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Measure Theory and Integration

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CONTENTS

ACKNOWLEDGMENTS

These notes were created fo provide an expanded version of a part of Folland's book (see [1]) on Measure Theory and Integration, and a little of L^p spaces. Here I prove many results that are left for the reader, correct some small mistakes present in the book, as well as provide solutions for many exercises. This is not an independently created material, and all the credit of the theory and exercises is due to [1]. I do not claim ownership of any content presented in these notes.

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For whoever is using these notes: if you wish to suggest corrections and comments, please e-mail me at `m.bortolan@ufsc.br`. Feel free to e-mail asking me for the .tex files of these notes if you wish.

Happy measuring and integrating!

"The problem is not the problem. The problem is your attitude about the problem."
Captain Jack Sparrow.

CHAPTER 1 MEASURE SPACES

1.1 | σ-ALGEBRAS

In this section X is a nonempty set, $\mathbb{N} = \{1, 2, 3, \cdots\}$ represents the positive integers and $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ represents the nonnegative integers.

Definition 1.1.1. *An* **algebra** *of sets in X is a nonempty collection $\mathcal{A}$ of subsets of X such that given $A, B \in \mathcal{A}$ we have*

$$A \cup B \in \mathcal{A} \qquad \textit{and} \qquad A^c = X \setminus A \in \mathcal{A}.$$

In other words, $\mathcal{A}$ is an algebra if it is closed under unions and complements.

It is clear that if $\mathcal{A}$ is an algebra then given $A, B \in \mathcal{A}$ we have $A \cap B \in \mathcal{A}$, since $A \cap B = (A^c \cup B^c)^c$. Now if n is a fixed positive integer and $E_1, \cdots, E_n \in \mathcal{A}$ then

$$\bigcup_{i=1}^{n} E_i \in \mathcal{A} \qquad \text{and} \qquad \bigcap_{i=1}^{n} E_i \in \mathcal{A}.$$

Moreover we have the following:

Proposition 1.1.2. *If $\mathcal{A}$ is an algebra in X then $\varnothing \in \mathcal{A}$ and $X \in \mathcal{A}$.*

Proof. Since $\mathcal{A}$ is nonempty, there exists a set $A \in \mathcal{A}$. Therefore we have $\varnothing = A \cap A^c \in \mathcal{A}$ and also $X = \varnothing^c \in \mathcal{A}$. ■

Definition 1.1.3. *A* **σ-algebra** *$\mathcal{A}$ in X is an algebra $\mathcal{A}$ which is closed under countable unions, that is, if $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ and $E = \bigcup\limits_{n=1}^{\infty} E_n$ then $E \in \mathcal{A}$.*

Clearly a σ-algebra $\mathcal{A}$ is also closed by countable intersections, since $\bigcap_n E_n = (\bigcup_n E_n^c)^c$.

Remark 1.1.4. *It is worth to point out that an algebra $\mathcal{A}$ is a σ-algebra if it is closed under disjoint unions. In fact let $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$. Then given $k \in \mathbb{N}$ we have*

$$F_k = E_k \setminus \left(\bigcup_{n=1}^{k-1} E_n \right) = E_k \cap \left(\bigcup_{n=1}^{k-1} E_n \right) \in \mathcal{A}.$$

Hence the sequence $\{F_k\}_{k\in\mathbb{N}}$ is in $\mathcal{A}$, is pairwise disjoint and

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{k=1}^{\infty} F_k \in \mathcal{A}.$$

Example 1.1.5. *Given a nonempty set X, are σ-algebras:*

1. $\mathcal{A} = \{\varnothing, X\}$.
2. $\mathcal{A} = \mathcal{P}(X)$, *the collection of all subsets of X.*
3. *If X is uncountable:*

$$\mathcal{A} = \{E \subset X \colon E \text{ is countable or } E^c \text{ is countable}\}.$$

In fact, given $A, B \in \mathcal{A}$ then $A \cup B$ is countable if both A and B are countable and $(A \cup B)^c = A^c \cap B^c$ is countable if at least one of them has countable complement. Also $A^c \in \mathcal{A}$, thus $\mathcal{A}$ is an algebra. A similar argument shows that $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{A}$ if $E_n \in \mathcal{A}$ for all $n \in \mathbb{N}$, and hence $\mathcal{A}$ is a σ-algebra.

This σ-algebra is called the **σ-algebra of countable or co-countable sets**.

Proposition 1.1.6. *Let $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ be a collection of σ-algebras in X, indexed over a set Λ. Then*

$$\bigcap_{\lambda\in\Lambda} \mathcal{A}_\lambda = \{E \subset X \colon E \in \mathcal{A}_\lambda \text{ for all } \lambda \in \Lambda\}$$

is also a σ-algebra in X.

Proof. It is straightforward. ∎

Let $\mathcal{E} \subset \mathcal{P}(X)$. Then, using Proposition 1.1.6, there is a unique smallest σ-algebra $\mathcal{M}(\mathcal{E})$ which contains $\mathcal{E}$, namely, the intersection of all σ-algebras in X which contain $\mathcal{E}$ (there is always at least one, namely $\mathcal{P}(X)$).

$\mathcal{M}(\mathcal{E})$ is called the σ-algebra **generated** by $\mathcal{E}$.

Lemma 1.1.7. *If $\mathcal{E} \subset \mathcal{M}(\mathcal{F})$ then $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F})$.*

Proof. Since $\mathcal{M}(\mathcal{F})$ is a σ-algebra containing $\mathcal{E}$, by definition, it contains $\mathcal{M}(\mathcal{E})$. ■

1.2 | Product σ-algebras

Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a collection of nonempty sets and consider the **product space**

$$X = \prod_{\lambda\in\Lambda} X_\lambda = \Big\{ f\colon \Lambda \to \bigcup_{\lambda\in\Lambda} X_\lambda \text{ such that } f(\lambda) \in X_\lambda \text{ for each } \lambda \in \Lambda \Big\}.$$

We denote for simplicity the function f by $(x_\lambda)_{\lambda\in\Lambda}$, where $x_\lambda = f(\lambda)$ for each $\lambda \in \Lambda$. Consider also for each $\alpha \in \Lambda$ the **coordinate map** $\pi_\alpha\colon X \to X_\alpha$, given by

$$\pi_\alpha((x_\lambda)_{\lambda\in\Lambda}) = x_\alpha.$$

Now consider a σ-algebra $\mathcal{M}_\lambda$ in X_λ for each $\lambda \in \Lambda$. For a fixed $\alpha \in \Lambda$ and $E \in \mathcal{M}_\alpha$, consider the set

$$\pi_\alpha^{-1}(E) = \{(x_\lambda)_{\lambda\in\Lambda}\colon x_\alpha \in E\},$$

and also the collection of sets

$$\mathcal{E} = \bigcup_{\alpha\in\Lambda} \bigcup_{E\in\mathcal{M}_\alpha} \pi_\alpha^{-1}(E).$$

The σ-algebra generated by $\mathcal{E}$ is called the **product σ-algebra** in X and is denoted by $\otimes_{\lambda\in\Lambda}\mathcal{M}_\lambda$. If $\Lambda = \{1, \cdots, n\}$ we denote $\otimes_{\lambda\in\Lambda}\mathcal{M}_\lambda = \otimes_{i=1}^n \mathcal{M}_i = \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$.

Proposition 1.2.1. *If Λ is countable, them $\otimes_{\lambda\in\Lambda}\mathcal{M}_\lambda$ is the σ-algebra generated by*

$$\mathcal{F} = \Big\{ \prod_{\lambda\in\Lambda} E_\lambda \colon E_\lambda \in \mathcal{M}_\lambda \Big\}.$$

Proof. Given $E_\lambda \in \mathcal{M}_\lambda$, define $E_\beta = X$ for all $\beta \neq \lambda$. Hence

$$\pi_\lambda^{-1}(E_\lambda) = \{(x_\beta)_{\beta\in\Lambda}\colon x_\lambda \in E_\lambda\} = \prod_{\beta\in\Lambda} E_\beta,$$

which means that $\mathcal{E} \subset \mathcal{F}$ and hence $\otimes_{\lambda\in\Lambda}\mathcal{M}_\lambda = \mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F})$.

On the other hand given $\prod_{\lambda\in\Lambda} E_\lambda \in \mathcal{F}$ we have

$$\prod_{\lambda\in\Lambda} E_\lambda = \bigcap_{\lambda\in\Lambda} \pi_\lambda^{-1}(E_\lambda) \overset{(*)}{\in} \mathcal{M}(\mathcal{E}) = \otimes_{\lambda\in\Lambda}\mathcal{M}_\lambda,$$

where in $(*)$ we used the fact that Λ is countable and $\otimes_{\lambda\in\Lambda}\mathcal{M}_\lambda$ is a σ-algebra. Hence $\mathcal{F} \subset \otimes_{\lambda\in\Lambda}\mathcal{M}_\lambda$, and by Lemma 1.1.7 we obtain $\mathcal{M}(\mathcal{F}) \subset \otimes_{\lambda\in\Lambda}\mathcal{M}_\lambda$, which concludes the proof. ■

Proposition 1.2.2. *Suppose that $\mathcal{M}_\lambda$ is generated by $\mathcal{E}_\lambda$. Then*

(a) *$\otimes_{\lambda\in\Lambda}\mathcal{M}_\lambda$ is generated by*

$$\mathcal{E}_1 = \bigcup_{\lambda\in\Lambda}\bigcup_{E\in\mathcal{E}_\lambda}\pi_\lambda^{-1}(E).$$

(b) *if Λ is countable then $\otimes_{\lambda\in\Lambda}\mathcal{M}_\lambda$ is generated by*

$$\mathcal{F}_1 = \Big\{\prod_{\lambda\in\Lambda}E_\lambda \colon E_\lambda \in \mathcal{E}_\lambda\Big\}.$$

Proof. **(a).** Clearly $\mathcal{M}(\mathcal{E}_1) \subset \otimes_{\lambda\in\Lambda}\mathcal{M}_\lambda$. For the converse, fix $\lambda\in\Lambda$ and define the set $\tilde{M}_\lambda$ as

$$\tilde{\mathcal{M}}_\lambda = \{E \subset X_\lambda \colon \pi_\lambda^{-1}(E) \in \mathcal{M}(\mathcal{E}_1)\}.$$

We claim that $\tilde{\mathcal{M}}_\lambda$ is a σ-algebra that contains $\mathcal{E}_\lambda$. In fact, if $\{A_n\}_{n\in\mathbb{N}} \subset \tilde{\mathcal{M}}_\lambda$ we have

$$\pi_\lambda^{-1}(A_n) \in \mathcal{M}(\mathcal{E}_1) \text{ for each } n\in\mathbb{N}.$$

Hence

$$\pi_\lambda^{-1}\Big(\bigcup_{n\in\mathbb{N}}A_n\Big) = \bigcup_{n\in\mathbb{N}}\pi_\lambda^{-1}(A_n) \in \mathcal{M}(\mathcal{E}_1),$$

and also

$$\pi_\lambda^{-1}(A_1^c) = (\pi_\lambda^{-1}(A_1))^c \in \mathcal{M}(\mathcal{E}_1),$$

which proves that $\tilde{\mathcal{M}}_\lambda$ is a σ-algebra. Now if $E \in \mathcal{E}_\lambda$ then $\pi_\lambda^{-1}(E) \in \mathcal{E}_1$, and hence $\pi_\lambda^{-1}(E) \in \mathcal{M}(\mathcal{E}_1)$, which shows that $\mathcal{E}_\lambda \subset \tilde{\mathcal{M}}_\lambda$, and concludes the proof of our claim.

Now, using Lemma 1.1.7, we have $\mathcal{M}_\lambda \subset \tilde{\mathcal{M}}_\lambda$. This proves that $\pi_\lambda^{-1}(E) \in \mathcal{M}(\mathcal{E}_1)$ for all $E \in \mathcal{M}_\lambda$, which means that $\mathcal{E} \subset \mathcal{M}(\mathcal{E}_1)$, and another application of Lemma 1.1.7 gives us $\otimes_{\lambda\in\Lambda}\mathcal{M}_\lambda \subset \mathcal{M}(\mathcal{E}_1)$, and proves (a).

(b). This follows from (a) as in the proof of Proposition 1.2.1. ■

1.3 | Borel σ-algebras on topological spaces

Definition 1.3.1. *Let (X, τ) be a topological space and $\mathcal{B}_X$ be the σ-algebra generated by τ, that is, $\mathcal{B}_X = \mathcal{M}(\tau)$. This σ-algebra is called the* **Borel σ-algebra** *in X and its sets are called* **Borel sets** *or simply* **borelians**.

It is clear that, from the definition of σ-algebra, $\mathcal{B}_X$ is also the σ-algebra generated by the closed subsets of X.

On $\mathcal{B}_X$ we have sets of the form:

0. all the open and closed sets of X;

1. countable intersection of open sets, which are called **G_δ-sets**;

2. countable union of closed sets, which are called **F_σ-sets**;

3. countable union of G_δ-sets, which are called **$G_{\delta\sigma}$-sets**;

4. countable intersection of F_σ-sets, which are called **$F_{\sigma\delta}$-sets**;

5. and so on...

