Non-quadratic Regularization of the Inverse Problem Associated to the Black-Scholes PDE
UNDER CAPRICORN [1]
Thanks to Antonio Leitao!!!

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[1] credit to Alvaro De Pierro

Outline

Options, Derivatives, Futures

- Why?

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- Why? HEDGING

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- Who? LOTS OF PEOPLE! (Derivative markets are bigger than the underlying ones!)
- Examples

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  - Insurance Markets
  - Pension Funds
  - Currency Markets

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Remark: Good estimation of the local volatility is crucial for the consistent pricing of other contracts (in particular exotic derivatives).

Derivative Contracts

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Its payoff is given by

$$h(X_T) = \begin{cases} X_T - K & \text{if } X_T > K, \\ 0 & \text{if } X_T \leq K. \end{cases}$$

European Put Option: a forward contract that gives the holder the right to sell a unit of the asset for a strike price K at the maturity date T. Its payoff is

$$h(X_T) = \begin{cases} K - X_T & \text{if } X_T < K, \\ 0 & \text{if } X_T \geq K. \end{cases}$$

At other times, the contract has a value known as the *derivative price*. The option price at time t with stock price $X_t = x$ is denoted by $P(t, x)$.

Call and Put Payoffs

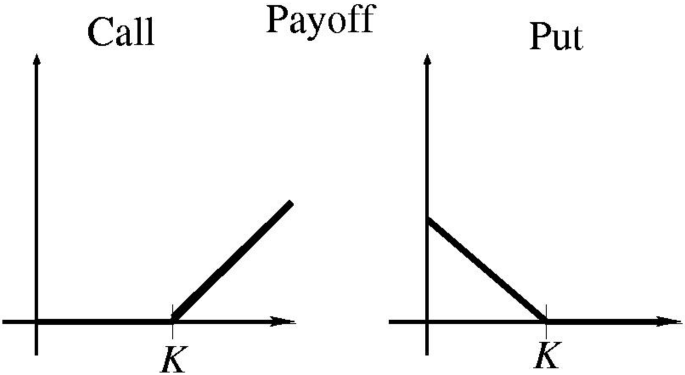

Figure: The payoff associated to call and put options

Fundamental Question: *How to price such an obligation fairly given today's information?*

How to address the pricing problem?

Black-Scholes Market Model

Assume two assets: a risky stock and a riskless bond.

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t,$$

where W_t is a standard Brownian Motion and volatility σ is constant

$$\mathrm{d}\beta_t = r\beta_t \mathrm{d}t.$$

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Under a number of assumptions one gets: The Black-Scholes Equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P}{\partial x^2} + r\left(x\frac{\partial P}{\partial x} - P\right) = 0$$

$$P(T,x) = h(x)$$

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Note 2: Final Value Problem

Main Contributions

Figure: L. Bachelier

- L. Bachelier (Paris)
- P. Samuelson
- F. Black
- M. Scholes
- R. Merton

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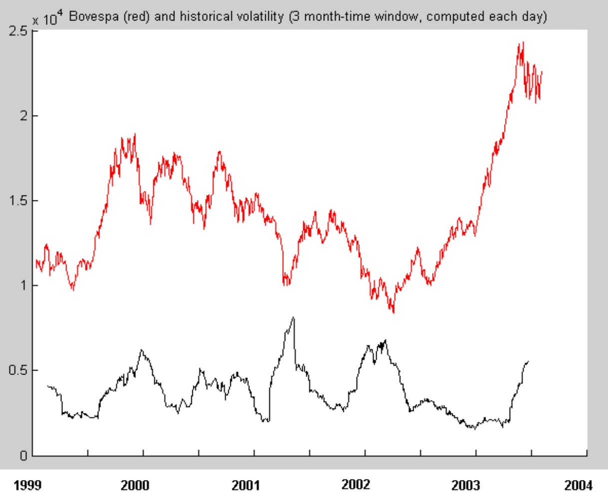

Figure: Example of Data from IBOVESPA

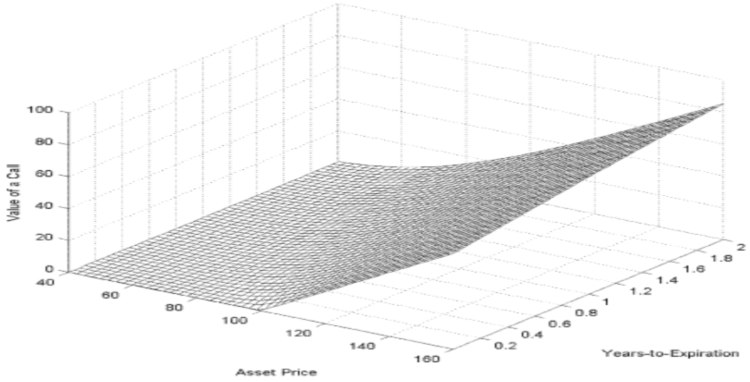

Figure: Example of the Solution to Black-Scholes

Motivation and Goals

Good model selection is crucial for modern sound financial practice.

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Application

Volatility surface calibration is crucial in many applications. E.G.: risk management, hedging, and the evaluation of exotic derivatives.

Main Features

- Address in a general and rigorous way the key issue of convergence and sensitivity of the regularized solution when the noise level of the observed prices goes to zero.
- Our approach relates to different techniques in volatility surface estimation. e.g.: the Statistical concept of exponential families and entropy-based estimation.
- Our framework connects with the Financial concept of Convex Risk Measures.

