverted velocity models depend strongly on regularization
 - Pure curvature constraints produced worst results
 - Migrations results are less sensitive to regularization than velocity models
- **Regularization along the dip of possible reflectors**
 - Implements in a natural way in slope tomography
 - Reduces the differences between layer based and grid based velocity model parameterizations
 - Highlights structural features in the velocity model
 - Improves the velocity model in areas of poor ray coverage in a geologically plausible way

Contents

- 1 Introduction
- 2 Reflector-oriented regularization in slope tomography
- 3 Decomposition of sensitivity kernels in full-waveform inversion
- 4 Conclusions

Forward problem

Non-linear problem,

$$p = \mathcal{F}(\mathbf{m}).$$

Small perturbations in the model parameters allow linearization,

$$\delta p = \Phi \delta \mathbf{m}.$$

Frechét derivatives for the acoustic wave equation

$$\delta p = \Phi \delta \mathbf{m} = \begin{bmatrix} \mathbf{U}_f & \mathbf{V}_f \end{bmatrix} \begin{bmatrix} \delta K \\ \delta \rho \end{bmatrix}.$$

Inverse problem

Adjoint Frechét derivatives → back-project perturbations in the wavefield (data residual) onto model domain.

$$\delta \mathbf{m}^k = \begin{bmatrix} \delta \mathbf{K}^k \\ \delta \rho^k \end{bmatrix} = \begin{bmatrix} \mathbf{U}_f^* \\ \mathbf{V}_f^* \end{bmatrix} \delta p = \Phi^* \delta p.$$

What a back-projection is needed for?

$$\mathbf{m}_{k+1} = \mathbf{m}_k + \alpha \underbrace{\Phi^* \delta p_k}_{\delta \mathbf{m}^k}$$

Secondary or adjoint sources

Want to know the Frechét derivatives?

Look for the **secondary sources**.

Secondary sources

Sources that will give rise to data residuals due to perturbations in the model parameters.

Secondary sources are derived from the wave equation.

Sensitivity kernels from secondary sources

For the acoustic impulse response

$$\mathcal{L}[p(\mathbf{x}, t; \mathbf{x}_s)] = \delta(\mathbf{x} - \mathbf{x}_s) S(t),$$

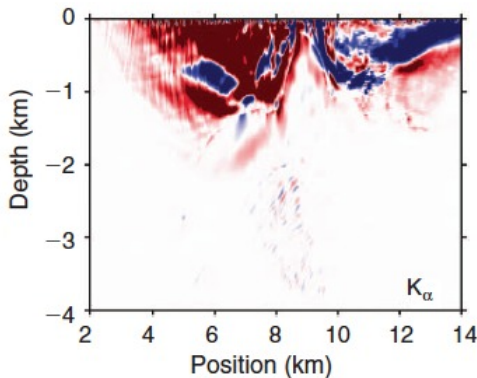
the secondary sources are (Tarantola, 1984, *Geophysics*, **48**)

$$\mathcal{L}[\delta p(\mathbf{x}, t; \mathbf{x}_s)] = \underbrace{-\delta \mathcal{L}[p(\mathbf{x}, t; \mathbf{x}_s)]}_{\text{secondary sources}}$$

Wavefield perturbation

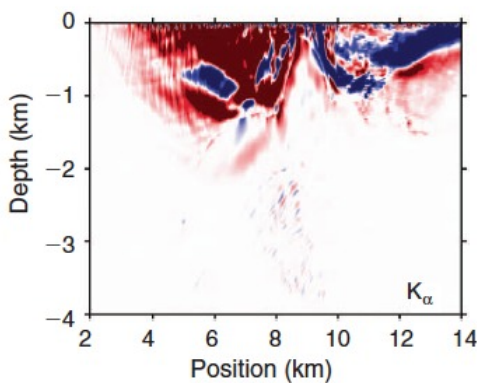
$$\delta p(\mathbf{x}, t; \mathbf{x}_s) = - \int_{\mathbb{V}} d^3 \mathbf{x}' G(\mathbf{x}, t; \mathbf{x}') * \delta \mathcal{L}[p(\mathbf{x}', t; \mathbf{x}_s)].$$