Proposition 1.3.2. *Let (X_1, τ_i) be topological spaces, for $i = 1, \cdots, n$ and consider the product space $\prod\limits_{i=1}^{n} X_i$ with the product topology. Then $\otimes_{i=1}^{n} \mathcal{B}_{X_i} \subset \mathcal{B}_X$. Moreover, if each X_i has a countable basis for τ_i, then $\otimes_{i=1}^{n} \mathcal{B}_{X_i} = \mathcal{B}_X$.*

Proof. By Proposition 1.2.2 item (b), we know that $\otimes_{i=1}^{n} \mathcal{B}_{X_i}$ is generated by the family

$$\mathcal{F}_1 = \left\{ \prod_{i=1}^{n} E_i \colon E_i \text{ is open in } X_i \right\},$$

which is a family of open sets in X with the product topology, hence $\otimes_{i=1}^{n} \mathcal{B}_{X_i} \subset \mathcal{B}_X$.

Now assume that each X_i has a countable basis $\mathcal{E}_i$ for τ_i. Hence each $\mathcal{B}_{X_i}$ is generated by $\mathcal{E}_i$ and again, using Proposition 1.2.2, $\otimes_{i=1}^{n} \mathcal{B}_{X_i}$ is generated by $\mathcal{F}_2 = \{\prod_{i=1}^{n} E_i \colon E_i \in \mathcal{E}_i\}$. But an open set E in X is a set of the form $E = \prod_{i=1}^{n} E_i$, where each E_i is open in X_i. Each E_i can be written as a countable union of elements in $\mathcal{E}_i$, which implies that E is a countable union of elements in $\mathcal{F}_2$ and hence $E \in \mathcal{M}(\mathcal{F}_2) = \otimes_{i=1}^{n} \mathcal{B}_{X_i}$. By Lemma 1.1.7, we have $\mathcal{B}_X \subset \otimes_{i=1}^{n} \mathcal{B}_{X_i}$, and we conclude the proof. ∎

In $\mathbb{R}$ with the usual topology, the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ has several sets that generate it.

Proposition 1.3.3. *The Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ is generated by any of the following collections:*

$$\mathcal{E}_1 = \{(a,b)\colon a<b\}, \quad \mathcal{E}_2 = \{[a,b]\colon a<b\}, \quad \mathcal{E}_3 = \{[a,b)\colon a<b\},$$
$$\mathcal{E}_4 = \{(a,b]\colon a<b\}, \quad \mathcal{E}_5 = \{(-\infty,a)\colon a\in\mathbb{R}\}, \quad \mathcal{E}_6 = \{(a,\infty)\colon a\in\mathbb{R}\},$$
$$\mathcal{E}_7 = \{(-\infty,a]\colon a\in\mathbb{R}\} \qquad \textit{and} \qquad \mathcal{E}_8 = \{[a,\infty)\colon a\in\mathbb{R}\}.$$

Proof. Since each open set in $\mathbb{R}$ is the countable union of open intervals, we have $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1)$. We will show now that each element of the collections $\mathcal{E}_j$, $j=2,\cdots,8$, can be written using only elements of $\mathcal{E}_1$ and operations that are closed for σ-algebras.

We have

$$[a,b] = \bigcap_{n=1}^{\infty}(a-\tfrac{1}{n}, b+\tfrac{1}{n}), \quad [a,b) = \bigcap_{n=1}^{\infty}(a-\tfrac{1}{n}, b), \quad (a,b] = \bigcap_{n=1}^{\infty}(a, b+\tfrac{1}{n}),$$
$$(-\infty,a) = \bigcap_{n=\lceil a+1\rceil}^{\infty}(-n,a), \qquad (a,\infty) = \bigcap_{n=\lceil a+1\rceil}^{\infty}(a,n)$$
$$(\infty,-a] = \bigcap_{n=\lceil a+1\rceil}^{\infty}(-n,a+\tfrac{1}{n}) \qquad \text{and} \qquad [a,\infty) = \bigcap_{n=\lceil a+1\rceil}^{\infty}(a-\tfrac{1}{n},n),$$

and thus we conclude that each $\mathcal{E}_j$ generates the same σ-algebra as $\mathcal{E}_1$, which is $\mathcal{B}_{\mathbb{R}}$. ■

As a direct consequence of Proposition 1.3.2, we have

$$\mathcal{B}_{\mathbb{R}^n} = \otimes_{i=1}^{n}\mathcal{B}_{\mathbb{R}}. \tag{1.3.1}$$

1.3.1 | The Extended Real Numbers

We define the set of the **extended real numbers** as

$$\overline{\mathbb{R}} = \mathbb{R}\cup\{-\infty,\infty\} = [-\infty,\infty],$$

which is the usual real line $\mathbb{R}$, together with the symbols $-\infty$ and ∞.

The order in $\overline{\mathbb{R}}$ is the usual order of $\mathbb{R}$, together with the relations

$$-\infty < x < \infty \text{ for all } x\in\mathbb{R}.$$

As for operations, we have the usual sum $(+)$ and product $(\cdot)$ on $\mathbb{R}$, together with the operations

$$\infty \cdot x = \begin{cases} \infty \text{ if } x > 0 \text{ or } x = \infty, \\ -\infty \text{ if } x < 0 \text{ or } x = -\infty. \end{cases} \quad \text{and} \quad -\infty \cdot x = \begin{cases} -\infty \text{ if } x > 0 \text{ or } x = \infty, \\ \infty \text{ if } x < 0 \text{ or } x = -\infty. \end{cases}$$

Here we will set, by convention, that $\infty \cdot 0 = -\infty \cdot 0 = 0$. The expressions $\infty - \infty$ and $-\infty + \infty$ are indeterminations.

The topology τ in $\overline{\mathbb{R}}$ is generated by the sets of the form $[-\infty, a)$ and $(a, \infty]$, for each $a \in \mathbb{R}$. Clearly, since $(a, b) = [-\infty, b) \cap (a, \infty]$ for $a < b$, we can see that each open set in $\mathbb{R}$ is also open in $\overline{\mathbb{R}}$.

Let $\mathcal{B}_{\overline{\mathbb{R}}}$ be the Borel σ-algebra of $\overline{\mathbb{R}}$, which is generated by the sets of the form $[-\infty, a)$ or by the sets of the form $(a, \infty]$, with $a \in \mathbb{R}$. Clearly $\mathcal{B}_{\mathbb{R}} \subset \mathcal{B}_{\overline{\mathbb{R}}}$. Furthermore, we have the following:

Theorem 1.3.4. *$A \in \mathcal{B}_{\overline{\mathbb{R}}}$ if and only if $A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$.*

Proof. First note that $\{\infty\} = \bigcap_{n=1}^{\infty} (n, \infty] \in \mathcal{B}_{\overline{\mathbb{R}}}$ and $\{-\infty\} = \bigcap_{n=1}^{\infty} [-\infty, n) \in \mathcal{B}_{\overline{\mathbb{R}}}$, and we can decompose each subset A of $\overline{\mathbb{R}}$ as $A = (A \cap \mathbb{R}) \cup (A \cap \{\infty\}) \cup (A \cap \{-\infty\})$. Hence, if $A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$ then $A \in \mathcal{B}_{\overline{\mathbb{R}}}$.

Now define

$$\mathcal{M} = \{E \subset \overline{\mathbb{R}} \colon E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}.$$

From the above, $\mathcal{M} \subset \mathcal{B}_{\overline{\mathbb{R}}}$. To prove the converse inclusion, we prove that $\mathcal{M}$ is a σ-algebra that contains all subsets of the form $[-\infty, a)$, for $a \in \mathbb{R}$. Thus $\mathcal{M}$ will contain $\mathcal{B}_{\overline{\mathbb{R}}}$.

It is clear that $[-\infty, a) \in \mathcal{M}$, since $[-\infty, a) \cap \mathbb{R} = (-\infty, a) \in \mathcal{B}_{\mathbb{R}}$. In particular, $\mathcal{M}$ is nonempty. If $\{E_i\} \subset \mathcal{M}$, we have

$$\left(\bigcup_{i=1}^{\infty} E_i \right) \cap \mathbb{R} = \bigcup_{i=1}^{\infty} \underbrace{(E_i \cap \mathbb{R})}_{\in \mathcal{B}_{\mathbb{R}}} \in \mathcal{B}_{\mathbb{R}},$$

and this $\bigcup\limits_{i=1}^{\infty} E_i \in \mathcal{M}$.

Now if $E \in \mathcal{M}$, then

$$(\overline{\mathbb{R}} \setminus E) \cap \mathbb{R} = \mathbb{R} \setminus E = \mathbb{R} \setminus \underbrace{(E \cap \mathbb{R})}_{\in \mathcal{B}_{\mathbb{R}}} \in \mathbb{R},$$

and thus $\overline{\mathbb{R}} \setminus E \in \mathcal{M}$, and concludes the proof.

■

We can turn $\overline{\mathbb{R}}$ into a metric space. The function $d\colon \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to [0, \infty)$ given by

$$d(x, y) = |\arctan(x) - \arctan(y)| \qquad \text{for all } x, y \in \overline{\mathbb{R}}, \tag{1.3.2}$$

is a metric. Clearly, if $x, y \in \mathbb{R}$, then

$$d(x, y) \leqslant \sup_{z \in \mathbb{R}} \frac{1}{1+z^2} \cdot |x - y| \leqslant |x - y|, \tag{1.3.3}$$

using the Mean Value Theorem. If $\tilde{\tau}$ the topology in $\overline{\mathbb{R}}$ generated by the metric d, we have:

Proposition 1.3.5. *$\tilde{\tau} = \tau$, that is, the topology induced by the metric d is τ.*

Proof. This result follows from the fact that $\arctan\colon \overline{\mathbb{R}} \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is continuous with continuous inverse, defining obviously $\tan(-\frac{\pi}{2}) = -\infty$ and $\tan(\frac{\pi}{2}) = \infty$. ■

1.4 Elementary families

Definition 1.4.1. *Let X be a nonempty set. A collection $\mathcal{E}$ of subsets of X is called an* **elementary family** *if it satisfies:*

(i) $\varnothing \in \mathcal{E}$;

(ii) *if* $E, F \in \mathcal{E}$ *then* $E \cap F \in \mathcal{E}$;

(iii) *if* $E \in \mathcal{E}$ *then* E^c *is a finite disjoint union of elements of* $\mathcal{E}$.

Let $\mathcal{E}$ be an elementary family in X. Define

$$\mathcal{A} = \{A \subset X \colon A \text{ is a finite disjoint union of elements of } \mathcal{E}\}. \tag{1.4.1}$$

Lemma 1.4.2. *If $A \in \mathcal{A}$ and $B \in \mathcal{E}$ then $A \cup B \in \mathcal{A}$.*

Proof. We have $A = \bigcup_{i=1}^n E_i$ and $B^c = \bigcup_{j=1}^m F_j$ where $E_i, F_j \in \mathcal{E}$ and the unions are disjoint. Now

$$E_i \setminus B = E_i \cap B^c = \bigcup_{j=1}^m E_i \cap F_j \text{ for each } i = 1, \cdots, n,$$

and therefore

$$A \setminus B = \bigcup_{i=1}^n (E_i \setminus B) = \bigcup_{i=1}^n \bigcup_{j=1}^m E_i \cap F_j,$$

which, since $E_i \cap F_j \in \mathcal{E}$, is a finite disjoint union of elements of $\mathcal{E}$ and therefore is in $\mathcal{A}$.

But $A \cup B = (A \setminus B) \cup B$, which again is a finite disjoint unions of elements of $\mathcal{E}$ (recall that $B \in \mathcal{E}$), and thus $A \cup B \in \mathcal{A}$. ■

Corollary 1.4.3. *If $A_1, \cdots, A_n \in \mathcal{E}$ then $\bigcup_{i=1}^n A_i \in \mathcal{A}$.*

Proof. Assume that the result is true for $n \geqslant 2$ (the case $n = 2$ is Lemma 1.4.2), and we have

$$\bigcup_{i=1}^{n+1} A_i = \Big(\bigcup_{i=1}^{n} A_i\Big) \cup A_{n+1},$$

and Lemma 1.4.2 proves the result, since $\bigcup_{i=1}^n A_i \in \mathcal{A}$, by induction, and $A_{n+1} \in \mathcal{E}$ by hypothesis. ■

Proposition 1.4.4. *If $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$.*

Proof. Write $A = \bigcup_{i=1,\cdots,n} E_i$ and $B = \bigcup_{j=1}^m F_j$, $E_i, F_j \in \mathcal{E}$ and both are disjoint unions. Defining $G_k = E_k$ for $k = 1, \cdots, n$ and $G_k = F_{k-n}$ for $k = n+1, \cdots, n+m$, we have

$$A \cup B = \bigcup_{k=1}^{n+m} G_k \in \mathcal{A},$$

by Corollary 1.4.3, since $G_k \in \mathcal{E}$ for $k = 1, \cdots, n+m$.

■

Proposition 1.4.5. *If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$.*

Proof. We write $A = \bigcup_{i=1}^n E_i$ with $E_i \in \mathcal{E}$ and $E_i \cap E_j = \varnothing$ for $i \neq j$. Now we can write

$$E_i^c = \bigcup_{j=1}^{m_i} F_{i,j} \text{ where } F_{i,j} \in \mathcal{E},$$

and the union is disjoint. Therefore

$$A^c = \bigcap_{i=1}^{n} E_i^c = \bigcap_{i=1}^{n}\bigcup_{j=1}^{m_i} F_{i,j} = \bigcup \Big\{ F_{1,j_1} \cap \cdots \cap F_{n,j_n} : j_i = 1, \cdots, m_i,\ i = 1, \cdots, n \Big\},$$

which is a finite disjoint union of elements in $\mathcal{E}$, and is henceforth in $\mathcal{A}$. ■

With these results, the proof of the following theorem is immediate.

