Level-set approaches of L_2 type for recovering shape and contrast in ill-posed problems

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Inverse Problems under Capricorn

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Plans

- Look in the model

$$F(u) = y \qquad\qquad F \text{ operator}. \tag{1}$$

- The direct problem:
  given the parameter u, solve (1)
- The inverse problem:
  given a set of data y, recover u in (1).
- set of data $y \in Y$ is obtained by indirect measurements of the parameter
- measurements $\Longrightarrow$ data corrupted by noise $y^\delta \in Y$ satisfying

$$\|y - y^\delta\|_Y \leq \delta. \tag{2}$$

- Direct problems: are well posed $\longleftrightarrow$ existence, uniqueness, stability.
- Inverse problem: are ill posed
- Inverse Problems call for regularization strategies !!!!!

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The inverse problems that we are looking ...

- Several inverse problems of interest consist of identifying an unknown physical quantity $u \in X$
- u represented by a piecewise constant real function over a bounded given domain $\Omega \subset \mathbb{R}^2$
- Ex: image problems (u represent the grey scale) , EIT problems, inverse potential problems, ...
- The relation between the unknown parameter function and the problem data is described by the model

$$F(u) = y \qquad F : \mathcal{D}(F) \subset X \longrightarrow Y \tag{3}$$

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- If the unknown function u is piecewise constant distinguishing between two or several given values, level-set approaches were considered by several authors (Osher, Tai, Burger, ...)
- In this case, we only need to identify the level sets of $u \iff$ shape identification problem.
- If the level values of u are also unknown, the inverse problem becomes harder, since one has to identify both the level sets as well as the level values of the unknown parameter u.

Plans

- assumption that the parameter function u is a piecewise constant function assuming two distinct unknown values
- $u(x) \in \{c^1, c^2\}$ a.e. in Ω
- In this case one can assume the existence of an open mensurable set $D \subset \Omega$, s.t.

$$u(x) = c^1 \,, x \in D =: D_1 \qquad u(x) = c^2 \,, x \in \Omega / D =: D_2$$

- consists in introducing the level set function $\phi \in L_2(\Omega)$ that acts as a regularization in the parameter space

  u can be represented as

$$u = c^2 H(\phi) + c^1 (1 - H(\phi)) =: P_s(\phi, c^j)$$

- $u(x) = c^j$, $x \in D_j$, where $D_2 = \{x \in \Omega : \phi(x) > 0\}$ and $D_1 = \{x \in \Omega : \phi(x) \leq 0\}$

  Some remarks:

i) The operator P_s establishes a straightforward relation between the level sets of ϕ and the sets D_j

ii) representing our a priori knowledge about the solution u.

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ii) representing our a priori knowledge about the solution u.

$$F(u) = y \quad (3) \qquad \|y - y^\delta\|_Y \leq \delta \quad (2)$$

- With the (sLS) approach the inverse problem (3), with data given by (2), can be written in the form of the operator equation

$$F(P_s(\phi, c^j)) = y^\delta . \tag{4}$$

- To obtain an approximated solution of (4), we propose the minimization of the Tikhonov functional

$$\mathcal{G}_{s,\alpha}(\phi, c^j) = \|F(P_s(\phi, c^j)) - y^\delta\|_Y^2 + \alpha R_s(\phi, c^j) \tag{5}$$

with a TV-L_2 regularization

$$R_s(\phi, c^j) := \beta_1 |H(\phi)|_{BV} + \beta_2 \|\phi\|_{L_2(\Omega)}^2 + \beta_2 \|c^j\|_{\mathbb{R}^2}^2 .$$

