A Double Regularization Approach for Inverse Problems with Noisy Data and Inexact Operator

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JKU
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Overview

☐ Introduction

☐ Proposed method: DBL-RTLS

☐ Computational aspects

☐ Numerical illustration

☐ Outline and future work
Overview

- Introduction
- Proposed method: DBL-RTLS
- Computational aspects
- Numerical illustration
- Outline and future work
Inverse problems

“Inverse problems are concerned with determining causes for a desired or an observed effect” [Engl, Hanke, and Neubauer, 2000]

Consider a linear operator equation

\[ Ax = y. \]

Inverse problems most oft do not fulfill Hadamard’s postulate [1902] of well posedness (existence, uniqueness and stability).

Computational issues: observed effect has measurement errors or perturbations caused by noise.
1st Case: noisy data

Solve $Ax = y_0$ out of the measurement $y_\delta$ with $\|y_0 - y_\delta\| \leq \delta$. Need apply some regularization technique

$$\minimize_x ||Ax - y_\delta||^2 + \alpha ||Lx||^2.$$ 

**Tikhonov** regularization
- fidelity term (based on LS);
- regularization parameter $\alpha$;
- stabilization term (quadratic).

1st Case: noisy data

Solve $Ax = y_0$ out of the measurement $y_δ$ with $\|y_0 - y_δ\| \leq δ$. Need apply some regularization technique

$$\min_x \|Ax - y_δ\|^2 + \alpha R(x).$$

**Tikhonov-type** regularization

- fidelity term (based on LS);
- regularization parameter $\alpha$;
- $R$ is a proper, convex and weakly lower semicontinuous functional.

[Burger and Osher, 2004, Resmerita, 2005]
Subgradient

The *Fenchel subdifferential* of a functional $\mathcal{R} : \mathcal{U} \to [0, +\infty]$ at $\bar{u} \in \mathcal{U}$ is the set

$$\partial^F \mathcal{R} (\bar{u}) = \{ \xi \in \mathcal{U}^* \mid \mathcal{R}(v) - \mathcal{R}(\bar{u}) \geq \langle \xi , v - \bar{u} \rangle \ \forall v \in \mathcal{U} \}.$$ 


**Optimality condition:**

If $\bar{u}$ minimizes $\mathcal{R}$ then

$$0 \in \partial^F \mathcal{R} (\bar{u})$$
Example

Consider the function $\mathcal{R}(u) = |u|$

Figure: Function (left) and its subdifferential (right).
2nd Case: inexact operator and noisy data

Solve $A_0 x = y_0$ under the assumptions

(i) noisy data $\|y_0 - y_\delta\| \leq \delta$.
(ii) inexact operator $\|A_0 - A_\epsilon\| \leq \epsilon$.

What have been done so far?

- **Linear case** - based on **TLS** [Golub and Van Loan, 1980]:
  - **R-TLS**: Regularized TLS [Golub et al., 1999];
  - **D-RTLS**: Dual R-TLS [Lu et al., 2007].

- **Nonlinear case**: no publication (?)

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<th>LS: $y_\delta$ and $A_0$</th>
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<td>minimize $y$ $|y - y_\delta|_2$</td>
<td>minimize $|[A, y] - [A_\epsilon, y_\delta]|_F$</td>
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**LS**: $y_\delta$ and $A_0$

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**TLS**: $y_\delta$ and $A_\epsilon$

| minimize $\| [A, y] - [A_\epsilon, y_\delta] \|_F$ | subject to $y \in \mathbb{R}(A)$ |
Illustration

Solve 1D problem: \( am = b \), find the slope \( m \).