Theorem 1.4.6. *Given an elementary family $\mathcal{E}$, the collection $\mathcal{A}$ defined in* (1.4.1) *is an algebra.*

Proof. Since $\mathcal{E} \subset \mathcal{A}$ we have $\varnothing \in \mathcal{A}$. Now Propositions 1.4.4 and 1.4.5 show that $\mathcal{A}$ is closed under union and complements, and hence it is an algebra. ∎

1.5 | σ-RINGS

The section is the Exercise 1 in Page 24 of [1].

DEFINITION 1.5.1. *A collection $\mathcal{R}$ of subsets of a nonempty set X is called a* **ring** *if given $A, B \in \mathcal{R}$ we have*

$$A \cup B \in \mathcal{R} \qquad \text{and} \qquad A \setminus B \in \mathcal{R}.$$

DEFINITION 1.5.2. *A collection $\mathcal{R}$ of subsets of a nonempty set X is called a* **σ-ring** *if it is a ring and given $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{R}$ we have*

$$A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{R}.$$

Clearly if $\mathcal{R}$ is a nonempty ring, then $\varnothing \in \mathcal{R}$, since given $A \in \mathcal{R}$ we have $\varnothing = A \setminus A \in \mathcal{R}$.

PROPOSITION 1.5.3. *Given a ring $\mathcal{R}$ and $A, B \in \mathcal{R}$ we have $A \cap B \in \mathcal{R}$.*

Proof. If $A, B \in \mathcal{R}$ then $A \setminus B \in \mathcal{R}$ and $A \cap B = A \setminus (A \setminus B) \in \mathcal{R}$. ∎

PROPOSITION 1.5.4. *If $\mathcal{R}$ is a σ-ring and $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{R}$ then $A = \bigcap_{n=1}^{\infty} A_n \in \mathcal{R}$.*

Proof. Just write $A = A_1 \setminus \left(\bigcup_{n=2}^{\infty}(A_1 \setminus A_n)\right) \in \mathcal{R}$. ∎

PROPOSITION 1.5.5. *Let $\mathcal{R}$ be a nonempty ring (σ-ring). Then $\mathcal{R}$ is an algebra (σ-algebra) if and only if $X \in \mathcal{R}$.*

Proof. It is clear that if $X \in \mathcal{R}$ then for each $A \in \mathcal{R}$ we have $A^c = X \setminus A \in \mathcal{R}$ and hence $\mathcal{R}$ is an algebra (or σ-algebra).

Now for the converse if $\mathcal{R}$ is a nonempty algebra (or σ-algebra) we have $\varnothing \in \mathcal{R}$ and hence $X = \varnothing^c \in \mathcal{R}$. ∎

PROPOSITION 1.5.6. *If $\mathcal{R}$ is a nonempty σ-ring then the collection*

$$\mathcal{M} = \{A \subset X \colon A \in \mathcal{R} \text{ or } A^c \in \mathcal{R}\}$$

is a σ-algebra.

Proof. First we note that $\varnothing \in \mathcal{R}$, hence $\varnothing \in \mathcal{M}$. Now let $A, B \in \mathcal{M}$. If $A, B \in \mathcal{R}$ then $A \cup B \in \mathcal{R}$ and hence $A \cup B \in \mathcal{M}$. When A^c and B^c are in $\mathcal{R}$, and we get $(A \cup B)^c = A^c \cap B^c \in \mathcal{R}$ and again $A \cup B \in \mathcal{M}$. Now, for the last case, if $A^c \in \mathcal{R}$ and $B \in \mathcal{R}$ we have $(A \cup B)^c = A^c \setminus B \in \mathcal{R}$, and so $A \cup B \in \mathcal{M}$.

Now if $A \in \mathcal{M}$ and $A \in \mathcal{R}$ then $(A^c)^c = A \in \mathcal{R}$ and hence $A^c \in \mathcal{M}$. If $A^c \in \mathcal{R}$ then directly we obtain $A^c \in \mathcal{M}$, and hence $\mathcal{M}$ is a σ-algebra.

Now consider a countable collection $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$. The case where all $A_n \in \mathcal{R}$ or all $A_n^c \in \mathcal{R}$ is analogous to the case with only two sets. We will focus in the case where there exists a disjoint decomposition of $\mathbb{N}$ into two subsequences $\{n_k\}_{k\in\mathbb{N}}$ and $\{m_k\}_{k\in\mathbb{N}}$ such that $A_{n_k} \in \mathcal{R}$ and $A_{m_k}^c \in \mathcal{R}$ for all k. We write

$$\Big(\bigcup_{n=1}^{\infty} A_n\Big)^c = \underbrace{\Big(\bigcup_{k=1}^{\infty} A_{m_k}^c\Big)}_{\in\mathcal{R}} \setminus \underbrace{\Big(\bigcup_{k=1}^{\infty} A_{n_k}\Big)}_{\in\mathcal{R}} \in \mathcal{R},$$

and therefore $\mathcal{M}$ is a σ-algebra. ∎

PROPOSITION 1.5.7. *If $\mathcal{R}$ is a nonempty σ-ring then the collection*

$$\mathcal{M} = \{A \subset X \colon A \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$$

is a σ-algebra.

Proof. Since $\mathcal{R}$ is nonempty, we have $\varnothing \in \mathcal{R}$ and since $\varnothing \cap F = \varnothing \in \mathcal{R}$ for all $F \in \mathcal{R}$, we have $\varnothing \in \mathcal{M}$. In this case we also have $X \in \mathcal{M}$, since $X \cap F = F \in \mathcal{F}$ for all $F \in \mathcal{M}$.

If $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ then given $F \in \mathcal{R}$ we have

$$\Big(\bigcup_{n=1}^{\infty} A_n\Big) \cap F = \bigcup_{n=1}^{\infty} \underbrace{(A_n \cap F)}_{\in\mathcal{R}} \in \mathcal{R},$$

and hence $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

Now if $A \in \mathcal{M}$, we write $A^c \cap F = (X \setminus A) \cap F = F \setminus (A \cap F) \in \mathcal{R}$, since F and $A \cap F$ are in $\mathcal{R}$, hence $A^c \in \mathcal{M}$, and concludes the proof that $\mathcal{M}$ is a σ-algebra. ∎

1.6 | SOLVED EXERCISES FROM [1, PAGE 24]

EXERCISE 1.

Solution. This exercise is completely solved in Section 1.5.

Exercise 2.

Solution. This exercise is completely done in Proposition 1.3.3.

Exercise 3. In this subsection we show that if $\mathcal{M}$ is an infinite σ-algebra then

(a) $\mathcal{M}$ contains an infinite sequence of nonempty disjoint sets.

(b) $\operatorname{card}(\mathcal{M}) \geqslant \mathfrak{c}$.

Solution to (a). Let $\{A_n\}_{n\in\mathbb{N}}$ a sequence with distinct nonempty sets in $\mathcal{M}$, which exist since $\mathcal{M}$ is infinite.

We can assume that $A_{n+1} \setminus A_n \neq \varnothing$, for all $n \in \mathbb{N}$. In fact, if $A_{n+1} \subset A_n$, then $A_n \setminus A_{n+1} \neq \varnothing$ since otherwise we would have $A_n = A_{n+1}$ which would contradict the assumption that the A_n's are all distinct, and we replace A_n with $A_n \setminus A_{n+1}$ and we have

$$A_{n+1} \setminus (A_n \setminus A_{n+1}) = A_{n+1} \neq \varnothing.$$

Now define $B_n = \bigcup_{i=1}^n A_i$ for each $n \in \mathbb{N}$. Since $\mathcal{M}$ is a σ-algebra, $B_n \in \mathcal{M}$ for each $n \in \mathbb{N}$, and moreover $B_n \subset B_{n+1}$ for each $n \in \mathbb{N}$, and from the previous assumption, we have $B_{n+1} \setminus B_n \neq \varnothing$. Setting now $C_1 = B_1$ and $C_n = B_n \setminus B_{n-1}$ for each $n \geqslant 2$. Thus $C_n \in \mathcal{M}$ for each $n \in \mathbb{N}$. Also, if $x \in C_i \cap C_j$ for $i > j$ then we have $x \notin B_{i-1}$ and $x \in B_j$, where $i-1 \geqslant j$, which gives us a contradiction and proves that $\{C_n\}_{n\in\mathbb{N}}$ is an infinite sequence of nonempty disjoint sets in $\mathcal{M}$.

Solution to (b). Assume that $\mathcal{M}$ is countable. By (a), we can assume that there exists a sequence $\mathcal{E} = \{A_n\}_{n\in\mathbb{N}}$ of disjoint nonempty elements of $\mathcal{M}$.

Now we will construct the following function: given a nonempty subset $J \subset \mathbb{N}$ we define

$$\psi(J) = \bigcup_{j\in J} A_j \in \mathcal{M}(\mathcal{E}) \subset \mathcal{M},$$

and we complete the definition setting $\psi(\varnothing) = \varnothing$. Thus we have constructed a function $\psi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{M}$, and since the family $\mathcal{E}$ is made of pairwise disjoint sets, we can see that ψ is injective, which shows that $\operatorname{card}(\mathcal{M}) \geqslant \mathfrak{c}$.

Exercise 4. We show that an algebra $\mathcal{A}$ is a σ-algebra if and only if it is closed under countable increasing unions, that is, if $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ and $E_n \subset E_{n+1}$ for each $n \in \mathbb{N}$ then

$\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$.

Solution. In fact, is it clear that if $\mathcal{A}$ is a σ-algebra, then it is closed by any countable unions, and in particular, it is closed by countable increasing unions. Now for the converse, assume that $\{A_n\}_{n\in\mathbb{N}}$ is any countable sequence of elements in $\mathcal{A}$.

Define $B_n = \bigcup_{k=1}^{n} A_k$ for each $n \in \mathbb{N}$. Since $\mathcal{A}$ is an algebra, it is closed by finite unions, and hence $B_n \in \mathcal{A}$ for each $n \in \mathbb{N}$. Also, $\{B_n\}_{n\in\mathbb{N}}$ is an increasing sequence and we have

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \in \mathcal{A},$$

and thus $\mathcal{A}$ is a σ-algebra.

Exercise 5. Let $\mathcal{M}$ be the σ-algebra generated by a collection $\mathcal{E}$ of subsets of a nonempty set X. Then

$$\mathcal{M} = \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F}),$$

where $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$.

Hint: *Show that the latter object is a σ-algebra.*

Solution. We note that since $\mathcal{F} \subset \mathcal{E}$, we have $\mathcal{M}(\mathcal{F}) \subset \mathcal{M}(\mathcal{E}) = \mathcal{M}$, and hence

$$\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F}) \subset \mathcal{M}.$$

For the converse, note that $\mathcal{E} \subset \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$. Thus, if we prove that $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ is a σ-algebra, then we obtain the other inclusion.

We denote $\tilde{\mathcal{M}} = \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$. Clearly $\varnothing \in \tilde{\mathcal{M}}$. Now if $\{A_n\}_{n\in\mathbb{N}} \subset \tilde{\mathcal{M}}$, for each $n \in \mathbb{N}$ there exists a countable subset $\mathcal{F}_n$ of $\mathcal{E}$ such that $A_n \in \mathcal{M}(\mathcal{F}_n)$.

The union $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ is also a countable subset of $\mathcal{E}$ and

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}(\mathcal{F}),$$

and hence $\bigcup_{n=1}^{\infty} A_n \in \tilde{\mathcal{M}}$. This shows that $\tilde{\mathcal{M}}$ is a σ-algebra and completes the proof.

CHAPTER 2

MEASURES

2.1 | Basic notions and definitions

Let X be a nonempty set with a σ-algebra $\mathcal{M}$.

Definition 2.1.1. *A* **measure** μ *on* $(X, \mathcal{M})$ *(or simply on* $\mathcal{M}$*, or simply on* X *if* $\mathcal{M}$ *is understood) is a function* $\mu\colon \mathcal{M} \to [0, \infty]$ *such that*

1. $\mu(\varnothing) = 0$.

2. *if* $\{E_n\}$ *is a pairwise disjoint sequence in* $\mathcal{M}$ *then*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Condition (2) is called **σ-additivity** or **countable additivity**. This condition implies **finite additivity**:

2^*. if $E_1, \cdots, E_n$ are pairwise disjoint sets in $\mathcal{M}$ then

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i),$$

just by taking $E_j = \varnothing$ for $j > n$.

A function μ that satisfies (1) and (2^*), but not necessarily (2), is called a **finitely additive measure**.

Definition 2.1.2. *If* X *is a nonempty set and* $\mathcal{M}$ *is a* σ*-algebra on* X*, the pair* $(X, \mathcal{M})$ *is called a* **measurable space***, and the sets in* $\mathcal{M}$ *are called* **measurable sets***.*

If μ is a measure on $(X, \mathcal{M})$, the triple $(X, \mathcal{M}, \mu)$ is called a **measure space**.