Limitations of Classical Black-Scholes

- log-normality of asset prices is not verified by statistical tests
- option prices are subjet to the smile effects
- volatility of the prices tends fluctuate with time and revert to a mean value

Local Volatility Models

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or in the case you have dividends:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma(t,x)^2 x^2 \frac{\partial^2 P}{\partial x^2} + (r-d)x\frac{\partial P}{\partial x} - rP = 0$$

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The Direct Problem:

Given $\sigma = \sigma(t, x)$ and the payoff information, determine $P = P(t, x, T, K; \sigma)$

The Inverse Problem

Given a set of observed prices

$$\{P = P(t, x, T, K; \sigma)\}_{(T,K)\in\mathcal{S}}$$

find the volatility $\sigma = \sigma(t, x)$.
The set $\mathcal{S}$ is taken typically as $[T_1, T_2] \times [K_1, K_2]$.
Caveat: The data is not realistic at all!!!

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Caveat: The data is not realistic at all!!!
In Practice: Very limited and scarce data
Note: To price in a consistent way the so-called exotic derivatives one has to know σ and not only the transition probabilities

Assuming that there exists a local volatility function $\sigma = \sigma(S,t)$ for which (1) holds Dupire(1994) showed that the call price satisfies

$$\begin{cases} \partial_T C - \frac{1}{2}\sigma^2(K,T)K^2\partial_K^2 C + rS\partial_K C = 0 \ , \quad S > 0 \ , t < T \\ C(K, T = 0) = (S - K)^+ \ , \end{cases} \tag{3}$$

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$$\sigma(K,T) = \sqrt{2\left(\frac{\partial_T C + rK\partial_K C}{K^2\partial_K^2 C}\right)} \tag{4}$$

In practice To estimate σ from (3), limited amount of discrete data and thus interpolate.

The Smile Curve and Dupire's Equation

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In practice To estimate σ from (3), limited amount of discrete data and thus interpolate. Numerical instabilities! Even to keep the argument positive is hard.

Many interpretations of Local Vol Models

1. Stochastic Clock (time of trading)
2. Local vol as a weighted average of the implied volatility over all possible scenarios. (IMPORTANT RESULT!!!)

$$\sigma^2(K, T, S_0) = \mathbb{E}\left[v_T \middle| S_T = K\right] ,$$

where v_T is the implied variance.

Remark: Good estimation of the local volatility is crucial for the consistent pricing of exotics. In fact, prices of exotics based on constant volatility can lead to pretty wrong results.

Problem Statement

Starting Point: Dupire forward equation [Dup94]

$$-\partial_T U + \frac{1}{2}\sigma^2(T,K)K^2\partial_K^2 U - (r-q)K\partial_K U - qU = 0\,, \quad T > 0\,, \tag{5}$$

$$K = S_0 e^y\,,\ \tau = T - t\,,\ b = q - r\,, \quad u(\tau, y) = e^{q\tau}U^{t,S}(T,K) \tag{6}$$

and

$$a(\tau, y) = \frac{1}{2}\sigma^2(T - \tau; S_0 e^y)\,, \tag{7}$$

Set $q = r = 0$ for simplicity to get:

$$u_\tau = a(\tau, y)(\partial_y^2 u - \partial_y u) \tag{8}$$

and initial condition

$$u(0, y) = S_0(1 - e^y)^+ \tag{9}$$

Problem Statement

The Vol Calibration Problem

Given an observed set

$$\{u = u(t, S, T, K; \sigma)\}_{(T,K)\in\mathcal{S}}$$

find $\sigma = \sigma(t, S)$ that best fits such market data

Noisy data: $u = u^{\delta}$

Admissible *convex* class of calibration parameters:

$$\mathcal{D}(F) := \{a \in a_0 + H^{1+\varepsilon}(\Omega) : \underline{a} \leq a \leq \overline{a}\}. \quad (10)$$

where, for $0 \leq \varepsilon$ fixed, $U := H^{1+\varepsilon}(\Omega)$ and $\overline{a} > \underline{a} > 0$.

Parameter-to-solution operator

$F : \mathcal{D}(F) \subset H^{1+\varepsilon}(\Omega) \to L^2(\Omega)$

$$F(a) = u(a) \quad (11)$$

Literature

Very vast!!!

- Avellaneda et al. [ABF$^+$00, Ave98c, Ave98b, Ave98a, AFHS97]
- Bouchev & Isakov [BI97]
- Crepey [Cré03]
- Derman et al. [DKZ96]
- Egger & Engl [EE05]
- Hofmann et al. [HKPS07, HK05]
- Jermakyan [BJ99]
- Achdou & Pironneau (2004)

- Abken et al. (1996)
- Ait Sahalia, Y & Lo, A (1998)
- Berestycki et al. (2000)
- Buchen & Kelly (1996)
- Coleman et al. (1999)
- Cont, Cont & Da Fonseca (2001)
- Jackson et al. (1999)
- Jackwerth & Rubinstein (1998)
- Jourdain & Nguyen (2001)
- Lagnado & Osher (1997)
- Samperi (2001)
- Stutzer (1997)

Well-Posed and Ill-Posed Problems

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Requirements:

- Stability: Computed solution should depend continuously on data. Stability bounds for the solution.
- Approximation: Computed solution should be close to the solution of equation for noise-free data

Regularization

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Nonlinear Problems: Tikhonov regularization.

Classical Theory: Add a quadratic regularization term.