Sensitivity kernel for the scattered field



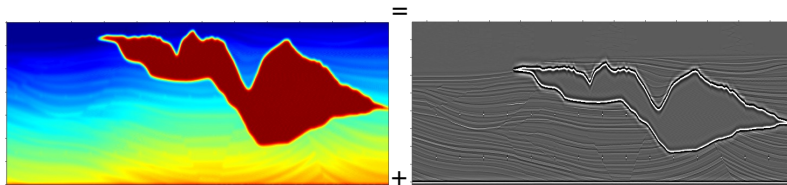
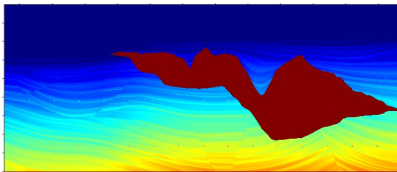
Zhu et al, 2009, *Geophysics*, **74**

Decomposition of sensitivity kernel



$$= \left\{ \begin{array}{c} ? \\ + \\ ? \\ \vdots \\ ? \end{array} \right.$$

Decomposition of the model



Smooth part

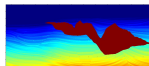
Velocity model from velocity analysis

singular part (sharp contrasts)

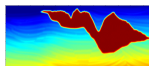
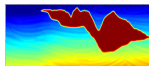
Migrated image

Decomposition of the wavefield

$$\mathcal{L}[p(\mathbf{x}, t)] = \delta(\mathbf{x} - \mathbf{x}_s)S(t)$$



$$\begin{cases} \mathcal{L}^B[p_0(\mathbf{x}, t)] = \delta(\mathbf{x} - \mathbf{x}_s)S(t) \\ \mathcal{L}[p_s(\mathbf{x}, t)] = -\mathcal{V}[p_0(\mathbf{x}, t)] \end{cases}$$



$\mathcal{V} = \mathcal{L} - \mathcal{L}^B$: Scattering potential

Conventionally: $p_s = \delta p$ is perturbation of $p_0 = p$,

$\mathcal{V} = \delta \mathcal{L}$ is perturbation of \mathcal{L}

Here: Both contributions are perturbed $\rightarrow \delta p_0, \delta p_s, \delta \mathcal{L}^B, \delta \mathcal{V}$

Reparametrization

Conventionally:

$$\mathbf{m} = \begin{bmatrix} \mathbf{K} \\ \rho \end{bmatrix} \Rightarrow \delta \mathbf{m} = \begin{bmatrix} \delta \mathbf{K} \\ \delta \rho \end{bmatrix}$$

Here:

$$\mathbf{m} = \begin{bmatrix} \mathbf{K}_B \\ \rho_B \\ \mathbf{K}_S \\ \rho_S \end{bmatrix} \Rightarrow \delta \mathbf{m} = \begin{bmatrix} \delta \mathbf{K}_B \\ \delta \rho_B \\ \delta \mathbf{K}_S \\ \delta \rho_S \end{bmatrix}$$

Reparametrization

Conventionally:

$$\delta \hat{\mathbf{p}} = \begin{bmatrix} \mathbf{U}_f & \mathbf{V}_f \end{bmatrix} \begin{bmatrix} \delta \mathbf{K} \\ \delta \rho \end{bmatrix}$$

Here:

$$\begin{bmatrix} \delta \hat{\mathbf{p}}_0 \\ \delta \hat{\mathbf{p}}_s \end{bmatrix} = \begin{bmatrix} \mathbf{U} & \mathbf{V} & \mathbf{0} & \mathbf{0} \\ \mathbf{U}_B & \mathbf{V}_B & \mathbf{U}_S & \mathbf{V}_S \end{bmatrix} \begin{bmatrix} \delta \mathbf{K}_B \\ \delta \rho_B \\ \delta \mathbf{K}_S \\ \delta \rho_S \end{bmatrix}$$

Reference wavefield residual and sensitivity kernel

Residual evaluated from reference secondary sources

$$\delta \hat{p}_0(\mathbf{x}; \mathbf{x}_s) = - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_0(\mathbf{x}; \mathbf{x}') \delta \mathcal{L}^B [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] .$$