Let $(X, \mathcal{M}, \mu)$ be a measure space. We say that

(a) μ is **finite** if $\mu(X) < \infty$.

(b) μ is **σ-finite** if $X = \bigcup_{j=1}^{\infty} E_j$ where $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$ for all j.

(c) **semifinite** if for each $E \in \mathcal{M}$ with $\mu(E) = \infty$ then there exists $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$.

Following item (b) of this definition, we say that a subset E of X is called **σ-finite for μ** if $E = \bigcup_{j=1}^{\infty} E_j$ where $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$ for all j. If μ is finite then $\mu(E) < \infty$ for each $E \in \mathcal{M}$ since $\mu(X) = \mu(E) + \mu(E^c)$.

Theorem 2.1.3. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then*

(a) **(monotonicity)** *if $E, F \in \mathcal{M}$ and $E \subset F$ then $\mu(E) \leqslant \mu(F)$.*

(b) **(subadditivity)** *if $\{E_j\} \subset \mathcal{M}$ then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leqslant \sum_{j=1}^{\infty} \mu(E_j)$.*

Proof. **(a)** If $E, F \in \mathcal{M}$ and $E \subset F$ we have

$$\mu(F) = \mu[E \cup (F \setminus E)] = \mu(E) + \mu(F \setminus E),$$

and since $\mu(F \setminus E) \geqslant 0$ we have $\mu(F) \geqslant \mu(E)$.
(b) Let $F_1 = E_1$ and $F_j = E_j \setminus \left(\bigcup_{k=1}^{j-1} E_k\right)$. Then $\{F_j\}$ is a pairwise disjoint sequence in $\mathcal{M}$, $\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} F_j$ and

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \mu\left(\bigcup_{j=1}^{\infty} F_j\right) = \sum_{j=1}^{\infty} \mu(F_j) \leqslant \sum_{j=1}^{\infty} \mu(E_j).$$

■

Proposition 2.1.4. *If μ is a σ-finite measure on $(X, \mathcal{M})$ then there exists a sequence $\{F_j\}$ of pairwise disjoint sets in $\mathcal{M}$ such that $X = \bigcup_{j=1}^{\infty} F_j$ and $\mu(F_j) < \infty$ for all j.*

Proof. Since μ is σ-finite, we have $X = \bigcup_{j=1}^{\infty} E_j$ where $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$ for all j. Now we set $F_1 = E_1$ and $F_j = E_j \setminus \left(\bigcup_{k=1}^{j-1} E_k\right)$. Then the sequence $\{F_f\}$ is pairwise disjoint, $\bigcup_{j=1}^{\infty} F_j = \bigcup_{j=1}^{\infty} E_j = X$ and $\mu(F_j) \leqslant \mu(E_j) < \infty$ by Theorem 2.1.3 item (a), which concludes the proof of this result. ■

Example 2.1.5. *Let X be a nonempty set, $\mathcal{M} = \mathcal{P}(X)$ and $f\colon X \to [0,\infty]$ a function. Consider $A_f = \{x \in X\colon f(x) > 0\}$ and construct a measure μ on $\mathcal{M}$ as follows:*

(i) $\mu_f(\varnothing) = 0$;

(ii) *if $E \in \mathcal{P}(X)$ then we have two cases to consider: if $E \cap A_f$ is uncountable, then $\mu_f(E) = \infty$. Otherwise, we set*

$$\mu_f(E) = \sum_{x \in E \cap A_f} f(x).$$

To prove that μ_f is a measure, it remains to prove the σ-additivity property. To that end consider $\{E_j\} \subset \mathcal{P}(X)$ and $E = \bigcup_{j=1}^{\infty} E_j$ with disjoint unions. If $E_j \cap A_f$ is uncountable for some j, then $E \cap A_j$ is also uncountable, hence

$$\mu_f(E) = \infty = \sum_{j=1}^{\infty} \mu(E_j).$$

Assume that $E_j \cap A_f$ is countable, then $E \cap A_f$ is also countable, and by the absolute convergence and rearrangements property we have

$$\mu_f(E) = \sum_{x \in E \cap A_f} f(x) = \sum_{j=1}^{\infty} \sum_{x \in E_j \cap A_f} f(x) = \sum_{j=1}^{\infty} \mu_f(E_j),$$

since $\{E_j \cap A\}$ is a pairwise disjoint sequence.

We obtain properties on the measure μ_f if we have properties of f.

Property 1. *If $f(x) < \infty$ for all $x \in X$ then μ_f is semifinite.*

Assume that $\mu_f(E) = \infty$. If $f(x) = 0$ for all $x \in E$ then $\mu_f(E) = 0$ by definition, which is a contradiction. Hence there exists $x \in E$ such that $f(x) > 0$ and considering $F = \{x\}$ we have $F \subset E$ and $0 < \mu_f(F) = f(x) < \infty$. Thus μ_f is semifinite.

Property 2. *μ_f is σ-finite if and only if μ is semifinite and A_f is countable.*

Assume that μ_f is σ-finite. Using Proposition 2.1.4 there exists a sequence of pairwise disjoint sets $\{F_j\}$ such that $\mu_f(F_j) < \infty$ for all j and $X = \bigcup_{j=1}^{\infty} F_j$. If E is such that $\mu_f(E) = \infty$. If $\mu_f(E \cap F_j) = 0$ for all j, then

$$\mu_f(E) = \mu_f(E \cap X) = \mu_f\left(\bigcup_{j=1}^{\infty} (E \cap F_j)\right) = \sum_{j=1}^{\infty} \mu_f(E \cap F_j) = 0,$$

which is a contradcition, hence there exists j such that $\mu_f(E \cap F_j) > 0$. Setting $F = E \cap F_j$

then $F \subset E$ and $0 < \mu_f(F) \leqslant \mu_f(F_j) < \infty$, which means that μ_f is semifinite.

Also if $A_f = \bigcup_{j=1}^{\infty} A_f \cap F_j$ and since $\mu_f(A_f \cap F_j) \leqslant \mu_f(F_j) < \infty$ we have $A_f \cap F_j$ countable for all j. Hence A_f is countable.

For the converse, if $\mu_f(X) < \infty$ there is nothing to prove (just take $E_1 = X$ and $E_j = \varnothing$ for $j \geqslant 2$). So consider that $\mu_f(X) = \infty$. Since A_f is countable, consider $A_f = \{x_j\}$ where the sequence consists of distinct elements and define $F_j = \{x_j\}$ for all j. Then $0 < \mu_f(F_j) = f(x_j) < \infty$ and setting $F_0 = A_f^c$ we have $\mu_f(F_0) = 0$ and

$$X = A_f \cup A_f^c = \bigcup_{j=0}^{\infty} F_j,$$

which shows that μ_f is σ-finite.

Two particular cases are very important. If $f(x) = 1$ for all $x \in X$, then μ_f is called **counting measure**. *If $f(x_0) = 1$ and $f(x) = 0$ for $x \neq x_0$ then μ_f is called* **point mass** *or the* **Dirac measure**.

Example 2.1.6. *Let X be an uncountable set and $\mathcal{M}$ the σ-algebra of countable or co-countable sets. Define μ on $\mathcal{M}$ by setting*

$$\mu(E) = \begin{cases} 0, & \text{if } E \text{ is countable,} \\ 1, & \text{if } E^c \text{ is countable.} \end{cases}$$

We prove that μ is a measure on $\mathcal{M}$. Clearly $\mu(\varnothing) = 0$. Now if $\{E_j\} \subset \mathcal{M}$ and $E = \bigcup_{j=1}^{\infty} E_j$ with disjoint union, then if E_j is countable for all j we have $\mu(E_j) = 0$ for all j, E is countable and

$$\mu(E) = 0 = \sum_{j=1}^{\infty} \mu(E_j).$$

On the other hand, if $E_{j_0}^c$ is countable for some E_{j_0} then $E^c = \bigcap_{j=1}^{\infty} E_j^c \subset E_{j_0}^c$ is also countable. Since the $\{E_j\}$ is a pairwise disjoint sequence $\bigcup_{j \neq j_0} E_j \subset E_{j_0}^c$ is also countable, hence E_j is countable for each $j \neq j_0$, and we have

$$\mu(E) = 1 = \mu(E_{j_0}) = \sum_{j=1}^{\infty} \mu(E_j).$$

Example 2.1.7. *Let X be an infinite set and $\mathcal{M} = \mathcal{P}(X)$. Define $\mu(E) = 0$ if E is finite and $\mu(E) = \infty$ if E is infinite. Then μ is a finitely additive measure but not a measure.*

Clearly if $E_1, \cdots, E_n$ is a finite sequence of subsets of X, then we have two cases: if all E_i are finite then $E = \bigcup_{i=1}^{n} E_i$ is also finite and hence $\mu(E) = 0 = \sum_{i=1}^{n} \mu(E_i)$. If, however, *one of them is infinite then E is infinite and $\mu(E) = \infty = \sum_{i=1}^{n} \mu(E_i)$. Hence μ is a finitely additive measure.*

Now if $\{x_n\}$ is an infinite sequence of distinct elements of X then defining $E_n = \{x_n\}$ and $E = \bigcup_{n=1}^{\infty} E_n$, we can easily see that μ is not a measure.

Theorem 2.1.8. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{E_j\}$ a sequence in $\mathcal{M}$.*

(a) **(Continuity from below)** *If $E_j \subset E_{j+1}$ for all j then*

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j).$$

(b) **(Continuity from above)** *If $E_j \supset E_{j+1}$ and $\mu(E_1) < \infty$ then*

$$\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j).$$

Proof. **(a)** If $\mu(E_{j_0}) = \infty$ for some j_0 then $\mu(E_j) = \infty$ for all $j \geqslant j_0$ and $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \infty$ hence $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \infty = \lim_{j\to\infty} \mu(E_j)$.

Now assume that $\mu(E_j) < \infty$ for all j. Hence setting $E_0 = \varnothing$ we have

$$\mu(E_j) = \mu(E_{j-1}) + \mu(E_j \setminus E_{j-1}),$$

and hence, by the finiteness of $\mu(E_j)$ for all j, we have $\mu(E_j \setminus E_{j-1}) = \mu(E_j) - \mu(E_{j-1})$. Thus we have

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \mu\left(\bigcup_{j=1}^{\infty} (E_j \setminus E_{j-1})\right) = \sum_{j=1}^{\infty} \mu(E_j \setminus E_{j-1}) = \lim_{j\to\infty} \sum_{k=1}^{j} \mu(E_k \setminus E_{k-1}) = \lim_{j\to\infty} \mu(E_j).$$

(b) Let $F_j = E_1 \setminus E_j$, then $F_j \subset F_{j+1}$ for all j, $\mu(E_1) = \mu(F_j) \setminus \mu(E_j)$ and $\bigcup_{j=1}^{\infty} F_j = E_1 \left(\bigcap_{j=1}^{\infty} E_j\right)$. Using item (a) we have

$$\mu(E_1) = \mu\left(\bigcap_{j=1}^{\infty} E_j\right) + \lim_{j\to\infty} \mu(F_j) = \mu\left(\bigcap_{j=1}^{\infty} E_j\right) + \lim_{j\to\infty} [\mu(E_1) - \mu(E_j)],$$

and since $\mu(E_1) < \infty$ we obtain the result by subtracting $\mu(E_1)$ from both sides. ■

Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \in \mathcal{M}$ such that $\mu(E) = 0$ is called a **null set**. If more precision is needed to specify the measure, we say a **μ-null set**. Using the subadditivity property, we know that any countable union of null sets is a null set.

Let P be a property, and define $P(x)$ as meaning: x satisfies the property P. If there exists a μ-null set N such that $P(x)$ for all $x \in X \setminus N$, we say that property P holds **almost everywhere** (abbreviated by **a.e.**) or for **almost every** x. If more precision is needed for the measure, we use **μ-almost everywhere**.

Assume that P holds μ-almost everywhere. If $\sim P(x)$ means that x does not satisfy the property P, we point out that $F = \{x \in X \colon \sim P(x)\}$ doesn't need to have zero measure, since it is not required that $F \in \mathcal{M}$. However, it does imply that $F \subset N$ for some $N \in \mathcal{M}$ with $\mu(N) = 0$.

If $\mu(E) = 0$ and $F \subset E$ then $\mu(F) = 0$ $\underline{\text{if } F \in \mathcal{M}}$. But this last statement does not need to be true in general. A measure that contains all subsets of null sets is called **complete**. Complete measures simplify the theoretical results, and can be always achieved by enlarging (if necessary) the domain of the measure μ, as follows:

Theorem 2.1.9. *Suppose that $(X, \mathcal{M}, \mu)$ is a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} \colon \mu(N) = 0\}$ and*

$$\overline{\mathcal{M}} = \{E \cup F \colon E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}.$$

Then $\overline{\mathcal{M}}$ is a σ algebra, and there exists a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

Proof. We prove first that $\overline{\mathcal{M}}$ is a σ-algebra. Since $\varnothing \in \mathcal{M}$ and $\mu(\varnothing) = 0$ we have $\mathcal{M} \subset \overline{\mathcal{M}}$. Assume that $\{A_j\}$ is a sequence in $\overline{\mathcal{M}}$ with pairwise disjoint sets. Then $A_j = E_j \cup F_j$ with $E_j \in \mathcal{M}$ and $F_j \subset N_j$ with $N_j \in \mathcal{N}$. Then

$$\bigcup_{j=1}^{\infty} A_j = \underbrace{\left(\bigcup_{j=1}^{\infty} E_j\right)}_{\in \mathcal{M}} \cup \underbrace{\left(\bigcup_{j=1}^{\infty} F_j\right)}_{\subset \bigcup_{j=1}^{\infty} N_j}, \tag{2.1.1}$$

and $\bigcup_{j=1}^{\infty} N_j \in \mathcal{N}$, by the subadditivity property.