Approach

Theorem (Egger-Engel[EE05] Crepey[Cré03])

The parameter to solution map

$$F : H^{1+\varepsilon}(\Omega) \to L^2(\Omega)$$

is

- *weak sequentialy continuous*
- *compact and weakly closed*

Consequences:

- The inverse problem is ill-posed.
- We can prove that the inverse problem satisfies the conditions to apply the regularization theory.

Convex Tikhonov Regularization

For given convex f minimize the Tikhonov functional

$$\mathcal{F}_{\beta,u^\delta}(a) := ||F(a) - u^\delta||^2_{L^2(\Omega)} + \beta f(a) \tag{12}$$

over $\mathcal{D}(F)$, where, $\beta > 0$ is the regularization parameter.

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$$||\bar{u} - u^\delta||_{L^2(\Omega)} \leq \delta\,, \tag{13}$$

where $\bar{u}$ is the data associated to the actual value $\hat{a} \in \mathcal{D}(F)$.

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Assumption (very general!)

Let $\varepsilon \geq 0$ be fixed. $f : \mathcal{D}(f) \subset H^{1+\varepsilon}(\Omega) \longrightarrow [0, \infty]$ is a convex, proper and sequentially weakly lower semi-continuous functional with domain $\mathcal{D}(f)$ containing $\mathcal{D}(F)$.

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- Does there exist a minimizer of the regularized problem?

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Main Theoretical Result

$F(a) = u(a)$ (11) $\mathcal{F}_{\beta,u^\delta}(a) := ||F(a) - u^\delta||^2_{L^2(\Omega)} + \beta f(a)$ (12)

Theorem (Existence, Stability, Convergence)

For the regularized inverse problem

$$F(a) = u \tag{14}$$

we have:

- $\exists$ *minimizer of* $\mathcal{F}_{\beta,u^\delta}$.
- *If* $(u_k) \to u$ *in* $L^2(\Omega)$, *then* $\exists$ *a seq.* (a_k) *s.t.*

  $$a_k \in \text{argmin}\{\mathcal{F}_{\beta,u_k}(a) : a \in \mathcal{D}\}$$

  has a subsequence which converges weakly to $\tilde{a}$
- $\tilde{a} \in \text{argmin}\{\mathcal{F}_{\beta,u_k}(a) : a \in \mathcal{D}\}$

Theorem (cont.) NOISY CASE

Take $\beta = \beta(\delta) > 0$ and assume

- $\beta(\delta)$ satisfies

$$\beta(\delta) \to 0 \text{ and } \frac{\delta^2}{\beta(\delta)} \to 0\,, \text{ as } \delta \to 0\,. \qquad (15)$$

- The seq. (δ_k) converges to 0, and that $u_k := u^{\delta_k}$ satisfies $\|\bar{u} - u_k\| \leq \delta_k$.

Then,

1. Every seq. $(a_k) \in \operatorname{argmin} \mathcal{F}_{\beta_k,u_k}$, has weak-convergent subseq. $(a_{k'})$.
2. The limit $a^\dagger := w - \lim a_{k'}$ is an f-minimizing solution of (11), and $f(a_k) \to f(a^\dagger)$.
3. If the f-minimizing solution $a^\dagger$ is unique, then $a_k \to a^\dagger$ weakly.

Bregman distance

Let f be a convex function. For $a \in \mathcal{D}(f)$ and $\partial f(a)$ the subdifferential of the functional f at a.

We denote by $\mathcal{D}(\partial f) = \{\tilde{a} : \partial f(\tilde{a}) \neq \emptyset\}$ the domain of the subdifferential.

The Bregman distance w.r.t $\zeta \in \partial f(a_1)$ is defined on $\mathcal{D}(f) \times \mathcal{D}(\partial f)$ by

$$D_\zeta(a_2, a_1) = f(a_2) - f(a_1) - \langle \zeta, a_2 - a_1 \rangle \ .$$

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Assumption (1)

We assume that

1. $\exists$ *an f-minimizing sol.* $a^\dagger$ *of* (11), $a^\dagger \in \mathcal{D}_B(f)$.
2. $\exists \beta_1 \in [0,1)$, $\beta_2 \geq 0$, *and* $\zeta^\dagger \in \partial f(a^\dagger)$ *s.t.*

$$\langle \zeta^\dagger, a^\dagger - a \rangle \leq \beta_1 D_{\zeta^\dagger}(a, a^\dagger) + \beta_2 \left\| F(a) - F(a^\dagger) \right\|_{L}^{2}(\Omega) \ \ \textit{for } a \in \mathcal{M}_{\beta_{max}}(\rho) \ , \tag{16}$$

where $\rho > \beta_{max} f(a^\dagger) > 0$.

Convergence rates [SGG$^+$08]

Theorem (Convergence rates [SGG$^+$08])

Let F, f, $\mathcal{D}$, $H^{1+\varepsilon}(\Omega)$, and $L^2(\Omega)$ satisfy Assumption 1. Moreover, let $\beta : (0,\infty) \to (0,\infty)$ satisfy $\beta(\delta) \sim \delta$. Then

$$D_{\zeta^\dagger}(a^\delta_\beta, a^\dagger) = O(\delta)\,, \quad \left\| F(a^\delta_\beta) - u^\delta \right\|_{L^2(\Omega)} = O(\delta)\,,$$

and there exists $c > 0$, such that $f(a^\delta_\beta) \le f(a^\dagger) + \delta/c$ for every δ with $\beta(\delta) \le \beta_{max}$.