Explicit bulk modulus contribution

$$\delta \hat{p}_0^K(\mathbf{x}_g; \mathbf{x}_s) = \int_{\mathbb{V}} d^3 \mathbf{x}' \left[- \frac{\omega^2}{K_B^2(\mathbf{x}') } \hat{G}_0(\mathbf{x}'; \mathbf{x}_g) \hat{p}_0(\mathbf{x}'; \mathbf{x}_s) \right] \delta K_B(\mathbf{x}').$$

Scattered wavefield residual

The residual evaluated from scattered secondary sources is given by

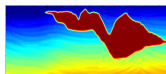
$$\begin{aligned}
 \delta \hat{p}_S(\mathbf{x}; \mathbf{x}_s) = & \\
 & - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \mathcal{V} [\delta \hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_0(\mathbf{x}'; \mathbf{x}) \mathcal{V} [\delta \hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] \\
 & - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_0(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] \\
 & - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_S(\mathbf{x}'; \mathbf{x}_s)] - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_0(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_S(\mathbf{x}'; \mathbf{x}_s)] \\
 & + \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L}^B [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] + \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_0(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L}^B [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)]
 \end{aligned}$$

Scattered wavefield residual

The residual evaluated from scattered secondary sources is given by

$$\begin{aligned} \delta \hat{p}_S(\mathbf{x}; \mathbf{x}_s) = & - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \mathcal{V} [\delta \hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_0(\mathbf{x}'; \mathbf{x}) \mathcal{V} [\delta \hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] \\ & - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_0(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] \\ & - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_S(\mathbf{x}'; \mathbf{x}_s)] - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_0(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_S(\mathbf{x}'; \mathbf{x}_s)] \\ & + \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L}^B [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] + \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_0(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L}^B [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] \end{aligned}$$

Smooth part of δm



Scattered wavefield residual

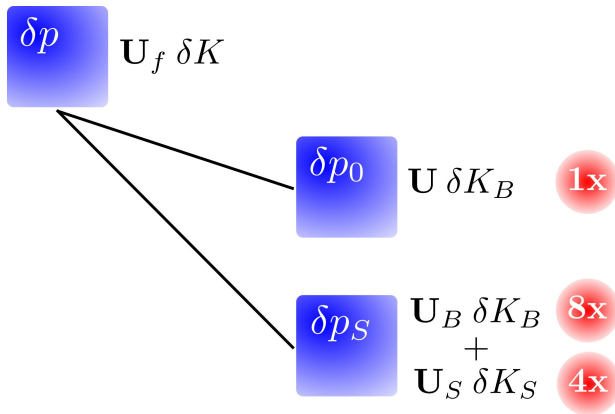
The residual evaluated from scattered secondary sources is given by

$$\begin{aligned} \delta \hat{p}_S(\mathbf{x}; \mathbf{x}_s) = & - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \mathcal{V} [\delta \hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_0(\mathbf{x}'; \mathbf{x}) \mathcal{V} [\delta \hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] \\ & - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_0(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] \\ & - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_S(\mathbf{x}'; \mathbf{x}_s)] - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_0(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_S(\mathbf{x}'; \mathbf{x}_s)] \\ & + \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L}^B [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] + \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_0(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L}^B [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] \end{aligned}$$

Singular part of δm



Kernel decomposition



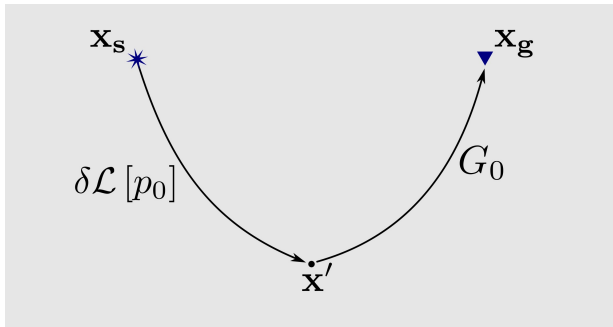
Different levels of non-linearity

$$\begin{aligned}
 \delta \hat{p}_S(\mathbf{x}; \mathbf{x}_s) = & \\
 & - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \mathcal{V} [\delta \hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_0(\mathbf{x}'; \mathbf{x}) \mathcal{V} [\delta \hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] \\
 & - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_0(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] \\
 & - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_S(\mathbf{x}'; \mathbf{x}_s)] - \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_0(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_S(\mathbf{x}'; \mathbf{x}_s)] \\
 & + \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L}^B [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)] + \int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_0(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L}^B [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)].
 \end{aligned}$$