Now if $A = E \cup F \in \overline{\mathcal{M}}$ with $E \in \mathcal{M}$ and $F \subset N \in \mathcal{N}$, then considering $F_1 = F \setminus E$ and $N_1 = N \setminus E$ we have $F_1 \subset N_1 \in \mathcal{N}$ and $A = E \cup F_1$ with $E \cap F_1 = \varnothing$. Hence we can always assume that $E \cap N = \varnothing$ (and hence $E \cap F = \varnothing$). Thus we have

$$A^c = (E \cup F)^c = \underbrace{(E \cup N)^c}_{\in \mathcal{M}} \cup \underbrace{(N \setminus F)}_{\subset N \in \mathcal{N}} \in \overline{\mathcal{M}},$$

therefore $\overline{\mathcal{M}}$ is a σ-algebra.

Now for $A = E \cup F$, with $E \in \mathcal{M}$ and $F \subset N$, where $N \in \mathcal{N}$, we define

$$\overline{\mu}(A) = \mu(E).$$

First we have to prove that $\overline{\mu}$ is well defined, that is, if $A = E_1 \cup F_1 = E_2 \cup F_2$ where $E_1, E_2 \in \mathcal{M}$ and $F_i \subset N_i \in \mathcal{N}$ for $i = 1, 2$. We have $E_1 \subset E_2 \cup F_2 \subset E_2 \cup N_2$ and hence

$$\mu(E_1) \leqslant \mu(E_2 \cup N_2) \leqslant \mu(E_2) + \underbrace{\mu(N_2)}_{=0} = \mu(E_2).$$

Analogously we have $\mu(E_2) \leqslant \mu(E_1)$ and therefore $\mu(E_1) = \mu(E_2)$.

If $E \in \mathcal{M}$ then $\overline{\mu}(E) = \mu(E)$, hence $\overline{\mu}$ is an extension of μ. Now assume that $\{A_j\}$ is a sequence of pairwise disjoint sets in $\overline{\mathcal{M}}$. Then $A_j = E_j \cup F_j$ where $E_j \in \mathcal{M}$ and $F_j \subset N_j$, with $N_j \in \mathcal{N}$. Since they are disjoint, we have in particular that $\{E_j\}$ is a sequence of pairwise disjoint sequences in $\mathcal{M}$. Then (2.1.1) hold and

$$\overline{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right) = \mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty} \overline{\mu}(A_j),$$

which shows that $\overline{\mu}$ is a measure on $\overline{\mathcal{M}}$.

Now we have to show that $\overline{\mu}$ is complete. Assume that $A \subset B$ where $B \in \overline{\mathcal{M}}$ and $\overline{\mu}(B) = 0$. Then $B = E \cup F$ where $E \in \mathcal{M}$ with $\mu(E) = 0$ and $F \subset N$ where $N \in \mathcal{N}$. Since $\mu(E) = 0$ we have also $E \in \mathcal{N}$ and hence $A = \varnothing \cup [(E \cup F) \cap A]$, where $(E \cup F) \cap A \subset E \cup F \in \mathcal{N}$, and proves that $A \in \overline{\mathcal{M}}$.

Finally, it remains to prove the uniqueness of $\overline{\mu}$. To that end, assume that ν is a measure in $\overline{\mathcal{M}}$ which is an extension of $\overline{\mu}$, that is, $\nu = \mu$ in $\mathcal{M}$. First, note that if $F \subset N \in \mathcal{N}$ then $F = \varnothing \cup F \in \overline{\mathcal{M}}$ and $\nu(F) \leqslant \nu(N) = \mu(N) = 0$. Now if $A \in \mathcal{M}$, where $A = E \cup F$ with $E \in \mathcal{M}$, $F \subset N \in \mathcal{N}$, and we can assume that $E \cap F = \varnothing$. Therefore

$$\nu(A) = \nu(E \cup F) = \nu(E) + \nu(F) = \nu(E) = \mu(E) = \overline{\mu}(A),$$

which proves that $\nu = \overline{\mu}$, and completes the proof of the theorem. ■

The measure $\overline{\mu}$ of this previous theorem is called the **completion** of μ, and $\overline{\mathcal{M}}$ is called the **completion** $\mathcal{M}$ with respect to μ.

Proposition 2.1.10. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. If $E \in \overline{\mathcal{M}}$ is such that $\overline{\mu}(E) = 0$ then there exists $N \in \mathcal{M}$ with $\mu(N) = 0$ and $E \subset N$.*

Proof. If $E \in \overline{\mathcal{M}}$, then by definition, there exists $G \in \mathcal{M}$ and $F \subset N_1 \in \mathcal{N}$ with $E = G \cup F$. Since $0 = \overline{\mu}(E) = \mu(G)$, we also have $G \in \mathcal{N}$. Hence $E = G \cup F \subset N := G \cup N_1$ and $N \in \mathcal{N}$. ■

2.2| Solved exercises from [1, Page 27]

Exercise 6.

Solution. This is done in Theorem 2.1.9.

Exercise 7. If $\mu_1, \cdots, \mu_n$ are measure on $(X, \mathcal{M})$ and $a_1, \cdots, a_n \in [0, \infty)$ then $\sigma_{i=1}^n a_i \mu_i$ is a measure on $(X, \mathcal{M})$.

Solution. We set $\nu = \sum_{i=1}^n a_i \mu_i$. Clearly

$$\nu(\varnothing) = \sum_{i=1}^{n} \mu_i(\varnothing) = 0.$$

Now if $\{E_j\}$ is a sequence of pairwise disjoint elements of $\mathcal{M}$ and $E = \bigcup_{j=1}^\infty E_j$ then

$$\nu(E) = \sum_{i=1}^{n} a_i \mu_i(E) = \sum_{i=1}^{n} a_i \sum_{j=1}^{\infty} \mu_i(E_j) = \sum_{j=1}^{\infty} \sum_{i=1}^{n} a_i \mu_i(E_j),$$

where in the last equality we used the rearrangement properties of absolute convergence of series, and hence

$$\nu(E) = \sum_{j=1}^{\infty} \nu(E_j).$$

Exercise 8. **Definition:** If $\{E_j\}$ is a sequence of sets in X then we set $\liminf E_j = \bigcup_{j=1}^\infty \bigcap_{n=j}^\infty E_n$ and $\limsup E_j = \bigcap_{j=1}^\infty \bigcup_{n=j}^\infty E_n$.

Now if $(X, \mathcal{M}, \mu)$ is a measure space and $\{E_j\} \subset \mathcal{M}$ then $\mu(\liminf E_j) \leqslant \liminf_{j\to\infty} \mu(E_j)$. Also $\mu(\limsup E_j) \geqslant \limsup_{j\to\infty} \mu(E_j)$ provided that $\mu\left(\bigcup_{j=1}^\infty E_j\right) < \infty$.

Solution. Note that $\{\bigcap_{n=j}^\infty E_n\}_j$ is an increasing sequence of sets in $\mathcal{M}$. From the continuity from below (Theorem 2.1.8, item (a)), we have

$$\mu(\liminf E_j) = \mu\left(\bigcup_{j=1}^{\infty} \bigcap_{n=j}^{\infty} E_n\right) = \lim_{j\to\infty} \mu\left(\bigcap_{n=j}^{\infty} E_n\right) \leqslant \liminf_{j\to\infty} \mu(E_j), .$$

since $\bigcap_{n=j}^{\infty} E_n \subset E_j$ for each j.

For the other inequality, note that the sequence $\{\bigcup_{n=j}^{\infty} E_n\}_j$ is decreasing and $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) < \infty$. From the continuity from above (Theorem 2.1.8, item (b)) we have

$$\mu(\limsup E_j) = \mu\left(\bigcap_{j=1}^{\infty}\bigcup_{n=j}^{\infty} E_n\right) = \lim_{j\to\infty} \mu\left(\bigcup_{n=j}^{\infty} E_n\right) \geqslant \limsup_{j\to\infty} \mu(E_j),$$

since $\bigcup_{n=j}^{\infty} E_n \supset E_j$ for each j.

EXERCISE 9. If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$ then

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F). \tag{2.2.1}$$

Solution. In fact, if $\mu(E) = \infty$ then $\mu(E \cup F) = \infty$ and (2.2.1) is trivial, and the same is true if $\mu(F) = \infty$. Assume then both $\mu(E)$ and $\mu(F)$ are finite. Since $E \cup F = (E \setminus F) \cup (E \cap F) \cup (F \setminus E)$ we have

$$\mu(E \cup F) = \mu(E \setminus F) + \mu(E \cap F) + \mu(F \setminus E),$$

but $\mu(E \setminus F) = \mu(E) - \mu(E \cap F)$ and $\mu(F \setminus E) = \mu(F) - \mu(E \cap E)$ and hence

$$\mu(E \cup F) = \mu(E) + \mu(F) - \mu(E \cap F),$$

which proves the result.

EXERCISE 9. Given a measure $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for each $A \in \mathcal{M}$. Then μ_E is a measure.

Solution. Clearly $\mu_E(\varnothing) = \mu(A \cap \varnothing) = \mu(\varnothing) = 0$. Now if $\{A_j\}$ is a sequence of a pairwise disjoint sets in $\mathcal{M}$ then

$$\mu_E\left(\bigcup_{j=1}^{\infty} A_j\right) = \mu\left(E \cap \bigcup_{j=1}^{\infty} A_j\right) = \mu\left(\bigcup_{j=1}^{\infty}(E \cap A_j)\right) = \sum_{j=1}^{\infty} \mu(E \cap A_j),$$

since $\{E \cap A_j\}$ is a pairwise disjoint sequence in $\mathcal{M}$. Therefore μ_E is a measure on $\mathcal{M}$.

EXERCISE 11. Let μ be a finitely additive measure. Then

(a) μ is a measure if and only if μ is continuous from below.

(b) if $\mu(X) < \infty$, then μ is a measure if and only if it is continuous from above.

Solution to (a) If μ is a measure then the continuity from below follows from item (a) of Theorem 2.1.8. Now assume that μ is continuous from below and let $\{E_j\}$ be a pairwise disjoint sequence in $\mathcal{M}$. Set $F_j = \bigcup_{k=1}^{j} E_k$ for each j. Then $\{F_j\}$ is an increasing sequence of sets in $\mathcal{M}$ and by the finitely additive property of μ we have

$$\mu(F_j) = \sum_{k=1}^{j} \mu(E_k)$$

and

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \mu\left(\bigcup_{j=1}^{\infty} F_j\right) = \lim_{j\to\infty} \mu(F_j) = \lim_{j\to\infty} \sum_{k=1}^{j} \mu(E_k) = \sum_{j=1}^{\infty} \mu(E_j).$$

Solution to (b) Again, if μ is a measure, then it is continuous from above by item (b) of Theorem 2.1.8. Now assume that $\mu(X) < \infty$ and μ is a finitely additive continuous form above measure. We will show that μ is continuous from below and the conclusion will follow from item (a).

Let $\{E_j\}$ be an increasing sequence of measurable sets. Then setting $F_j = X \setminus E_j$ for all j we have a decreasing sequence $\{F_j\}$ of measurable sets, and by the continuity from above we obtain

$$\mu\left(X \setminus \bigcup_{j=1}^{\infty} E_j\right) = \mu\left(\bigcap_{j=1}^{\infty} F_j\right) = \lim_{j\to\infty} \mu(F_j) = \lim_{j\to\infty} \mu(X \setminus E_j),$$

and since $\mu(X) < \infty$ we have

$$\mu(X) - \mu\left(X \setminus \bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} [\mu(X) - \mu(E_j)],$$

which concludes the results.

Exercise 12. **Definition:** Define $E\Delta F = (E \setminus F) \cup (F \setminus E)$ for $E, F \subset X$.

Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

(a) If $E, F \in \mathcal{M}$ and $\mu(E\Delta F) = 0$ then $\mu(E) = \mu(F)$.

(b) Say that $E \sim F$ if $\mu(E\Delta F) = 0$. Then $\sim$ is an equivalence relation on $\mathcal{M}$.

(c) For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E\Delta F)$. Then ρ defines a metric on the space $\mathcal{M}/\sim$.

Solution to (a) Note first that $\mu(E\Delta F) = \mu(E \setminus F) + \mu(F \setminus E)$ and since $\mu(E\Delta F) = 0$ we have $\mu(E \setminus F) = \mu(F \setminus E) = 0$.

Also, writing $E = (E \setminus F) \cup (E \cap F)$ and $F = (F \setminus E) \cup (F \cap E)$ we have

$$\mu(E) = \mu(E \setminus F) + \mu(E \cap F) \quad \text{and} \quad \mu(F) = \mu(F \setminus E) + \mu(F \cap E),$$

and hence $\mu(E) = \mu(E \cap F) = \mu(F)$.

Solution to (b) We prove that $\sim$ is an equivalence relation of $\mathcal{M}$.