Example: The regularization functional f as the Boltzmann-Shannon entropy

$$f(a) = \int_\Omega a \log(a)\, dx\,, \qquad a \in \mathcal{D}(F)\,,$$

Putting it all together

NOTE: We have proved

We have also proved a tangential cone condition for this problem, which implies that the Landwever iteration converges in a suitable neighborhood. **Landweber Iteration [EHN96]:**

$$a_{k+1}^{\delta} = a_k^{\delta} + cF'(a_k^{\delta})^*(u^{\delta} - F(a_k^{\delta}))\,. \tag{17}$$

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Discrepancy Principle:

$$\left\| u^{\delta} - F(a^{\delta}_{k_*(\delta,y^{\delta})}) \right\| \leq r\delta < \left\| u^{\delta} - F(a_k^{\delta}) \right\| \,, \tag{18}$$

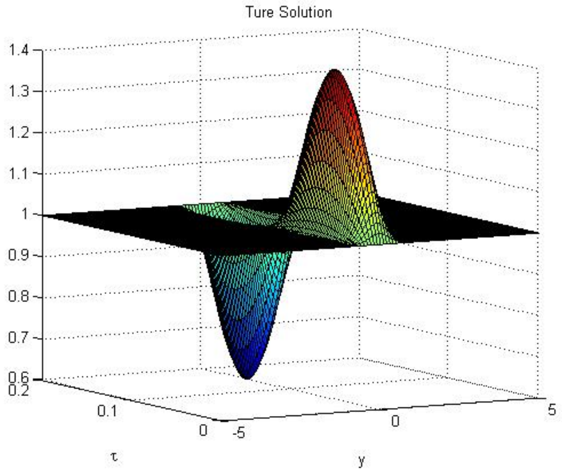
where

$$r > 2\frac{1+\eta}{1-2\eta} \,, \tag{19}$$

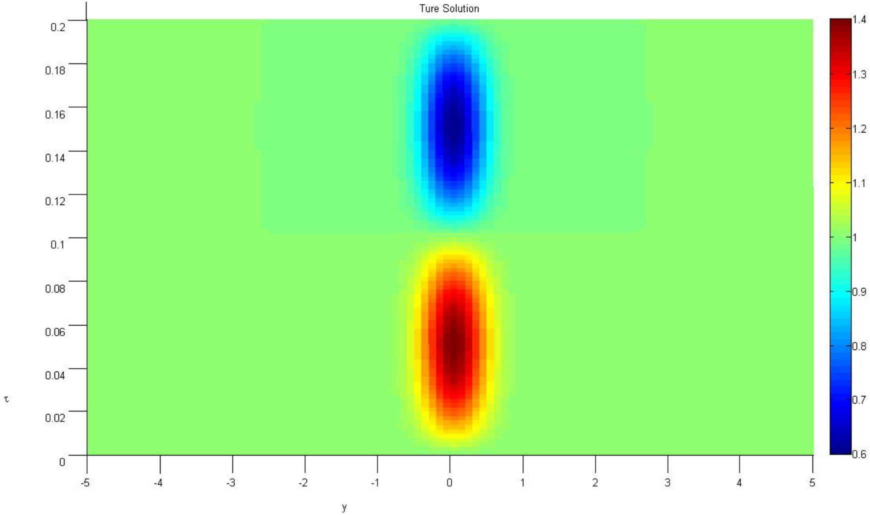
is a relaxation term.
If the iteration is stopped at index $k_*(\delta, y^{\delta})$ such that for the first time the residual becomes small compared to the quantity $r\delta$.

- Using a Landweber iteration technique we implemented the calibration.
- Produced for different test variances a the option prices and added different levels of multiplicative noise.
- The examples consisted of perturbing $a = 1$ during a period of $T = 0, \cdots, 0.2$ and log-moneyness y varying between -5 and 5.
- Initial guess: Constant volatility.