Scattering: **single**, **multiple**, **strong multiple**

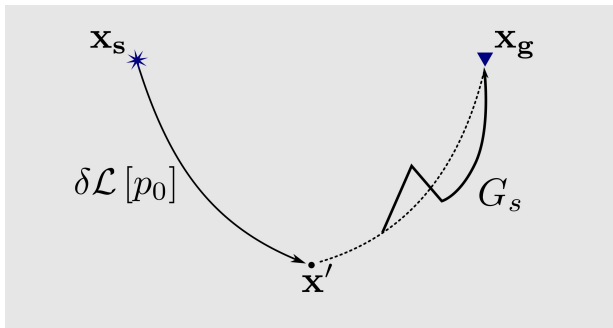
Different levels of non-linearity

Single scattering: $-\int_{\mathbb{V}} d^3\mathbf{x}' \hat{G}_0(\mathbf{x}'; \mathbf{x}) \delta\mathcal{L}[\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)]$



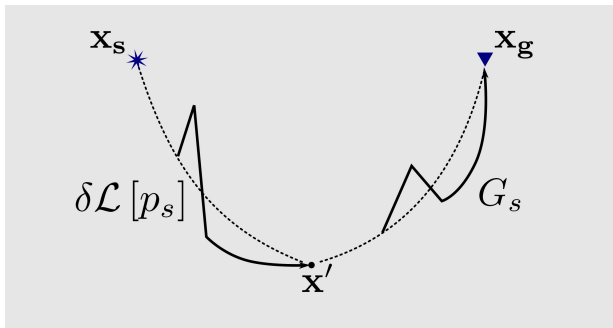
Different levels of non-linearity

Multiple scattering: $-\int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_0(\mathbf{x}'; \mathbf{x}_s)]$



Different levels of non-linearity

Strong multiple scattering: $-\int_{\mathbb{V}} d^3 \mathbf{x}' \hat{G}_S(\mathbf{x}'; \mathbf{x}) \delta \mathcal{L} [\hat{p}_S(\mathbf{x}'; \mathbf{x}_s)]$



Forward and adjoint decomposition

Bulk modulus contribution:

$$\begin{bmatrix} \delta \hat{p}_0 \\ \delta \hat{p}_s \end{bmatrix} = \begin{bmatrix} \mathbf{U} & 0 \\ \sum_{i=1}^{n=8} \mathbf{U}_{B,i} & \sum_{i=3}^{n=6} \mathbf{U}_{S,i} \end{bmatrix} \begin{bmatrix} \delta \mathbf{K}_B \\ \delta \mathbf{K}_S \end{bmatrix}$$

The backprojection based on the above decomposition is

$$\begin{bmatrix} \delta \mathbf{K}_B^{\text{est}} \\ \delta \mathbf{K}_S^{\text{est}} \end{bmatrix} = \begin{bmatrix} \mathbf{U}^\dagger & \sum_{i=1}^{n=8} \mathbf{U}_{B,i}^\dagger \\ 0 & \sum_{i=3}^{n=6} \mathbf{U}_{S,i}^\dagger \end{bmatrix} \begin{bmatrix} \delta \hat{p}_0 \\ \delta \hat{p}_s \end{bmatrix}$$

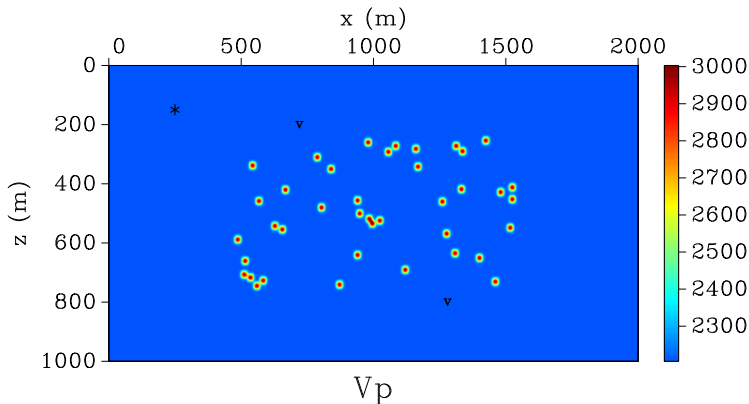
Numerical experiment

Residual-wavefield backprojection

Perturbation of the singular part

Numerical experiment

Unperturbed model:



Perturbation: Random change of the scatterer positions.