- consisting in introduce the piecewise constant level set function $\phi \in L_2(\Omega)$ such that $\phi(x) = j \qquad x \in D_j$
- then define auxiliary functions $\psi_1(t) = (2 - t)$ and $\psi_2(t) = (t - 1)$. Hence, we represent the characteristic function of D_j as $\chi_{D_j}(x) = \psi_j(\phi)$
- the solution of (3) can be written in the form

$$u = c^1 \psi_1(\phi) + c^2 \psi_2(\phi) = P_{pc}(\phi, c^j)\,. \tag{6}$$

$$F(u) = y \quad (3) \qquad \|y - y^\delta\|_Y \leq \delta \quad (2)$$

- Remark: The piecewise constant assumption on ϕ corresponds to the constraint

$$\mathcal{K}(\phi) = (\phi - 1)(\phi - 2) = 0\,. \tag{7}$$

- With this framework, the inverse problem (3), whit data given by (2), can be written as

$$F(P_{pc}(\phi, c^i)) = y^\delta\,, \tag{8}$$
$$s.t. \quad \phi \in L_2(\Omega) \,:\, \mathcal{K}(\phi) = 0\,.$$

- Approximate solutions to (8) can be obtained by minimizing the Tikhonov functional

$$\mathcal{G}_{\alpha,pc}(\phi, c^j) = \|F(P_{pc}(\phi, c^j)) - y^\delta\|_Y^2 + \beta_3 \|\mathcal{K}(\phi)\|_{L_1(\Omega)} + \alpha R_{pc}(\phi, c^j) \tag{9}$$

where

$$R_{pc}(\phi, c^j) := \beta_1 |P_{pc}(\phi, c^j)|_{BV} + \beta_2 \|c^j\|_{\mathbb{R}}^2 .$$

Notice that the minimization of the functional $\mathcal{G}_{\alpha,pc}$ furnishes a regularized solution to the system of operator equations:

$$\begin{bmatrix} F(P_{pc}(\phi, c^j)) \\ \mathcal{K}(\phi) \end{bmatrix} = \begin{bmatrix} y^\delta \\ 0 \end{bmatrix}$$

Plans

(A1) $\Omega \subset \mathbb{R}^2$ is bounded with piecewise C^1 boundary.

(A2) The operator $F : \mathcal{D}(F) \subset L_p(\Omega) \longrightarrow Y$ is continuous on $\mathcal{D}(F)$ with respect to the L_p-topology, where $1 \leq p < 2$.

(A3) α, β denote positive parameters.

(A4) Equation (3) has a solution, i.e. there exists $u \in L_\infty(\Omega)$ satisfying $F(u) = y$; there exists a function $\phi \in L_2(\Omega)$ satisfying $|\nabla\phi| \neq 0$, in a neighborhood of $\{\phi = 0\}$ such that $H(\phi) = z \in L_1(\Omega)$ and there exist constants values $c^j \in \mathbb{R}$ such that $P_s(z, c^j) = u$.

(A4') Equation (3) has a solution, i.e. there exists $u \in L_\infty(\Omega)$ satisfying $F(u) = y$; there exists a function $\phi \in BV(\Omega)$ and there exist constants values $c^1 \neq c_2 \in \mathbb{R}$ such that $P_{pc}(\phi, c^j) = u$.

Plans

- (sLS) approach

  The graph of $\mathcal{G}_{\alpha,s}$ is not closed!
- We look for generalized minimizers of $\mathcal{G}_{\alpha,s}$.
- Let $\varepsilon > 0$. Define the smooth approximation of H

$$H_\varepsilon(t) = \begin{cases} 1 + t/\varepsilon\,, & t \in [-\varepsilon, 0] \\ H(t)\,, & \text{otherwise}\,. \end{cases}$$

  and

$$P_{\varepsilon,s}(\phi) = c^1 H_\varepsilon(\phi) + c^2 (1 - H_\varepsilon(\phi))\,.$$

- Generalized Minimizers of $\mathcal{G}_{\alpha,s}$:

1) A **vector** $(z,\phi,c^j) \in L_\infty(\Omega) \times L_2(\Omega) \times \mathbb{R}^2$ is called admissible when there exists sequences

i) $\phi_k \in L_2(\Omega)$ with $\lim_{k\to\infty} \|\phi_k - \phi\|_{L_2(\Omega)} \to 0$;

ii) $\exists\ \varepsilon_k \to 0^+$ such that $\lim_{k\to\infty} \|H_{\varepsilon_k}(\phi_k) - z\|_{L_p(\Omega)} \to 0$;