Given:
1. \( b_\delta, a_\epsilon \) (red)

Solution:
1. LS solution (blue)
2. TLS solution (green)

Example: \( \arctan(1) = 45^\circ \) [Van Huffel and Vandewalle, 1991]
R-TLS

The R-TLS method [Golub, Hansen, and O’leary, 1999]

\[
\text{minimize } \| A - A_{\epsilon} \|^2 + \| y - y_{\delta} \|^2 \\
\text{subject to } \begin{cases} 
Ax = y \\
\|Lx\|^2 \leq M.
\end{cases}
\]

If the inequality constraint is active, then

\[
(A_{\epsilon}^T A_{\epsilon} + \alpha L^T L + \beta I) \hat{x} = A_{\epsilon}^T y_{\delta} \text{ and } \|L\hat{x}\| = M
\]

with \( \alpha = \mu(1 + \|\hat{x}\|^2) \), \( \beta = -\frac{\|A_{\epsilon}\hat{x} - y_{\delta}\|^2}{1 + \|\hat{x}\|^2} \) and \( \mu > 0 \) is the Lagrange multiplier.

Difficulty: requires a reliable bound \( M \) for the norm \( \|Lx^\dagger\|^2 \).
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Consider the operator equation

\[ B(k, f) = g_0 \]

where \( B \) is a bilinear operator \((\text{nonlinear})\)

\[ B : \mathcal{U} \times \mathcal{V} \rightarrow \mathcal{H} \]

\[ (k, f) \mapsto B(k, f) \]

and \( B \) is characterized by a function \( k_0 \).

- \( K \cdot = B(\tilde{k}, \cdot) \) compact linear operator for a fixed \( \tilde{k} \in \mathcal{U} \)
- \( F \cdot = B(\cdot, \tilde{f}) \) linear operator for a fixed \( \tilde{f} \in \mathcal{V} \)

\[ \| B(k_0, \cdot) \|_{\mathcal{V} \rightarrow \mathcal{H}} \leq C \| k_0 \|_{\mathcal{U}} ; \]
\[ \| B(k, f) \|_{\mathcal{H}} \leq C \| k \|_{\mathcal{U}} \| f \|_{\mathcal{V}} ; \]

Example:

\[ B(k, f)(s) := \int_\Omega k(s, t) f(t) dt . \]
Consider the operator equation

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\begin{itemize}
  \item \( K \cdot = B(\tilde{k}, \cdot) \) compact linear operator for a fixed \( \tilde{k} \in \mathcal{U} \)
  \item \( F \cdot = B(\cdot, \tilde{f}) \) linear operator for a fixed \( \tilde{f} \in \mathcal{V} \)
  \item \( \|B(k_0, \cdot)\|_{\mathcal{V} \rightarrow \mathcal{H}} \leq C\|k_0\|_{\mathcal{U}} \);
  \item \( \|B(k, f)\|_{\mathcal{H}} \leq C\|k\|_{\mathcal{U}}\|f\|_{\mathcal{V}} \);
\end{itemize}

\textbf{Example:}

\[ B(k, f)(s) := \int_{\Omega} k(s, t)f(t)dt. \]
We want to solve

\[ B(k_0, f) = g_0 \]

out of the measurements \( k_\epsilon \) and \( g_\delta \) with

(i) noisy data \( \| g_0 - g_\delta \|_H \leq \delta \).

(ii) inexact operator \( \| k_0 - k_\epsilon \|_U \leq \epsilon \).

We introduce the **DBL-RTLS**

\[
\min_{k,f} J(k,f) := T(k,f,k_\epsilon,g_\delta) + R(k,f)
\]

where

- **T** measures of accuracy (closeness/discrepancy)
- **R** promotes stability.
Proposed method: DBL-RTLS

\[ \text{minimize } J(k, f) := T(k, f, k_\epsilon, g_\delta) + R(k, f) \]  \hspace{1cm} (1)

where

\[
T(k, f, k_\epsilon, g_\delta) = \frac{1}{2} \| B(k, f) - g_\delta \|_H^2 + \frac{\gamma}{2} \| k - k_\epsilon \|_U^2
\]

\[
R(k, f) = \frac{\alpha}{2} \| Lf \|_V^2 + \beta R(k)
\]