Perturbations on the singular part of the model

No background perturbation means $\delta \mathbf{K}_B = 0$. Then

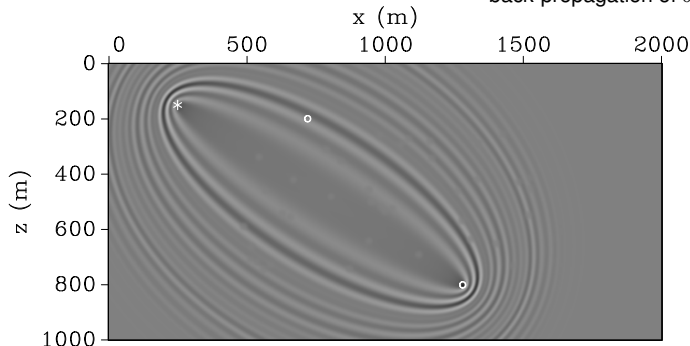
$$\delta p_0 = 0 \quad \text{and} \quad \delta p_s = \left(\sum_{i=3}^{n=6} \mathbf{U}_{s,i} \right) \delta \mathbf{K}_s$$

Backprojection of the scattered-wavefield residual yields

$$\delta \mathbf{K}_s^{\text{est}} = \left(\sum_{i=3}^{n=6} \mathbf{U}_{s,i}^\dagger \right) \delta p_s$$

Perturbations on the singular part of the model

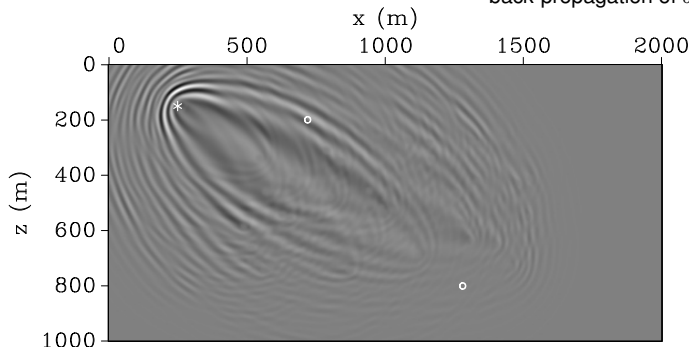
$$\delta K_{S,4}^{\text{est}}(x'_i) = - \int d\omega \frac{\omega^2}{K^2(x'_i)} \underbrace{\widehat{p}_0^*(x'_i, \omega; x_s)}_{\text{direct wavefield cross-correlation}} \underbrace{\widehat{G}_0^*(x'_i, \omega; x_g) \delta \widehat{p}_s(x_g, \omega; x_s)}_{\substack{\text{background extrapolator} \\ \text{back-propagation of } \delta \widehat{p}_s}}$$



dKestS-S4-cr680

Perturbations on the singular part of the model

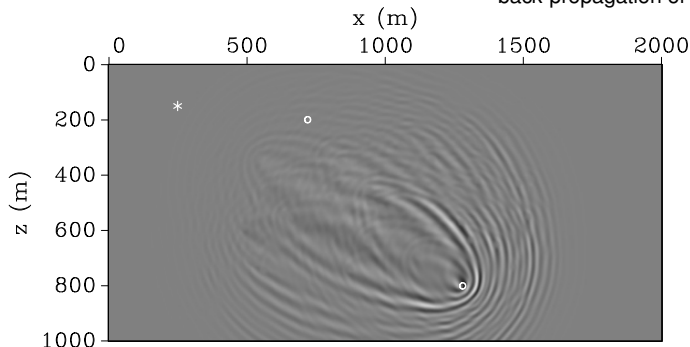
$$\delta \mathbf{K}_{\mathbf{S},3}^{\text{est}}(x'_i) = - \int d\omega \frac{\omega^2}{K^2(x'_i)} \underbrace{\widehat{p}_0^*(x'_i, \omega; x_s)}_{\text{direct wavefield cross-correlation}} \underbrace{\widehat{G}_{\mathbf{S}}^*(x'_i, \omega; x_g) \delta \widehat{p}_{\mathbf{S}}(x_g, \omega; x_s)}_{\substack{\text{scattered wave extrapolator} \\ \text{back-propagation of } \delta \widehat{p}_{\mathbf{S}}}}$$