2) A generalized minimizer of $\mathcal{G}_{\alpha,s}$ is considered as any admissible vector (z,ϕ,c^j) minimizing

$$G_\alpha(z,\phi,c^j) = \|F(q(z,c^j)) - y^\delta\|_Y^2 + \alpha R(z,\phi,c^j)\,, \tag{10}$$

where $q(z,c^j) = c^1 z + c^2(1-z)$ and

$$R(z,\phi,c^j) = \inf\{\liminf_{k\to\infty}(\beta_1 |H_{\varepsilon_k}(\phi_k)|_{BV} + \beta_2\|\phi_k\|^2_{L_2(\Omega)})\} + \beta_3\|c^j\|^2_{\mathbb{R}^2}\,.$$

Theorem

1) *Existence* $\mathcal{G}_{\alpha,s}$ *attain a minimizer in the set of admissible vector.*
2) *Convergence for exact data* $\delta = 0$ *Let* $(z_\alpha, \phi_\alpha, c_\alpha^j) \in \operatorname{argmin} G_\alpha$. *If* $\alpha_k \to 0$ *than the corresponding sequence* $(z_{\alpha_k}, \phi_{\alpha_k}, c_{\alpha_k}^j) \in \operatorname{argmin} G_{\alpha_k}$ *is strongly convergent in* $L_p(\Omega) \times L_2(\Omega) \times \mathbb{R}$. *Moreover, the limit is a solution of* (3).
2) *Convergence for noisy data* $\delta \neq 0$ *Let* $\alpha = \alpha(\delta)$ *satisfying* $\alpha \to 0$ *and* $\delta^2/\alpha \to 0$ *as* $\delta \to 0$. *For* $\delta_k \to 0$, *there exists a sequence* $(z_{\alpha_k}, \phi_{\alpha_k}, c_{\alpha_k}^j) \in \operatorname{argmin} G_{\alpha_k}$ *is strongly convergent in* $L_p(\Omega) \times L_2(\Omega) \times \mathbb{R}$. *Moreover, the limit is a solution of* (3).

- pcLS approach:
  The admissible solution is not in a generalized setting!!!
- Let $\tau > 0$. A pair $(\phi, c^j) \in L_2(\Omega) \times \mathbb{R}^2$ is called admissible if $\phi \in BV(\Omega)$ and $|c^1 - c^2| \geq \tau$.

Theorem

1) *$\mathcal{G}_{\alpha,pc}$ attain an admissible minimizer.*
2) *Convergence for exact data $\delta = 0$*
3) *Convergence for noisy data $\delta \neq 0$*

Plans

- The Direct Problem: solving the Poisson boundary value problem

$$\Delta w = u \,, \in \Omega \qquad w = 0 \,, \in \partial\Omega \,, \tag{11}$$

  with $u \in L_2(\Omega)$.

- The Inverse Potential Problem (IPP): consists of recovering the function u, from measurements of the Cauchy data of its corresponding potential w (the measurements are available only on the boundary of Ω).
- Operator equation

$$F : L_2(\Omega) \longrightarrow L_2(\partial\Omega) \qquad F(u) = w|_{\partial\Omega}$$

- $\Omega = (0,1) \times (0,1)$
- $u = 1 + \chi_D \,, \qquad D \subset\subset \Omega$
- For this class of parameters no unique identifiability result is known.
- Nevertheless, our methods prove the ability to detect the desired (piecewise constant) solutions.
- Remarks about the forward operator: $F : L_2(\Omega) \longrightarrow L_2(\partial\Omega)$ is continuous.
  Since $u = 1 + \chi_D$ the operator F is continuous in $L_p(\Omega)$ for $1 \leq p \leq 2$. Therefore the assumption (A2) is satisfied.

- sLS algorithm
- These optimality conditions for the approximation of the Tikhonov functional $\mathcal{G}_{\alpha,s}$ can be written in the form of the system

$$\alpha\phi = L_{\varepsilon,\alpha,\beta}(\phi, c^1, c^2) \qquad \alpha c^j = L^j_{\varepsilon,\alpha,\beta}(\phi, c^1, c^2) \tag{12}$$

were $L_{\varepsilon,\alpha,\beta}(\phi, c^1, c^2)$ and $L^j_{\varepsilon,\alpha,\beta}(\phi, c^1, c^2)$ are the formal derivative of the functional $G_{\alpha,s}$ composed with H_ε.