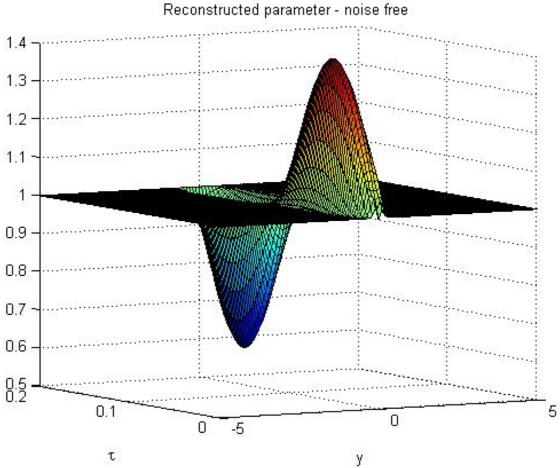
Numerical Examples - Exact Solution



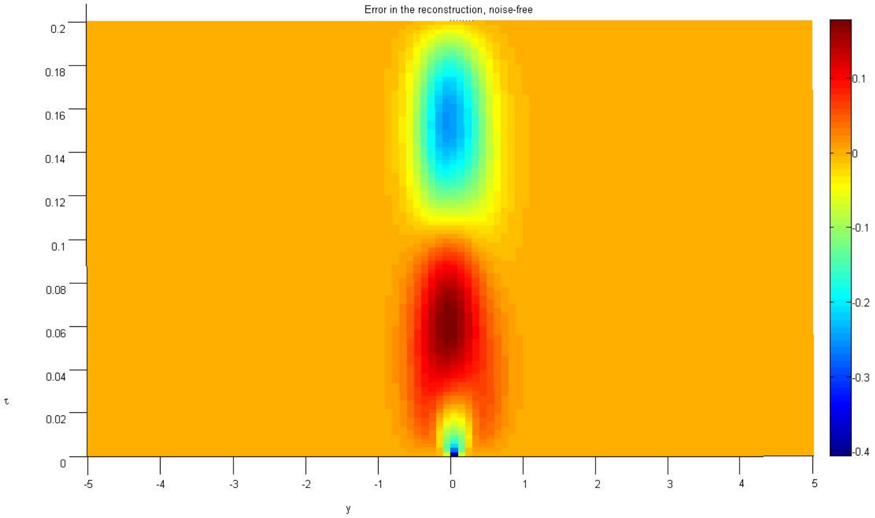
Numerical Examples - Exact Solution



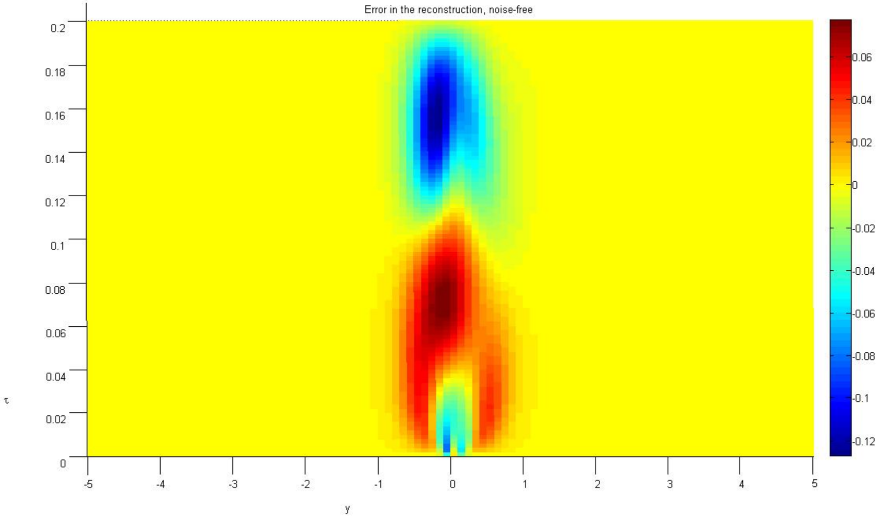
Numerical Examples 1 - noiseless - 4000 steps



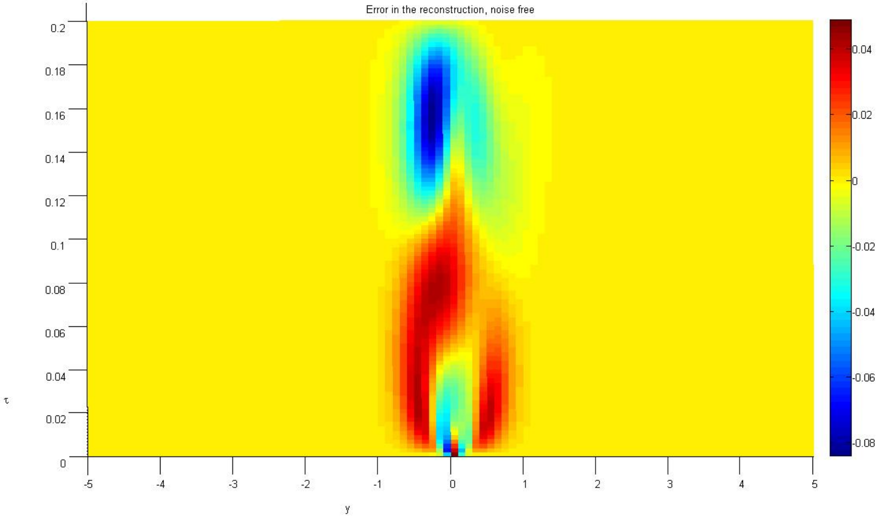
Numerical Examples 1 - error - 100 steps



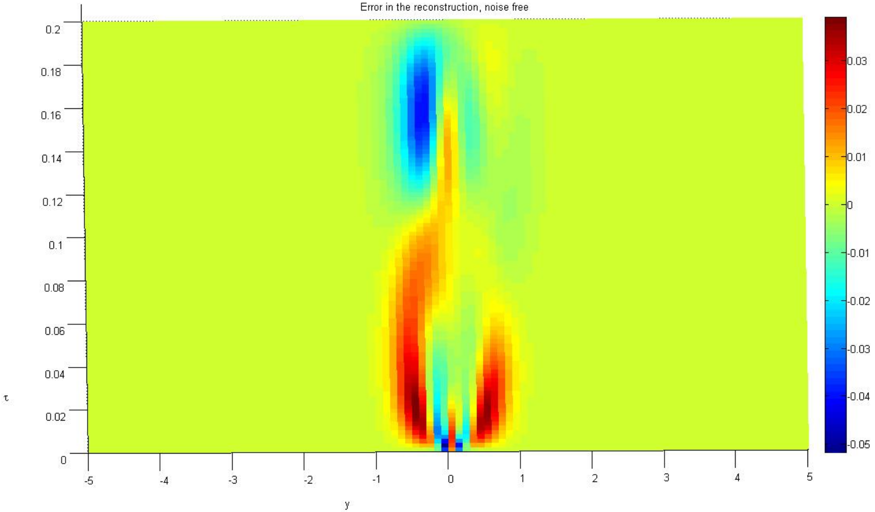
Numerical Examples 1 - error - 300 steps