(i) (Reflexive) Since $E\Delta E = \varnothing$, we have $\mu(E\Delta E) = 0$, hence $E \sim E$.

(ii) (Symmetric) If $E \sim F$, since $E\Delta F = F\Delta E$ then $F \sim E$.

(iii) (Transitive) Let $E \sim F$ and $F \sim G$. Now $E \setminus G \subset (E \setminus F) \cup (F \setminus G)$ and $G \setminus E \subset (G \setminus F) \cup (F \setminus E)$, hence

$$E\Delta G \subset (E\Delta F) \cup (F\Delta G), \tag{2.2.2}$$

and the subadditivity property of μ shows that

$$\mu(E\Delta G) \leqslant \mu(E\Delta F) + \mu(F\Delta G) = 0,$$

and therefore $E \sim G$.

Solution to (c) Consider the space $\mathcal{M}/\sim$ of equivalence classes of $\sim$. Clearly $\rho(E, F) = \rho(F, E)$ for each $E, F \in \mathcal{M}$.

Let us show that this metric is well defined: if $E_1 \sim E_2$ and $F \in \mathcal{M}$ then using (2.2.2) we have

$$E_1\Delta F \subset (E_1\Delta E_2) \cup (E_2\Delta F) \quad \text{and} \quad E_2\Delta F \subset (E_2\Delta E_1) \cup (E_1\Delta F),$$

and we obtain $\rho(E_1, F) = \rho(E_2, F)$. Now if $F_1 \sim F_2$ we have

$$\rho(E_1, F_1) = \rho(E_2, F_1) = \rho(E_2, F_2),$$

and proves that ρ is well defined. The symmetric property of ρ follows from the symmetric property of Δ. Lastly, in item (b) we have proven that $\mu(E\Delta G) \leqslant \mu(E\Delta F) + \mu(F\Delta G)$, and hence ρ satisfy the triangle property. Thus ρ is a metric in $\mathcal{M}/\sim$.

Exercise 13. If μ is a σ-finite measure then μ is semifinite.

Solution. We know that $X = \bigcup_{j=1}^{\infty} E_j$ where $\{E_j\}$ is a pairwise disjoint sequence in $\mathcal{M}$ with $\mu(E_j) < \infty$ for all j (see Proposition 2.1.4). If $\mu(E) = \infty$, we can write $E = E \cap X = \bigcup_{j=1}^{\infty}(E \cap E_j)$, with disjoint union and hence $\mu(E) = \sum_{j=1}^{\infty} \mu(E \cap E_j)$, with $\mu(E \cap E_j) < \infty$ for all j. If $\mu(E \cap E_j) = 0$ for all j then $\mu(E) = 0$, which is a contradiction, therefore there exists j_0 such that $\mu(E \cap E_{j_0}) > 0$, and hence $E \cap E_j \subset E$ and $0 < \mu(E \cap E_j) < \infty$, which means that μ is semifinite.

Exercise 14. If μ is a semifinite measure and $\mu(E) = \infty$, the for any given $c > 0$, there exists $F \subset E$ with $c < \mu(F) < \infty$.

Solution. Set $J(E) = \{F \colon F \subset E \text{ with } 0 < \mu(F) < \infty\}$. Clearly, since μ is semifinite, the set $J(E)$ is nonempty. Take $s = \sup_{F \in J(E)} \mu(F)$. The result is proven if we show that $s = \infty$.

To that end, suppose by absurd that $s < \infty$ and choose a sequence $\{F_j\}$ with $F_j \subset E$, $0 < \mu(F_j) < \infty$ and $\lim\limits_{j\to\infty} \mu(F_j) = s$. Define $G = \bigcup_{j=1}^{\infty} F_j$, then $\mu(G) \geqslant \mu(F_j)$ for all j and hence $\mu(G) \geqslant s$.

If $\mu(G) < \infty$ then $\mu(E \setminus G) = \infty$ and we can choose $G_1 \subset E \setminus G$ such that $0 < \mu(G_1) < \infty$. Then $G \cup G_1 \subset E$ and $\mu(G \cup G_1) = \mu(G) + \mu(G_1) > s$, which contradicts the fact that $s = \sup_{F \in J(E)} \mu(F)$, since $G \cup G_1 \in J(E)$.

If $\mu(G) = \infty$, then by the continuity from below, we have

$$\mu(G) = \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} F_j\right),$$

and hence there exists n such that $\mu\left(\bigcup_{j=1}^{n} F_j\right) > s$, which again contradicts the fact that $s = \sup_{F \in J(E)} \mu(F)$.

Thus, we must have $s = \infty$ and the result is proven.

Exercise 15. Given a measure space $(X, \mathcal{M}, \mu)$, define μ_0 on $\mathcal{M}$ by

$$\mu_0(E) = \sup\{\mu(F) \colon F \subset E \text{ and } \mu(F) < \infty\}.$$

(a) μ_0 is a semifinite measure, called the **semifinite part** of μ.

(b) If μ is semifinite, then $\mu = \mu_0$.

(c) There is a measure ν on $\mathcal{M}$ (in general, not unique) which assumes only values 0 and ∞ such that $\mu = \mu_0 + \nu$.

Solution to (a). Let us prove first that μ_0 is a measure on $\mathcal{M}$. Clearly $\mu_0(\varnothing) = 0$. If $\{E_j\}$ is a pairwise disjoint sequence in $\mathcal{M}$ and $E = \bigcup_{j=1}^{\infty} E_j$. Let $F \subset E$ with $\mu(F) < \infty$. Setting $F_j = F \cap E_j$ then $F \cap E_j \subset E_j$ and $\mu(F \cap E_j) < \infty$ then $\mu(F \cap E_j) \leqslant \mu_0(E_j)$ and hence

$$\mu(F) = \sum_{j=1}^{\infty} \mu(F \cap E_j) \leqslant \sum_{j=1}^{\infty} \mu_0(E_j).$$

Since this is true for all $F \subset E$ with $\mu(F) < \infty$ then

$$\mu_0(E) \leqslant \sum_{j=1}^{\infty} \mu_0(E_j).$$

Now if $F_j \subset E_j$ with $\mu(F_j) < \infty$ for all j, then $G_n = \bigcup_{j=1}^{n} F_j \subset E$ for each n and $G_n \subset E$ and $\mu(G_n) < \infty$ for alll n, thus

$$\mu_0(E) \geqslant \mu(G_n) = \sum_{j=1}^{n} \mu(F_j) \text{ for all } n.$$

Since this is true for all $F_j \subset E_j$ with $\mu(F_j) < \infty$, taking the supremum of such F_j for each $j = 1, \cdots, n$ we have

$$\mu_0(E) \geqslant \sum_{j=1}^{n} \mu_0(E_j) \text{ for all } n,$$

and making $n \to \infty$ we have $\mu_0(E) \geqslant \sum_{j=1}^{\infty} \mu_0(E_j)$, which concludes the proof that μ_0 is a measure on $\mathcal{M}$.

Now we prove that μ_0 is semifinite. To that end, we first prove that if $\mu(E) < \infty$ then $\mu_0(E) = \mu(E)$. Clearly if $\mu(E) < \infty$ then $\mu(E) \leqslant \mu_0(E)$. Now if $F \subset E$ then $\mu(F) \leqslant \mu_0(E)$ and hence, taking the supremum over F, we have $\mu_0(E) \leqslant \mu(E)$.

Assume that $\mu_0(E) = \infty$, then by definition of supremum, there exists $F \subset E$ with $0 < \mu(F) < \infty$, and since $\mu(F) = \mu_0(F)$ we obtain the result.

Solution to (b). We have already proven in item (a) that $\mu = \mu_0$ for measurable sets with finite μ-measure. Now assume that $\mu(E) = \infty$. Since μ is semifinite, by Exercise 14 above, for each positive integer n, there exists $F_n \subset E$ with $c < \mu(F_n) < \infty$. Hence $\mu_0(E) \geqslant \mu(F_n)$ for all n, which implies that $\mu_0(E) = \infty$ and concludes the result.

Solution to (c). We say that a set E is **σ-finite for μ** if $E = \bigcup_{j=1}^{\infty} E_j$ with disjoint union and $\mu(E_j) < \infty$ for all j. Set ν on $\mathcal{M}$ as follows:

$$\nu(E) = \begin{cases} 0, & \text{if } E \text{ is } \sigma\text{-finite for } \mu, \\ \infty, & \text{if } E \text{ is not } \sigma\text{-finite for } \mu. \end{cases}$$

We will prove first that if E is σ-finite for μ then $\mu(E) = \mu_0(E)$. If E is σ-finite for μ then

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty} \sum \mu_0(E_j) = \mu_0(E),$$

since $\mu = \mu_0$ for μ-finite measurable sets. Then $\mu(E) = \mu_0(E) = \mu_0(E) + \nu(E)$ if E is a σ-finite sets.

Now note that E is not σ-finite for μ then $\mu(E) = \infty$, and hence $\mu(E) = \nu(E) = \mu_0(E) + \nu(E)$. Therefore we have $\mu = \mu_0 + \nu$.

It only remains to prove that ν is a measure on $\mathcal{M}$. Clearly $\nu(\varnothing) = 0$. Now let $\{E_j\}$ be a pairwise disjoint sequence on $\mathcal{M}$ and $E = \bigcup_{j=1}^{\infty} E_j$. If E_j is σ-finite for μ for all j then E is also σ-finite, hence

$$\nu(E) = 0 = \sum_{j=1}^{\infty} \nu(E_j).$$

Assume that E_{j_0} is not σ-finite for μ for some j_0 and suppose that E is. Then $E = \bigcup_{k=1}^{\infty} F_k$ with disjoint union, $F_k \in \mathcal{M}$ and $\mu(F_k) < \infty$. Hence $E_{j_0} = E_{j_0} \cap E = \bigcup_{k=1}^{\infty} (E_{j_0} \cap F_k)$ and hence E_{j_0} is σ-finite for μ which is a contradiction, and hence E is not σ-finite for μ, thus

$$\nu(E) = \infty = \nu(E_{j_0}) \geqslant \sum_{j=1}^{\infty} \nu(E_j) = \infty,$$

hence $\nu(E) = \sum\limits_{j=1}^{\infty} \nu(E_j)$, and ν is a measure on $\mathcal{M}$.

Exercise 16. Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \subset X$ is called **locally measurable** if $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ such that $\mu(A) < \infty$. Let $\tilde{\mathcal{M}}$ be the collection of all locally measurable sets. We know that $\mathcal{M} \subset \tilde{\mathcal{M}}$. If $\mathcal{M} = \tilde{\mathcal{M}}$ then μ is called **saturated**.

(a) If μ is σ-finite then μ is saturated.

(b) $\tilde{\mathcal{M}}$ is a σ-algebra.

(c) Define $\tilde{\mu}$ on $\tilde{\mathcal{M}}$ by $\tilde{\mu}(E) = \mu(E)$ is $E \in \mathcal{M}$ and $\tilde{\mu}(E) = \infty$ otherwise. Then $\tilde{\mu}$ is a saturated measure on $\tilde{\mathcal{M}}$ called the **saturation** of μ.

(d) If μ is complete, so is $\tilde{\mu}$.

(e) Suppose that μ is semifinite. For $E \in \tilde{\mathcal{M}}$, define $\underline{\mu}(E) = \sup\{\mu(A) \colon A \in \mathcal{M} \text{ and } A \subset E\}$. Then $\underline{\mu}$ is a saturated measure on $\tilde{\mathcal{M}}$ that extends μ.

(f) Let X_1, X_2 be disjoint uncountable sets, $X = X_1 \cup X_2$ and $\mathcal{M}$ is the σ-algebra of the countable ou co-countable sets in X. Let μ_0 be the counting measure on $\mathcal{P}(X_1)$ and define μ on $\mathcal{M}$ by $\mu(E) = \mu_0(E \cap X_1)$. Then μ is a measure on $\mathcal{M}$, $\tilde{\mathcal{M}} = \mathcal{P}(X)$, and in the notation of parts (c) and (e), then $\tilde{\mu} \neq \underline{\mu}$.

Solution to (a). Let E be a locally measurable set. Since σ-finite is $X = \bigcup_{j=1}^{\infty} E_j$ with disjoint union and $\mu(E_j) < \infty$ for all j. Hence

$$E = E \cap X = E \cap \left(\bigcup_{j=1}^{\infty} E_j \right) = \bigcup_{j=1}^{\infty} \underbrace{(E \cap E_j)}_{\in \mathcal{M}} \in \mathcal{M},$$

by the locally measurability of E. Hence $\tilde{\mathcal{M}} \subset \mathcal{M}$ and μ is saturated.

Solution to (b). Clearly $\varnothing \in \tilde{\mathcal{M}}$. Now if $\{E_j\} \subset \tilde{\mathcal{M}}$ and $A \in \mathcal{M}$ with $\mu(A) < \infty$ then

$$\left(\bigcup_{j=1}^{\infty} E_j \right) \cap A = \bigcup_{j=1}^{\infty} \underbrace{(E_j \cap A)}_{\in \mathcal{M}} \in \mathcal{M},$$

and hence $\bigcup_{j=1}^{\infty} E_j \in \tilde{\mathcal{M}}$. If $E \in \tilde{\mathcal{M}}$ and $A \in \mathcal{M}$ with $\mu(A) < \infty$ then

$$E^c \cap A = A \setminus (E \cap A) \in \mathcal{M},$$

and hence $E^c \in \tilde{\mathcal{M}}$. Thus $\tilde{\mathcal{M}}$ is a σ-algebra.