Given a starting point (ϕ_0, c_0^j), each step of this iterative method consists of three parts

i) Evaluate the residual

$$r_k = F(P_{s,\varepsilon}(\phi_k, c_k^1, c_k^2)) - y^\delta = (w_k)_\nu|_{\partial\Omega} - y^\delta \,,$$

where w_k solves

$$\Delta w_k = P_{s,\varepsilon}(\phi_k, c_k^1, c_k^2) \ in\ \Omega \qquad w_k = 0\ \partial\Omega \,.$$

ii) Evaluate $h_k = F'(P_{s,\varepsilon}(\phi_k, c_k^1, c_k^2))^*(r_k) \in L_2(\Omega)$ solving $\Delta h_k = 0\ in\ \Omega \qquad h_k = r_k\ at\ \partial\Omega$

iii) Calculete $\mu\phi_k$ and μc_k^j as in (12).

- Update

$$\phi_{k+1} = \phi_k + \frac{1}{\alpha}\mu\phi_k \qquad and \qquad c_{k+1}^j = c_k^j + \frac{1}{\alpha}\mu c_k^j$$

- pcLS algorithm
- We consider an explicit iterative method based on the operator splitting technique and derived from the optimality conditions for the Tikhonov functional $\mathcal{G}_{\alpha,pc}$
- First $\mathcal{G}_{\alpha,pc}$ is splitted as

i) $\mathcal{G}^1_{\alpha,pc}(\phi, c^j) :=$
$\|F(P_{pc}(\phi, c^j) - y^\delta\|^2_Y + \alpha\{\beta_1 |P_{pc}(\phi, c^j)|_{BV} + \beta_2 \|c^j\|_{\mathbb{R}}\}$

ii) $\mathcal{G}^2_{\alpha,pc}(\phi) = \beta_3 \|\mathcal{K}(\phi)\|_{L_1(\Omega)}$

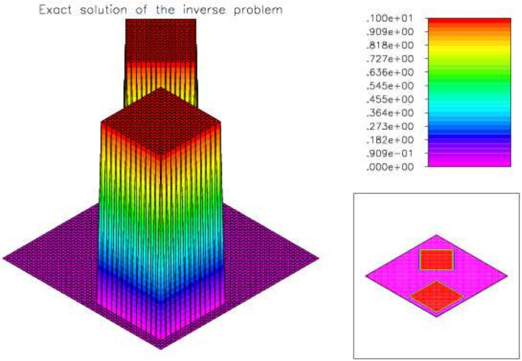
- Each step of the iterative method consists of two parts:

i) The iterate (ϕ_k, c_k^j) is updating using an explicit gradient step w.r.t. the operator $\mathcal{G}^1_{\alpha,pc}(\phi, c^j)$, i.e.,

$$\phi_{k+1/2} = \phi_k - \frac{\partial}{\partial \phi} \mathcal{G}^1_{\alpha,pc}(\phi_k, c_k^j) \qquad c^j_{k+1/2} = c_k^j - \frac{\partial}{\partial c_j} \mathcal{G}^1_{\alpha,pc}(\phi_k, c_k^j) .$$

ii) The obtained approximation $(\phi_{k+1/2}, c^j_{k+1/2})$ is improved by giving a gradient step w.r.t. the operator $\mathcal{G}^2_{\alpha,pc}(\phi_k, c_k^j)$, i.e.,

$$\phi_{k+1} = \phi_{k+1/2} - \frac{\partial}{\partial \phi} \mathcal{G}^2_{\alpha,pc}(\phi_{k+1/2}, c^j_{k+1/2}) \qquad c^j_{k+1} = c^j_{k+1/2} .$$