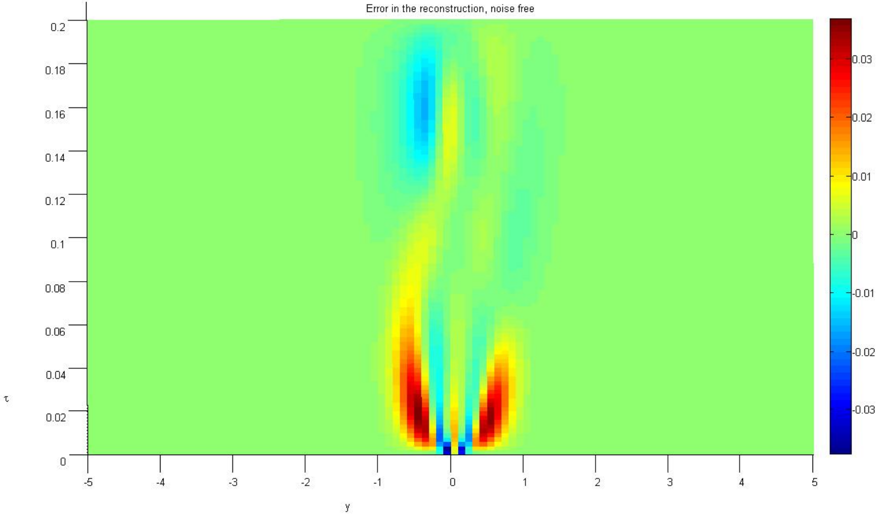
Numerical Examples 1 - error - 500 steps



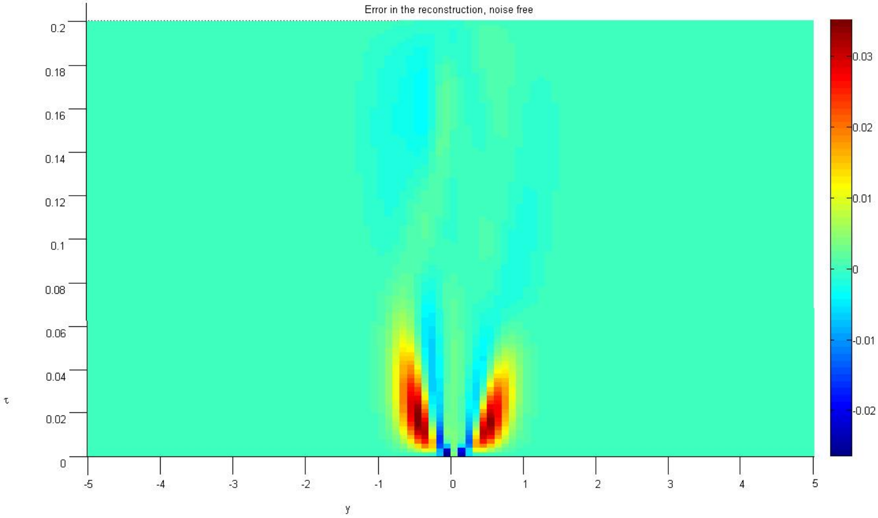
Numerical Examples 1 - error - 1000 steps

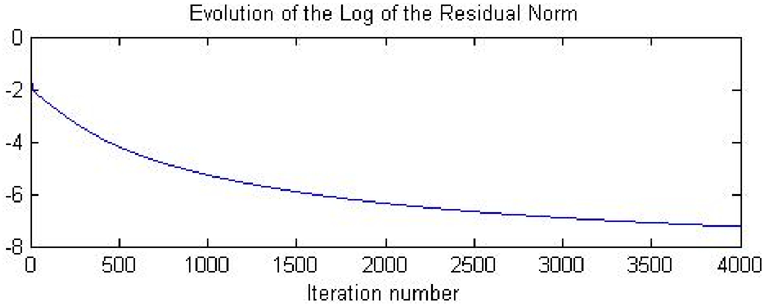


Numerical Examples 1 - error - 2000 steps

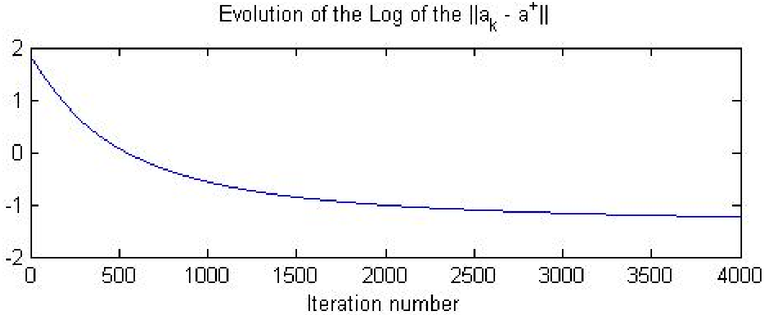


Numerical Examples 1 - error - 4000 steps



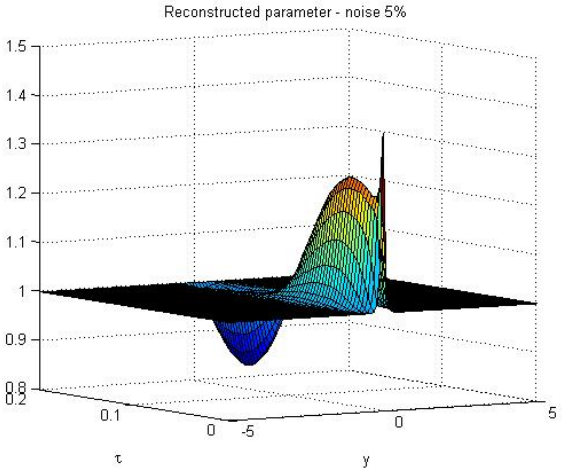


Evolution of the Log of the Residual Norm
0
-2
-4
-6
-8
0
500
1000
1500
2000
2500
3000
3500
4000
Iteration number