Solution to (c). We prove now that $\tilde{\mu}$ is a measure on $\tilde{\mathcal{M}}$. Let $E = \bigcup_{j=1}^{\infty} E_j$ with disjoint union, with $E_j \in \tilde{\mathcal{M}}$ for all j.

If $E \in \mathcal{M}$ and $\mu(E) < \infty$ then, since $E_j \in \tilde{\mathcal{M}}$, we have $E_j = E_j \cap E \in \mathcal{M}$, by definition of $\tilde{\mathcal{M}}$, for all j. Thus

$$\tilde{\mu}(E) = \mu(E) = \sum_{j=1}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty} \tilde{\mu}(E_j).$$

Now if $E \in \mathcal{M}$, $\mu(E) = \infty$ and $E_j \in \mathcal{M}$ then $\tilde{\mu}(E_j) = \mu(E_j)$ and $\tilde{\mu}(E) = \mu(E)$, and the result follows in this case.

Now assume that $E \in \mathcal{M} \setminus \mathcal{M}$. If $E_j \in \mathcal{M}$ for all j we have $E \in \mathcal{M}$, which implies that since $E \in \tilde{\mathcal{M}} \setminus \mathcal{M}$ there exists j_0 such that $E_{j_0} \in \tilde{\mathcal{M}} \setminus \mathcal{M}$. Therefore, in this case

$$\tilde{\mu}(E) = \infty = \tilde{\mu}(E_{j_0}) \leqslant \sum_{j=1}^{\infty} \tilde{\mu}(E_j) \leqslant \infty,$$

and hence $\tilde{\mu}(E) = \infty = \sum_{j=1}^{\infty} \tilde{\mu}(E_j)$. Therefore $\tilde{\mu}$ is a measure on $\tilde{\mathcal{M}}$.

It remains to prove that $\tilde{\mu}$ is saturated. To this end, we show that if $E \in \tilde{\tilde{\mathcal{M}}}$ then $E \in \tilde{\mathcal{M}}$. In fact, if $E \in \tilde{\tilde{\mathcal{M}}}$ then for each $A \in \tilde{\mathcal{M}}$ with $\tilde{\mu}(A) < \infty$ we have $E \cap A \in \tilde{\mathcal{M}}$. Now if $B \in \mathcal{M}$ is such $\mu(B) < \infty$ then $B \in \tilde{\mathcal{M}}$ and $\tilde{\mu}(B) = \mu(B) < \infty$, hence $E \cap B \in \tilde{\mathcal{M}}$ and $E \cap B = (E \cap B) \cap B \in \mathcal{M}$.

Hence $\tilde{\tilde{\mathcal{M}}} \subset \tilde{\mathcal{M}}$ and $\tilde{\mu}$ is saturated.

Solution to (d). Now assume that μ is complete, let $N \subset \tilde{\mathcal{M}}$ with $\tilde{\mu}(N) = 0$ and $F \subset N$. Since $\tilde{\mu}(N) = 0 < \infty$ we have $N \in \mathcal{M}$, and since μ is complete, $F \in \mathcal{M}$. This implies that $F \in \tilde{\mathcal{M}}$ and concludes the proof.

Solution to (e). We prove first that $\underline{\mu}$ is a measure on $\mathcal{M}$. Clearly $\underline{\mu}(\varnothing) = 0$. Now let $E = \bigcup_{j=1}^{\infty} E_j$ with disjoint union with $E_j \in \tilde{\mathcal{M}}$ for all j, $F \in \mathcal{M}$ with $F \subset E$. If $\mu(F) < \infty$ then, by definition of $\tilde{\mathcal{M}}$, we have $F \cap E_j \in \mathcal{M}$ for all j and

$$\mu(F) = \mu(F \cap E) = \sum_{j=1}^{\infty} \mu(F \cap E_j) \leqslant \sum_{j=1}^{\infty} \underline{\mu}(E_j).$$

If $\mu(F) = \infty$ the exists a sequence $\{F_n\} \subset \mathcal{M}$ such that $F_n \subset E$ and $n < \mu(F_n) < \infty$ for each n. Hence, for each fixed n, by the computation above, we have

$$n < \mu(F_n) \leqslant \sum_{j=1}^{\infty} \underline{\mu}(E_j),$$

and making $n \to \infty$ we obtain

$$\sum_{j=1}^{\infty} \underline{\mu}(E_j) = \infty = \mu(F).$$

Joining these two cases, we can write

$$\mu(F) \leqslant \sum_{j=1}^{\infty} \underline{\mu}(E_j),$$

and taking the supremum over F we get $\underline{\mu}(E) \leqslant \sum_{j=1}^{\infty} \underline{\mu}(E_j)$.

For the other inequality, note that if $F_j \in \mathcal{M}$ and $F_j \subset E_j$ for all j, then $G_n = \bigcup_{j=1}^{n} F_j \in \mathcal{M}$ and $G_n \subset E$ for each n, hence

$$\sum_{j=1}^{n} \mu(F_n) = \mu(G_n) \leqslant \underline{\mu}(E),$$

and taking the supremum over F_j for each $j = 1, \cdots, n$ we have

$$\sum_{j=1}^{n} \underline{\mu}(E_j) \leqslant \underline{\mu}(E),$$

and finally, making $n \to \infty$ we have $\sum_{j=1}^{\infty} \underline{\mu}(E_j) \leqslant \underline{\mu}(E)$, and concludes the proof that $\underline{\mu}$ is a measure on $\tilde{\mathcal{M}}$.

Now we prove that $\underline{\mu}$ extends μ. If $E \in \mathcal{M}$, then clearly $\mu(E) \leqslant \underline{\mu}(E)$. Now if $F \in \mathcal{M}$ and $F \subset E$ we have $\mu(F) \leqslant \mu(E)$, and taking the supremum of all such F we have $\underline{\mu}(E) \leqslant \mu(E)$. This completes the proof that $\mu = \underline{\mu}$ on $\mathcal{M}$.

It remains to prove that $\underline{\mu}$ is saturated. Let E be such that for all $A \in \tilde{\mathcal{M}}$ with $\underline{\mu}(A) < \infty$ then $E \cap A \in \tilde{\mathcal{M}}$. We have to show that $E \in \tilde{\mathcal{M}}$.

To that end let $B \in \mathcal{M}$ be such that $\mu(B) < \infty$. Since $B \in \mathcal{M}$ we have $\underline{\mu}(B) = \mu(B) < \infty$ and $E \cap B \in \mathcal{M}$ and $E \cap B = (E \cap B) \cap B \in \mathcal{M}$, which proves that $E \in \tilde{\mathcal{M}}$ and thus $\underline{\mu}$ is saturated.

Solution to (f). We prove first that μ is a measure. Clearly $\mu(\varnothing) = \mu_0(\varnothing) = 0$. Now let $E = \bigcup_{j=1}^{\infty} E_j$ with disjoint union and $E_j \in \mathcal{M}$ for all j. Then $\{E_j \cap X_1\}$ is a disjoint sequence on $\mathcal{P}(X_1)$ and hence

$$\mu(E) = \mu_0(E \cap X_1) = \mu_0\left(\bigcup_{j=1}^{\infty} (E_j \cap X_1)\right) = \sum_{j=1}^{\infty} \mu_0(E_j \cap X_1) = \sum_{j=1}^{\infty} \mu(E_j),$$

hence μ is a measure on $\mathcal{M}$.

Now fix $E \subset \mathcal{P}(X)$ and $A \in \mathcal{M}$ with $\mu(A) < \infty$. This implies that $\mu_0(A \cap X_1) < \infty$ and hence $A \cap X_1$ is finite. But since X_1 is uncountable, we have $X_1 \setminus A$ uncountable, and since $X \setminus A \supset X_1 \setminus A$, this implies that $X \setminus A$ is uncountable. Since $A \in \mathcal{M}$ we must have A countable. Therefore $E \cap A$ is countable, which means that $E \cap A \in \mathcal{M}$, and thus $\tilde{\mathcal{M}} = \mathcal{P}(X)$.

Now fix $x_1 \in X_1$ and consider $E = \{x_1\} \cup X_2 \subset X$. We have $\mu(E) = 1$ and thus $\underline{\mu}(E) = 1$. But E is neither countable nor co-countable, hence $E \notin \mathcal{M}$ and $\tilde{\mu}(E) = \infty$, and thus $\tilde{\mu} \neq \underline{\mu}$.

2.3 | Outer measures

Definition 2.3.1. *An* **outer measure** *on a nonempty set X is a function $\mu^* \colon \mathcal{P}(X) \to [0, \infty]$ that satisfies:*

(i) $\mu^*(\varnothing) = 0$,

(ii) $\mu^*(A) \leqslant \mu^*(B)$ *if* $A \subset B$,

(iii) $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leqslant \sum_{j=1}^{\infty} \mu^*(A_j)$.

Proposition 2.3.2. *Let $\mathcal{E} \subset \mathcal{P}(X)$ and $\mu\colon \mathcal{E} \to [0, \infty]$ be such that $\varnothing \in \mathcal{E}$, $X \in \mathcal{E}$ and $\mu(\varnothing) = 0$. For any $A \subset X$, define*

$$\mu^*(A) = \inf\left\{\sum_{j=1}^{\infty} \mu(E_j)\colon E_j \in \mathcal{E} \textit{ and } A \subset \bigcup_{j=1}^{\infty} E_j\right\}.$$

Then μ^* is an outer measure.

Proof. Given $A \subset X$, there exists $\{E_j\}_{j=1}^{\infty} \subset \mathcal{E}$ such that $A \subset \bigcup_{j=1}^{\infty} E_j$ (taking $E_j = X$ for all j, for instance), so the definition μ^* makes sense. Obviously $\mu^*(\varnothing) = 0$, just taking $E_j = \varnothing$ for all j.

If $A \subset B$, then each cover of B by subsets of $\mathcal{E}$ is also a cover of A, and hence $\mu^*(A) \leqslant \mu^*(B)$.

To prove the countable subadditivity, assume that $\{A_j\}_{j=1}^{\infty} \subset \mathcal{P}(X)$ and $\epsilon > 0$. For each j there exists a cover $\{E_{j,k}\}_{k=1}^{\infty} \subset \mathcal{E}$ such that

$$\sum_{k=1}^{\infty} \mu(E_{j,k}) \leqslant \mu^*(A_j) + \frac{\epsilon}{2^j},$$

but then if $A = \bigcup_{j=1}^{\infty} A_j$ then $A \subset \bigcup_{j,k=1}^{\infty} E_{j,k}$ and

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mu(E_{j,k}) \leqslant \sum_{j=1}^{\infty} \mu(A_j) + \epsilon,$$

hence $\mu^*(A) \leqslant \sum_{j=1}^{\infty} \mu^*(A_j) + \epsilon$. Since $\epsilon > 0$ is arbitrary, the result is proven. ∎

Given an outer measure μ^* on X, a set $A \subset X$ is called **μ^*-measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \qquad \text{for all } E \subset X.$$

Clearly, for all $E \subset X$ we have $\mu^*(E) \leqslant \mu^*(E \cap A) + \mu^*(E \cap A^c)$, hence A is μ^*-measurable if and only if $\mu^*(E) \geqslant \mu^*(E \cap A) + \mu^*(E \cap A^c)$, and this latter is trivial if $\mu^*(E) = \infty$. Hence

$A \subset X$ is μ^*-measurable if and only if

$$\mu^*(E) \geqslant \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \text{for all } E \subset X \text{ with } \mu^*(E) < \infty.$$

Theorem 2.3.3 (Carathéodory's Theorem)**.** *If μ^* is an outer measure on X, then the collection*

$$\mathcal{M} = \{A \subset X \colon A \textit{ is } \mu^*\textit{-measurable}\},$$

is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

Proof. We begin proving that $\mathcal{M}$ is an algebra. Clearly $\varnothing \in \mathcal{M}$. Now let $A, B \in \mathcal{M}$ and $E \subset X$. Since $A \in \mathcal{M}$, we have

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c),$$

and since $B \in \mathcal{M}$ we have

$$\mu^*(E \cap A) = \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c)$$

and

$$\mu^*(E \cap A^c) = \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c),$$

thus

$$\mu^*(E) = \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \underbrace{\mu^*(E \cap A^c \cap B^c)}_{=\mu^*(E \cap (A \cup B)^c)}.$$

But $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$, and hence $\mu^*(E \cap (A \cup B)) \leqslant \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B)$, which implies that

$$\mu^*(E) \geqslant \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c),$$

and thus $A \cup B \in \mathcal{M}$, and $\mathcal{M}$ is an algebra.

Now let $\{A_j\}$ be a pairwise disjoint sequence in $\mathcal{M}$ and $A = \bigcup_{j=1}^{\infty} A_j$. Set $B_j = \bigcup_{k=1}^{j} A_j$ for all j, and since $\mathcal{M}$ is an algebra, $\{B_j\} \subset \mathcal{M}$.