Exact solution of the inverse problem
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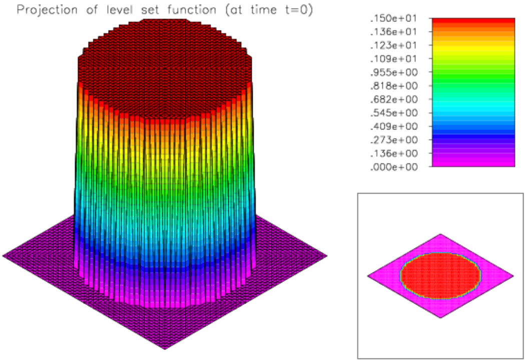
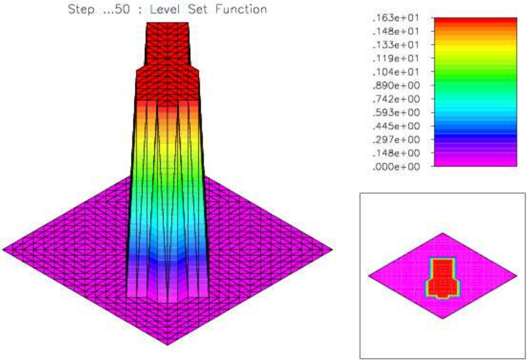
Level Set Algorithms



sLS iteration - $\delta = 0$



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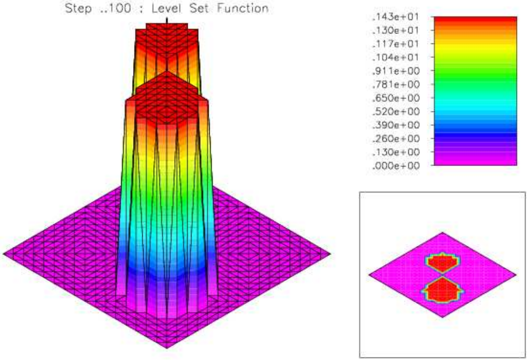

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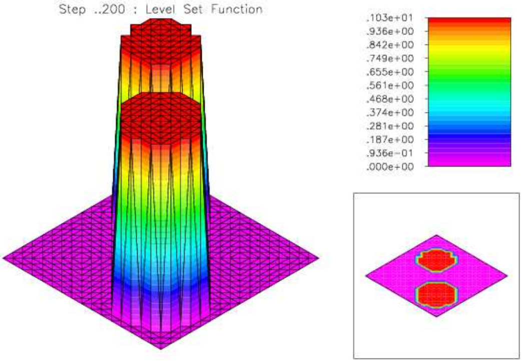

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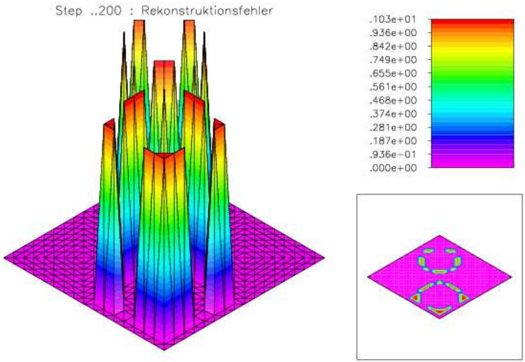
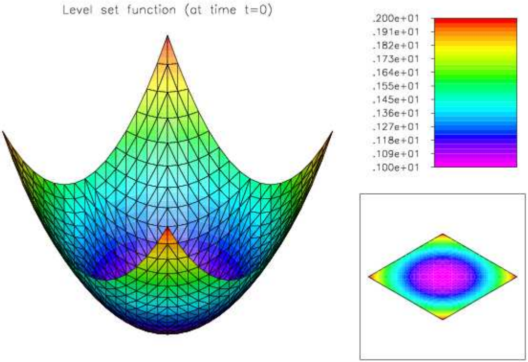


pcls iteration - $\delta = 0$



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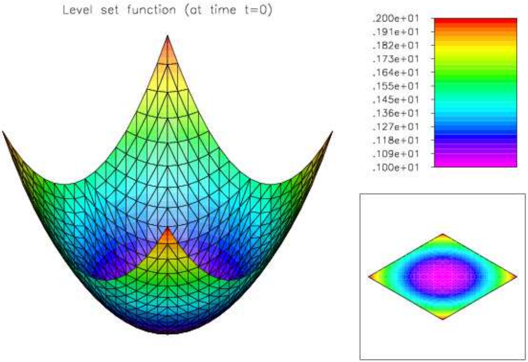