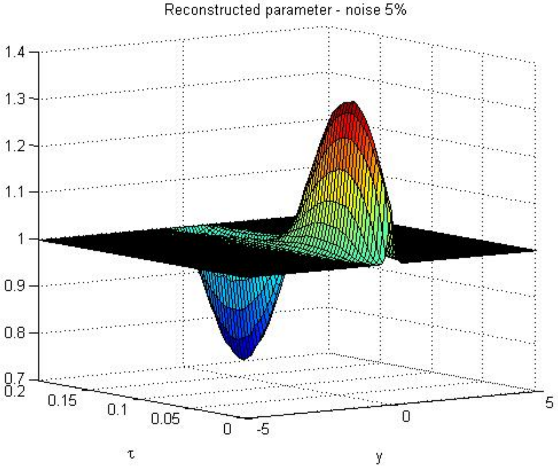


Evolution of the Log of the ||a_k - a^+||
2
1
0
-1
-2
0
500
1000
1500
2000
2500
3000
3500
4000
Iteration number

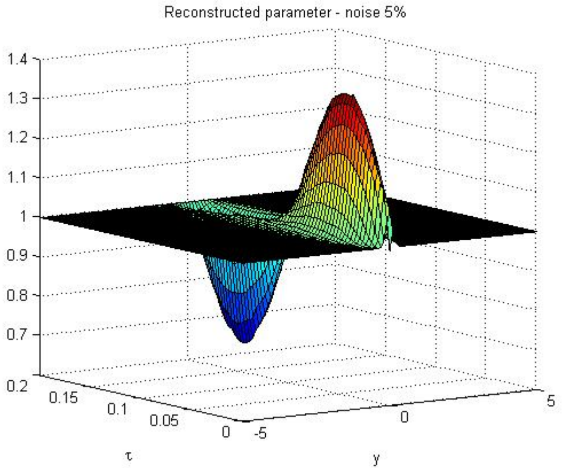
Numerical Examples 2 - 5% noise level - 100 steps



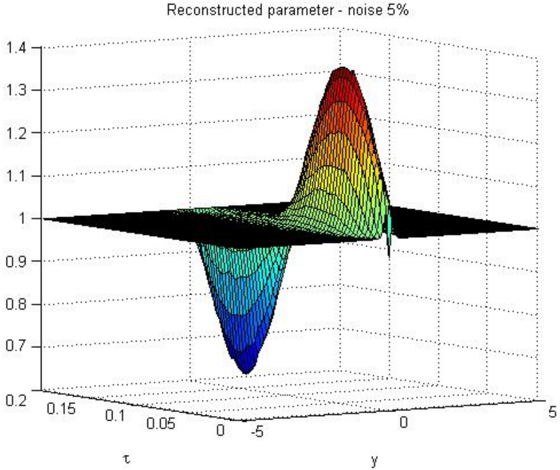
Numerical Examples 2 - 5% noise level - 200 steps



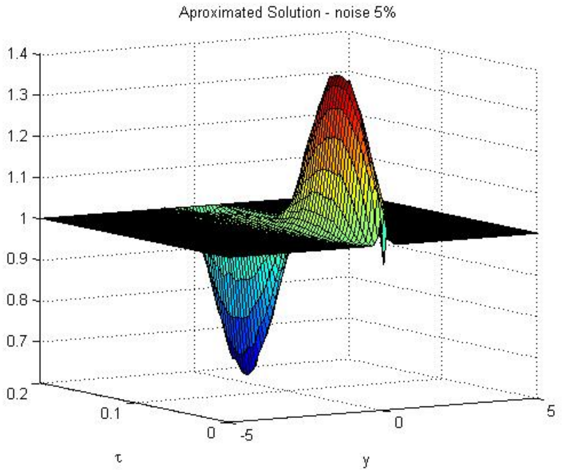
Numerical Examples 2 - 5% noise level - 300 steps