dKestS-S3-cr680

Perturbations on the singular part of the model

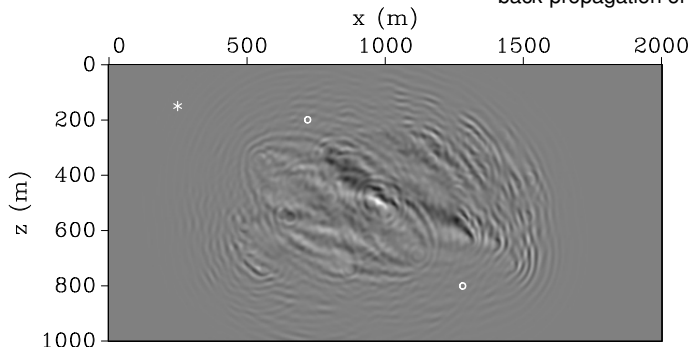
$$\delta K_{S,6}^{\text{est}}(x'_i) = - \int d\omega \frac{\omega^2}{K^2(x'_i)} \underbrace{\widehat{p}_S^*(x'_i, \omega; x_s)}_{\text{scattered wavefield cross-correlation}} \underbrace{\widehat{G}_0^*(x'_i, \omega; x_g) \delta \widehat{p}_S(x_g, \omega; x_s)}_{\substack{\text{background extrapolator} \\ \text{back-propagation of } \delta \widehat{p}_S}}$$



dKestS-S6-cr680

Perturbations on the singular part of the model

$$\delta K_{S,5}^{\text{est}}(x'_i) = - \int d\omega \frac{\omega^2}{K^2(x'_i)} \underbrace{\hat{p}_S^*(x'_i, \omega; x_s)}_{\text{scattered wavefield cross-correlation}} \underbrace{\hat{G}_S^*(x'_i, \omega; x_g) \delta \hat{p}_S(x_g, \omega; x_s)}_{\substack{\text{scattered wave extrapolator} \\ \text{back-propagation of } \delta \hat{p}_S}}$$



dKestS-S5-cr680

Discussion

- Successful kernel decomposition
 - Perturbation of background medium *and* singular part
 - Based on model building and migration-type imaging

Discussion

- Successful kernel decomposition
 - Perturbation of background medium *and* singular part
 - Based on model building and migration-type imaging
- Potential for better control over FWI optimization:
 - Contributions show different levels of non-linearity
 - Multiple scattering carries important information

Discussion

- Successful kernel decomposition
 - Perturbation of background medium *and* singular part
 - Based on model building and migration-type imaging
- Potential for better control over FWI optimization:
 - Contributions show different levels of non-linearity
 - Multiple scattering carries important information
- Practical challenges on separation of the model/data components
- Potential use in 4D-inversion problems

Contents

- 1 Introduction
- 2 Reflector-oriented regularization in slope tomography
- 3 Decomposition of sensitivity kernels in full-waveform inversion
- 4 Conclusions**

Conclusions

- Seismic Inverse problems are (partly) underdetermined
- Something has to be done about ambiguity

Conclusions

- Seismic Inverse problems are (partly) underdetermined
- Something has to be done about ambiguity
- Slope tomography
 - Model-geometry-based regularization
 - Led to more realistic velocity model

Conclusions

- Seismic Inverse problems are (partly) underdetermined
- Something has to be done about ambiguity
- Slope tomography
 - Model-geometry-based regularization
 - Led to more realistic velocity model
- Full-waveform inversion
 - Sensitivity kernel decomposition
 - Led to better understanding of contributions

Acknowledgements

Contributors to these topics were J. C. Costa, F. J. C. da Silva, E. N. S. Gomes, A. Mello, D. Amazonas, D. L. Macedo, and I. Vasconcelos.

We thank Gilles Lambaré for providing the Marmousoft data set and the picked events on this data set. This research was supported by CAPES, FINEP, CNPq, as well as Petrobras, Schlumberger, and the sponsors of the WIT consortium.

Thank you for your attention