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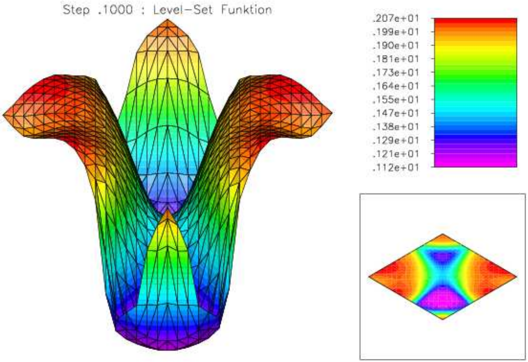

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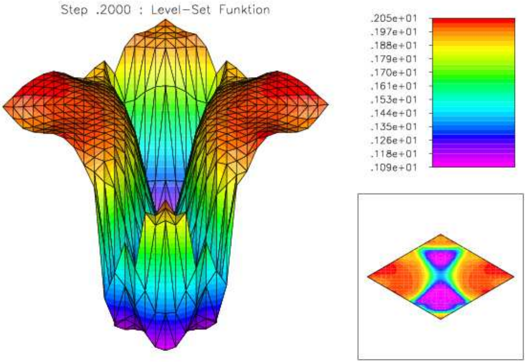

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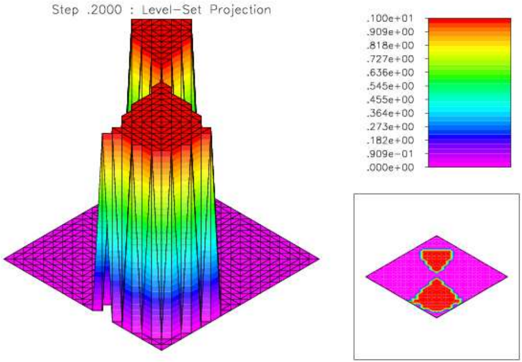

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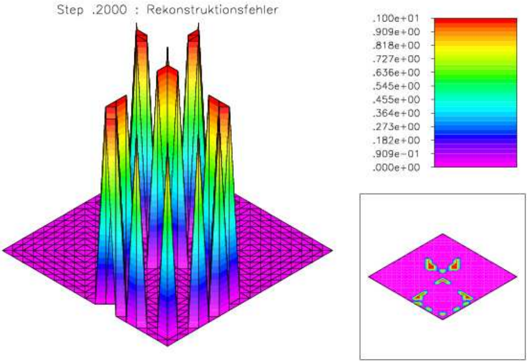
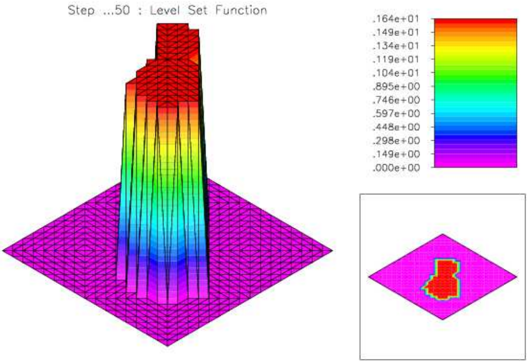


sLS iteration with random noise $= 25\%$



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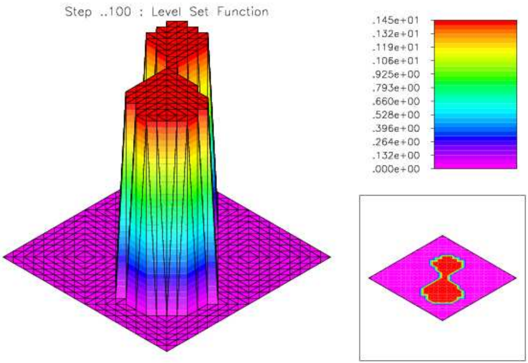