- \textbf{\textit{T}} is based on TLS method, measures the discrepancy on both data and operator;
- \textbf{\textit{L}} : \mathcal{V} \rightarrow \mathcal{V} is a linear bounded operator;
- \textbf{\textit{\alpha}}, \textbf{\textit{\beta}} are the regularization parameters and \textbf{\textit{\gamma}} is a scaling parameter;
- \textbf{double regularization} [You and Kaveh, 1996], \textbf{\textit{R}} : \mathcal{U} \rightarrow [0, +\infty] is proper \textbf{convex} function and \textbf{w-lsc}.
Main theoretical results

Assumption:
\( (A1) \) \( B \) is strongly continuous, ie, if \( (k^n, f^n) \rightharpoonup (\bar{k}, \bar{f}) \) then
\[ B(k^n, f^n) \to B(\bar{k}, \bar{f}) \]

<table>
<thead>
<tr>
<th>Proposition</th>
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<tbody>
<tr>
<td>Let ( J ) be the functional defined on (1) and ( L ) be a <strong>bounded and positive</strong> operator. Then ( J ) is <strong>positive, weak lower semi-continuous</strong> and <strong>coercive</strong> functional.</td>
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<th>Theorem (existence)</th>
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<td>Let the assumptions of Proposition 1 hold. Then there exists a <strong>global minimum</strong> of</td>
</tr>
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<td>minimize ( J(k, f) ).</td>
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Theorem (stability)

- \( \delta_j \to \delta \) and \( \epsilon_j \to \epsilon \)
- \( g_{\delta_j} \to g_\delta \) and \( k_{\epsilon_j} \to k_\epsilon \)
- \( \alpha, \beta > 0 \)
- \( (k^j, f^j) \) is a minimizer of \( J \) with \( g_{\delta_j} \) and \( k_{\epsilon_j} \)

Then there exists a convergent subsequence of \( (k^j, f^j)_j \)

\[
(k^{j_m}, f^{j_m}) \to (\bar{k}, \bar{f})
\]

where \( (\bar{k}, \bar{f}) \) is a minimizer of \( J \) with \( g_\delta, k_\epsilon, \alpha \) and \( \beta \).
Theorem (stability)

- $\delta_j \to \delta$ and $\epsilon_j \to \epsilon$
- $g\delta_j \to g\delta$ and $k_{\epsilon_j} \to k_\epsilon$
- $\alpha, \beta > 0$
- $(k^j, f^j)$ is a minimizer of $J$ with $g\delta_j$ and $k_{\epsilon_j}$

Then there exists a convergent subsequence of $(k^j, f^j)_j$

$$ (k^{jm}, f^{jm}) \to (\bar{k}, \bar{f}) $$

where $(\bar{k}, \bar{f})$ is a minimizer of $J$ with $g_\delta, k_\epsilon, \alpha$ and $\beta$. 
Consider the convex functional

$$\Phi(k, f) := \frac{1}{2} \| L f \|^2 + \eta R(k)$$

where the parameter $\eta$ represents the different scaling of $f$ and $k$.

For convergence results we need to define

**Definition**

We call $(k^\dagger, f^\dagger)$ a **$\Phi$-minimizing solution** if

$$(k^\dagger, f^\dagger) = \arg\min \{ \Phi(k, f) \mid B(k, f) = g_0 \}.$$
Theorem (convergence)