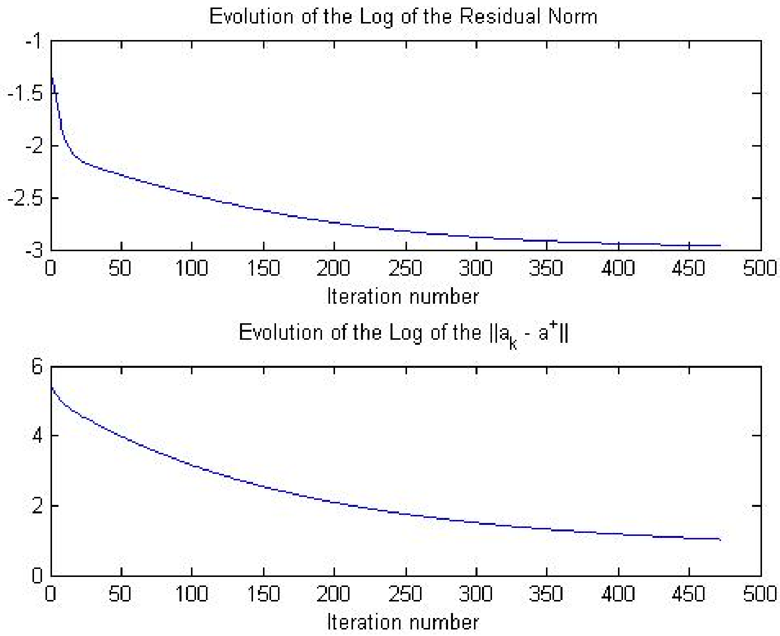


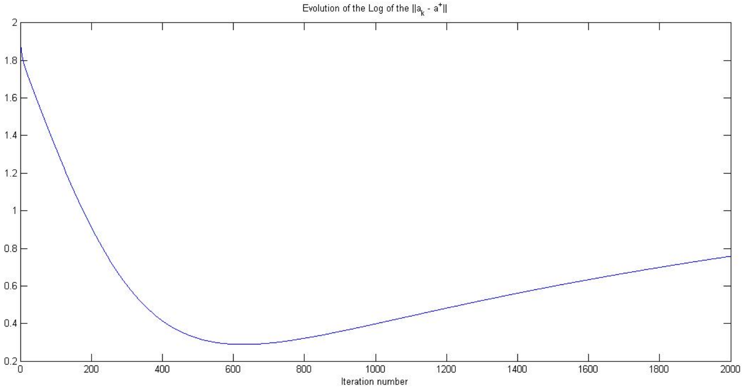
Numerical Examples 2 - 5% noise level - 400 steps



Numerical Examples 2 - 5% noise level - Stopping criteria

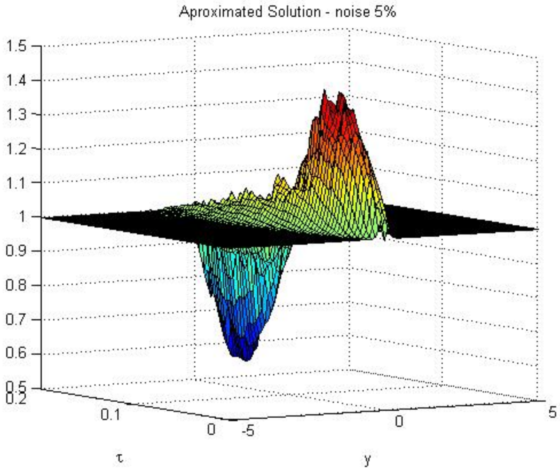



Evolution of the Log of the Residual Norm
-1
-1.5
-2
-2.5
-3
0
50
100
150
200
250
300
350
400
450
500
Iteration number
Evolution of the Log of the ||a_k - a^+||
6
4
2
0
0
50
100
150
200
250
300
350
400
450
500
Iteration number


Evolution of the Log of the ||a_k - a^+||
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
200
400
600
800
1000
1200
1400
1600
1800
2000
Iteration number

Numerical Examples 2 - 5% noise level - 2000 iterations

Too many!!!



Numerical Examples: with Real Data

Reconstruction of $a = \sigma^2/2$ with PBR Stock Data (implemented by Vinicius L. Albani/IMPA)

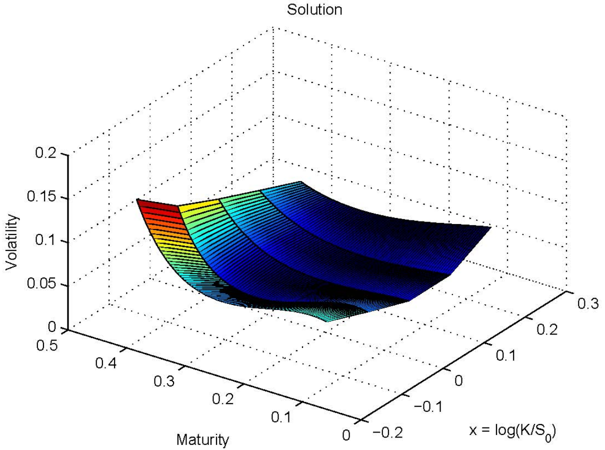


Figure: Minimal Entropy functional / Landweber Method / a priori Implied Vol / maturities: 2010-11

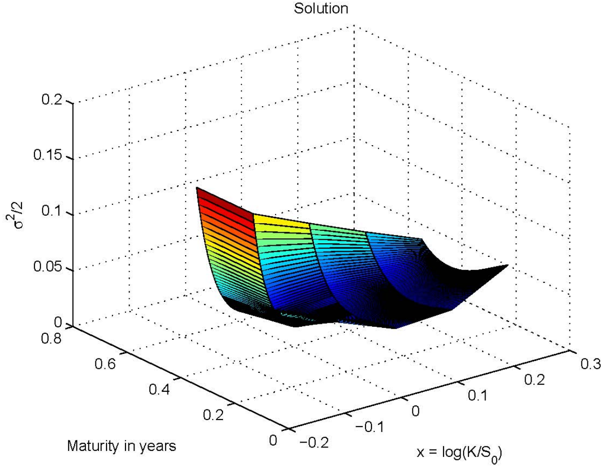


Figure: Minimal Entropy functional / Minimization (Levenberg-Marquadt) Method /

Conclusions

- The problem of volatility surface calibration is a classical and fundamental one in Quantitative Finance
- Unifying framework for the regularization that makes use of tools from Inverse Problem theory and Convex Analysis.
- Establishing convergence and convergence-rate results.
- Obtain convergence of the regularized solution with respect to the noise level in $L^1(\Omega)$
- The connection with exponential families opens the door to recent works on entropy-based estimation methods.
- The connection with convex risk measures required the use of techniques from Malliavin calculus.
- Implemented a Landweber type calibration algorithm.

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THANK YOU FOR YOUR ATTENTION!!!

Collaborators:

V. Albani (IMPA), A. de Cezaro (FURG), O. Scherzer (Vienna)

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