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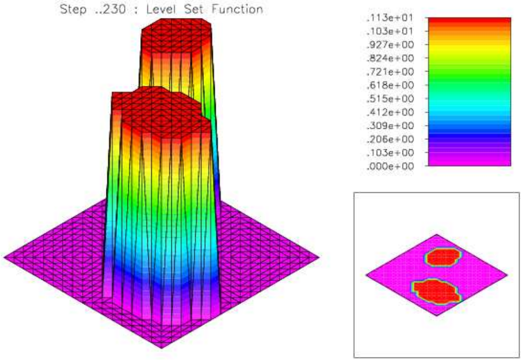

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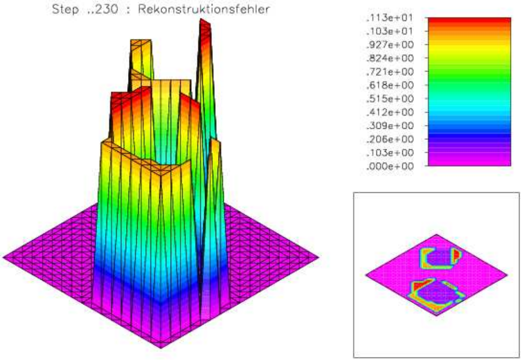

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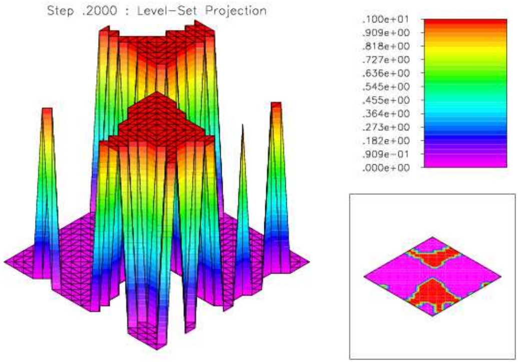

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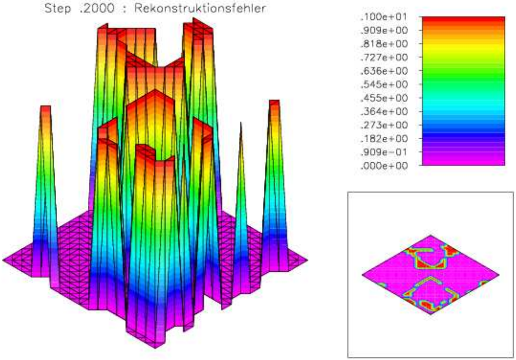

Plans

- Two distinct level-set type approaches for solving ill-posed problems are proposed, where the level-set functions are chosen in L_2-spaces.
- Based on each one of these two level-set approaches, corresponding Tikhonov functionals are derived. We provide convergence analysis for the resulting Tikhonov regularization methods.
- About the numerical???

- What concerns the numerical implementation of the level-set method based on the pcLS approach, some facts have to be observed:

1) Due to the operator splitting technique, we compute several times the step-part (i) before a single calculation of step-part (ii) is performed.
2) Step-part (i) aims to minimize the misfit in the iteration and is the most relevant component of the iteration step.
3) Step-part (ii) aims to drag the iterate ϕ_k to a piecewise constant (integer valued) function. If step-part (ii) is implemented too often, all the iterates ϕ_k become piecewise constant functions and the misfit becomes not monotony decreasing.

- On the other hand, if step-part (ii) is implemented only seldom, the iterates ϕ_k become too smooth and may be trapped in some local minimizer (due to the non -uniqueness of the inverse potential problem).
- Therefore, the determination of how often the step-part (ii) should be implemented is crucial for the good performance of the algorithm. In our numerical experiments the step-part (ii) was omitted in computation of the initial 100 iterations; then we started computing the step-part (ii) after every 20 iterations. For all test problems considered in our experiments, this strategy brought good results.

4) The constant β_3 should be chosen in such a way that $\beta_3 << 1$ in step- part (ii). This choice guarantees that the dragging effect resulting from step-part (ii) is not enforced too strongly.

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Obrigado pela atenção!

Antonio, obrigado pelo convite!