- $\delta_j \to 0 \text{ and } \epsilon_j \to 0$
- $\|g_{\delta_j} - g_0\| \leq \delta_j \text{ and } \|k_{\epsilon_j} - k_0\| \leq \epsilon_j$
- $\alpha_j = \alpha(\epsilon_j, \delta_j) \text{ and } \beta_j = \beta(\epsilon_j, \delta_j),$ s.t. $\alpha_j \to 0, \beta_j \to 0$,
  $$\lim_{j \to \infty} \frac{\delta_j^2 + \gamma \epsilon_j^2}{\alpha_j} = 0 \text{ and } \lim_{j \to \infty} \frac{\beta_j}{\alpha_j} = \eta$$
- $(k^j, f^j)$ is a minimizer of $J$ with $g_{\delta_j}, k_{\epsilon_j}, \alpha_j$ and $\beta_j$

Then there exists a convergent subsequence of $(k^j, f^j)_j$

$$(k^{jm}, f^{jm}) \rightarrow (k^\dagger, f^\dagger)$$

where $(k^\dagger, f^\dagger)$ is a $\Phi$-minimizing solution.
**Theorem (convergence)**

- \( \delta_j \to 0 \) and \( \epsilon_j \to 0 \)
- \( \|g_{\delta_j} - g_0\| \leq \delta_j \) and \( \|k_{\epsilon_j} - k_0\| \leq \epsilon_j \)
- \( \alpha_j = \alpha(\epsilon_j, \delta_j) \) and \( \beta_j = \beta(\epsilon_j, \delta_j) \), s.t. \( \alpha_j \to 0 \), \( \beta_j \to 0 \),

\[
\lim_{j \to \infty} \frac{\delta_j^2 + \gamma \epsilon_j^2}{\alpha_j} = 0 \quad \text{and} \quad \lim_{j \to \infty} \frac{\beta_j}{\alpha_j} = \eta
\]

- \((k^j, f^j)\) is a minimizer of \( J \) with \( g_{\delta_j}, k_{\epsilon_j}, \alpha_j \) and \( \beta_j \)

*Then there exists a convergent subsequence of \((k^j, f^j)\)*

\[
(k^{jm}, f^{jm}) \to (k^\dagger, f^\dagger)
\]

*where \((k^\dagger, f^\dagger)\) is a \( \Phi \)-minimizing solution.*
Optimality condition

If the pair \((\bar{k}, \bar{f})\) is a minimizer of \(J(k, f)\), then \((0, 0) \in \partial J(\bar{k}, \bar{f})\).

Theorem

Let \(J : \mathbb{U} \times \mathbb{V} \to \mathbb{R}\) be a nonconvex functional,

\[
J(u, v) = \varphi(u) + Q(u, v) + \psi(v)
\]

where \(Q\) is a nonlinear differentiable term and \(\varphi, \psi\) are lsc convex functions. Then

\[
\partial J(u, v) = \{\partial \varphi(u) + D_u Q(u, v)\} \times \{\partial \psi(v) + D_v Q(u, v)\}
\]

\[
= \{\partial_u J(u, v)\} \times \{\partial_v J(u, v)\}
\]
Remark:
- is difficult to solve wrt both \((k, f)\)
- \(J\) is bilinear and biconvex (linear and convex to each one)
- applied \textit{alternating minimization} method.

Alternating minimization algorithm

Require: \(g_\delta, k_\epsilon, L, \gamma, \alpha, \beta\)

1. \(n = 0\)
2. repeat
3. \(f^{n+1} \in \arg \min_f J(k, f | k^n)\)
4. \(k^{n+1} \in \arg \min_k J(k, f | f^{n+1})\)
5. until convergence
Remark:

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### Alternating minimization algorithm

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5: until convergence
Proposition

The sequence generated by the function $J(k^n, f^n)$ is non-increasing,

$$J(k^{n+1}, f^{n+1}) \leq J(k^n, f^{n+1}) \leq J(k^n, f^n).$$

Assumptions:

(A1) $B$ is strongly continuous, i.e., if $(k^n, f^n) \rightharpoonup (\bar{k}, \bar{f})$ then $B(k^n, f^n) \to B(\bar{k}, \bar{f})$

(A2) $B$ is weakly sequentially closed, i.e., if $(k^n, f^n) \rightharpoonup (\bar{k}, \bar{f})$ and $B(k^n, f^n) \rightharpoonup g$ then $B(\bar{k}, \bar{f}) = g$

(A3) the adjoint of $B'$ is strongly continuous, i.e., if $(k^n, f^n) \rightharpoonup (\bar{k}, \bar{f})$ then $B'(k^n, f^n)^* z \to B'(\bar{k}, \bar{f})^* z$, $\forall z \in \mathcal{D}(B')$
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Theorem

Given regularization parameters $0 < \alpha \leq \alpha$ and $\beta$, compute AM algorithm. The sequence $\{(k^{n+1}, f^{n+1})\}_{n+1}$ has a weakly convergent subsequence, namely $(k^{n_j+1}, f^{n_j+1}) \rightharpoonup (\bar{k}, \bar{f})$ and the limit has the property

$$J(\bar{k}, \bar{f}) \leq J(\bar{k}, f) \quad \text{and} \quad J(\bar{k}, \bar{f}) \leq J(k, \bar{f})$$

for all $f \in \mathcal{V}$ and for all $k \in \mathcal{U}$.

Proposition

Let $\{(k^n, f^n)\}_n$ be a weakly convergent sequence generated by AM algorithm, where $k^n \to \bar{k}$ and $f^n \to \bar{f}$. Then there exists a subsequence $\{k^{n_j}\}_{n_j}$ such that $k^{n_j} \to \bar{k}$ and there exists $\{\xi_{k^{n_j}}^{n_j}\}_{n_j}$ with $\xi_{k^{n_j}}^{n_j} \in \partial_k J(k^{n_j}, f^{n_j})$ such that $\xi_{k^{n_j}}^{n_j} \to 0$. 
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Proposition

Let \( \{n\} \) be a subsequence of \( \mathbb{N} \) such that the sequence \( \{(k^n, f^n)\}_n \) generated by AM algorithm satisfies \( k^n \to \bar{k} \) and \( f^n \rightharpoonup \bar{f} \). Then \( f^{n_j} \to \bar{f} \) and there exists \( \{\xi_f^{n_j}\}_{n_j} \) with \( \xi_f^{n_j} \in \partial_f J(k^{n_j}, f^{n_j}) \) such that \( \xi_f^{n_j} \to 0 \).

Remark: Graph of subdifferential mapping is sw-closed, i.e., if \( v^n \to \bar{v} \) and \( \xi_n \rightharpoonup \bar{\xi} \) with \( \xi_n \in \partial \varphi(v^n) \), then \( \bar{\xi} \in \partial \varphi(\bar{v}) \).

Theorem

Let \( \{(k^n, f^n)\}_n \) be the sequence generated by the AM algorithm, then there exists a subsequence converging towards to a critical point of \( J \), i.e.,

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(0, 0) \in \partial J(\bar{k}, \bar{f}) .
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First numerical result

Convolution in 1D

\[ \int_{\Omega} k(s - t)f(t)dt = g(s) \]

- **characteristic** kernel and **hat** function;
- space: \( \Omega = [0, 1] \), discretization: \( N = 2048 \) points;
- \( R(k) = \|k\|_{w,p} \) with \( p = 1 \)
- Haar wavelet for \( \{\phi\}_\lambda \) and \( J = 10 \);
- initial guess: \( k^0 = k_\epsilon \), \( \tau = 1.0 \);
  - 1st. relative error: 10% and 10%.
  - 2nd. relative error: 0.1% and 0.1%.
Numerical illustration
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- proved existence, stability and convergence;
- study of source conditions and convergence rates ($k$ and $f$);
- suggested an iterative implementation;
- proved convergence of AM algorithm to a critical point;

For further work:

- study variational inequalities;
- how to choose the best regularization parameter?
- a priori and a posteriori choice;
- implementations and numerical experiments (2D);
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Thank you for your kind attention!