Now, setting $B_0 = \varnothing$, for each fixed j and $E \subset X$, since $A_j \in \mathcal{M}$, we have

$$\begin{aligned}\mu^*(E \cap B_j) &= \mu^*(E \cap B_j \cap A_j) + \mu^*(E \cap B_j \cap A_j^c)\\ &= \mu^*(E \cap A_j) + \mu^*(E \cap B_{j-1}),\end{aligned}$$

and hence $\mu^*(E \cap B_j) = \sum_{k=1}^{j} \mu^*(E \cap A_k)$. Now

$$\begin{aligned}\mu^*(E) = \mu^*(E \cap B_j) + \mu^*(E \cap B_j^c) &= \sum_{k=1}^{j} \mu^*(E \cap A_k) + \mu^*(E \cap B_j^c) \\ &\geqslant \sum_{k=1}^{j} \mu^*(E \cap A_k) + \mu(E \cap A^c),\end{aligned}$$

and letting $n \to \infty$, we obtain

$$\begin{aligned}\mu^*(E) &\geqslant \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu(E \cap A^c) \\ &\geqslant \mu^*\Big(\bigcup_{j=1}^{\infty}(E \cap A_j)\Big) + \mu(E \cap A^c) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \geqslant \mu^*(E),\end{aligned}$$

which shows us that $A \in \mathcal{M}$. Moreover, taking $E = A$ in the above gives us

$$\mu^*(A) = \sum_{j=1}^{\infty} \mu^*(A_j),$$

and hence μ^* is countably additive in $\mathcal{M}$, hence μ^* restricted to $\mathcal{M}$ is a measure.

To show that it is complete, note that if $\mu^*(A) = 0$ then

$$\mu^*(E) \leqslant \mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*(E \cap A^c) \leqslant \mu^*(E),$$

and therefore $A \in \mathcal{M}$, and μ^* is a complete measure in $\mathcal{M}$. ■

The first application of CarathÃ©odory's Theorem is to extend measures from algebras to σ-algebras. More precisely, let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra. A function $\mu_0 \colon \mathcal{A} \to [0, \infty]$ is called a **premeasure** if

1. $\mu_0(\varnothing) = 0$,

2. if $\{A_j\}$ is a pairwise disjoint sequence of sets in $\mathcal{A}$ such that $\bigcup\limits_{j=1}^{\infty} A_j \in \mathcal{A}$ then

$$\mu_0\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu_0(A_j).$$

In particular, a premeasure if finitely additive, since one can take $A_j = \varnothing$ for all but a finite number of j. The notions of finite and σ-finite premeasures are defined just as for measures. If μ_0 is a premeasure on $\mathcal{A} \subset \mathcal{P}(X)$, it induces an outer measure on X, namely

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(A_j) \colon A_j \in \mathcal{A},\ E \subset \bigcup_{j=1}^{\infty} A_j \right\}. \tag{2.3.1}$$

Proposition 2.3.4. *If μ_0 is a premeasure on an algebra $\mathcal{A}$ and μ^* is defined as (2.3.1), then:*

(a) $\mu^*|_{\mathcal{A}} = \mu_0$.

(b) *every set in $\mathcal{A}$ is μ^*-measurable.*

Proof. **(a).** Clearly, if $E \in \mathcal{A}$ then $\mu^*(E) \leqslant \mu_0(E)$. Now, if $E \in \mathcal{A}$, let $\{A_j\}$ be a sequence in $\mathcal{A}$ with $E \subset \bigcup_{j=1}^{\infty} A_j$. Define $B_1 = A_1$ and $B_j = A_j \setminus \left(\bigcup_{k=1}^{j-1} A_k\right)$ for $j > 1$. Thus $\{B_j\}$ is a pairwise disjoint sequence in $\mathcal{A}$, $\bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} A_j$, and we have

$$\begin{aligned}\mu_0(E) = \mu_0\Big(E \cap \bigcup_{j=1}^{\infty} B_j\Big) = \mu_0\Big(\bigcup_{j=1}^{\infty} E \cap B_j\Big) &\overset{(\star)}{=} \sum_{j=1}^{\infty} \mu_0(E \cap B_j) \\ &\leqslant \sum_{j=1}^{\infty} \mu_0(E \cap A_j) \leqslant \sum_{j=1}^{\infty} \mu_0(A_j)\end{aligned}$$

where in $(\star)$ we used that fact that μ_0 is a premeasure. Since this holds for every cover of E in $\mathcal{A}$, we have $\mu_0(E) \leqslant \mu^*(E)$.

(b). Let $A \in \mathcal{A}$, $E \subset X$ and $\epsilon > 0$. By definition of μ^*, there exists $\{B_j\} \subset A$ such that $E \subset \bigcup_{j=1}^{\infty} B_j$ with $\sum_{j=1}^{\infty} \mu_0(B_j) \leqslant \mu^*(E) + \epsilon$. Hence

$$\begin{aligned}\mu^*(E) + \epsilon \geqslant \sum_{j=1}^{\infty} \mu_0(B_j) &\overset{(\star)}{=} \sum_{j=1}^{\infty} \Big[\mu_0(B_j \cap A) + \mu_0(B_j \cap A^c)\Big] \\ &= \sum_{j=1}^{\infty} \mu_0(B_j \cap A) + \sum_{j=1}^{\infty} \mu_0(B_j \cap A^c) \geqslant \mu^*(E \cap A) + \mu^*(E \cap A^c),\end{aligned}$$

where in $(\star)$ we used the fact that μ_0 is a premeasure in $\mathcal{A}$ and $B_j, A \in \mathcal{A}$. Since $\epsilon > 0$ is arbitrary, we obtain $\mu^*(E) \geqslant \mu^*(E \cap A) + \mu^*(E \cap A^c)$, and thus A is μ^*-measurable. ■

Theorem 2.3.5. *Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, μ_0 a premeasure on $\mathcal{A}$, and $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$. Then:*

(i) *$\mu = \mu^*|_{\mathcal{M}}$, where μ^* is given as in (2.3.1) is a measure on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0;*

(ii) *if ν is another measure on $\mathcal{M}$ that extends μ_0 then $\nu(E) \leqslant \mu(E)$ for all $E \in \mathcal{M}$, with equality when $\mu(E) < \infty$;*

(iii) *if μ_0 is σ-finite, then μ is the unique extension of μ_0 to a measure on $\mathcal{M}$.*

Proof. **(i).** Using CarathÃ©odory's Theorem, μ^* is a measure when restricted to the σ-algebra of the μ^*-measurable sets. By item (b) of the previous proposition, this σ-algebra contains $\mathcal{A}$ and hence it contains $\mathcal{M}$. Thus, the restriction of μ^* to $\mathcal{M}$ is a measure which, by item (a) of the previous proposition, extends μ_0.

(ii). Let ν be a measure on $\mathcal{M}$ that extends μ_0. Then if $E \in \mathcal{M}$ and $\{A_j\} \subset \mathcal{A}$ with $E \subset \bigcup_{j=1}^{\infty} A_j$ then

$$\nu(E) \leqslant \nu\Big(\bigcup_{j=1}^{\infty} A_j\Big) \leqslant \sum_{j=1}^{\infty} \nu(A_j) = \sum_{j=1}^{\infty} \mu_0(A_j),$$

and hence $\nu(E) \leqslant \mu^*(E) = \mu(E)$, since $E \in \mathcal{M}$.

Setting $A = \bigcup_{j=1}^{\infty} A_j$, we have $A \in \mathcal{M}$ and also

$$\nu(A) = \lim_{n\to\infty} \nu\Big(\underbrace{\bigcup_{j=1}^{n} A_j}_{\in\mathcal{A}}\Big) = \lim_{n\to\infty} \mu_0\Big(\bigcup_{j=1}^{n} A_j\Big) = \lim_{n\to\infty} \mu\Big(\bigcup_{j=1}^{n} A_j\Big) = \mu(A).$$

Now let $E \in \mathcal{M}$ with $\mu(E) < \infty$. Given $\epsilon > 0$, there exists a pairwise disjoint sequence $\{A_j\} \subset \mathcal{A}$ with $E \subset A = \bigcup_{j=1}^{\infty} A_j$ and $\sum_{j=1}^{\infty} \mu_0(A_j) < \mu^*(E) + \epsilon = \mu(E) + \epsilon$. Thus

$$\mu(A) = \sum_{j=1}^{\infty} \mu(A_j) = \sum_{j=1}^{\infty} \mu_0(A_j) < \mu(E) + \epsilon,$$

and since $\mu(E) < \infty$, we have $\mu(A \setminus E) < \epsilon$. Thus

$$\mu(E) \leqslant \mu(A) = \nu(A) = \nu(E) + \nu(A \setminus E) \leqslant \nu(E) + \mu(A \setminus E) < \nu(E) + \epsilon,$$

and since $\epsilon > 0$ is arbitrary, we obtain $\mu(E) \leqslant \nu(E)$.

(iii). Suppose that μ_0 is σ-finite, that is, there exists a pairwise disjoint sequence $\{A_j\} \subset \mathcal{A}$ with $\mu_0(A_j) < \infty$ and $X = \bigcup_{j=1}^{\infty} A_j$. Then if $E \in \mathcal{M}$ we have

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E \cap A_j) \overset{(\star)}{=} \sum_{j=1}^{\infty} \nu(E \cap A_j) = \nu(E),$$

where in $(\star)$ we used item (ii). Hence $\nu = \mu$ on $\mathcal{M}$. ■

This theorem shows more than the statement: in fact, μ_0 can be extended to a measure on the σ-algebra $\mathcal{M}^*$ of the μ^*-measurable sets.

2.4 | Solved exercises from [1, page 32])

Exercise 17. If μ^* is an outer measure on X and $\{A_j\}$ is a pairwise disjoint sequence of μ^*-measurable sets, then

$$\mu^*\Big(E \cap \bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) \qquad \text{for any } E \subset X.$$

Solution. We assume, without loss of generality, that $E \subset \bigcup_{j=1}^{\infty} A_j$ (if E is not a subset of $\bigcup\limits_{j=1}^{\infty} A_j$, apply the result to $F = E \cap (\bigcup\limits_{j=1}^{\infty} A_j)$. We thus have to prove that $\mu^*(E) = \sum\limits_{j=1}^{\infty} \mu^*(E \cap A_j)$.

Clearly, since $E = \bigcup\limits_{j=1}^{\infty} E \cap A_j$ and μ^* is an outer measure, we have $\mu^*(E) \leqslant \sum\limits_{j=1}^{\infty} \mu^*(E \cap A_j)$. Set $B_j = \bigcup\limits_{k=1}^{j} A_j$ for each j, then B_j is μ^*-measurable and

$$\mu^*(E \cap B_j) = \mu^*(E \cap B_j \cap A_j) + \mu^*(E \cap B_j \cap A_j^c) = \mu^*(E \cap A_j) + \mu^*(E \cap B_{j-1}),$$

since $B_j \cap A_j^c = B_{j-1}$. Using this argument j times, we obtain

$$\mu^*(E \cap B_j) = \sum_{k=1}^{j} \mu^*(E \cap A_k),$$

and hence for each j we have

$$\mu^*(E) \geqslant \mu^*(E \cap B_j) = \sum_{k=1}^{j} \mu^*(E \cap A_k),$$

and making $j \to \infty$ we obtain the result.

Exercise 18. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra. Let μ_0 be a premeasure in $\mathcal{A}$ and μ^* the induced outer measure.

(a) For any $E \subset X$ and $\epsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leqslant \mu^*(E) + \epsilon$.

(b) If $\mu^*(E) < \infty$, then E is μ^*-measurable if and only if there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.

(c) If μ_0 is σ-finite, the restriction $\mu^*(E) < \infty$ in (b) is superfluous.

Solution to (a). By definition of μ, given $\epsilon > 0$ there exists a sequence $\{A_j\} \subset \mathcal{A}$, which we can assume without loss of generality that it is pairwise disjoint, such that $\sum\limits_{j=1}^{\infty} \mu_0(A_j) \leqslant \mu^*(E) + \epsilon$. Setting $A = \bigcup\limits_{j=1}^{\infty} A_j$, we have $A \in \mathcal{A}_\sigma$ and $\mu^*(A) \leqslant \sum\limits_{j=1}^{\infty} \mu^*(A_j) = \sum\limits_{j=1}^{\infty} \mu_0(A_j)$, since μ^* extends μ_0. Hence $\mu^*(A) \leqslant \mu^*(E) + \epsilon$.

Solution to (b). Assume first that there exists $B \in \mathcal{A}_{\sigma\delta}$ such that $E \subset B$ and $\mu^*(B \setminus E) = 0$. If $F \subset X$, since $F \cap E^c \subset (F \cap (B \setminus E)) \cup (F \cap B^c)$ we have

$$\mu^*(F \cap E^c) \leqslant \mu^*(F \cap (B \setminus E)) + \mu^*(F \cap B^c) \leqslant \underbrace{\mu^*(B \setminus E)}_{=0} + \mu^*(F \cap B^c),$$

and since B is μ^*-measurable and $E \subset B$, we have

$$\mu^*(F) \geqslant \mu^*(F \cap B) + \mu(F \cap B^c) \geqslant \mu^*(F \cap E) + \mu^*(F \cap E^c)$$

and hence E is μ^*-measurable (this implication does not make use of the fact that $\mu^*(E) < \infty$, and it also holds without the assumption that $B \in \mathcal{A}_{\sigma\delta}$, B just need to be a μ^*-measurable set).

For the converse, assume that $\mu^*(E) < \infty$ and E is μ^*-measurable. For each $n \in \mathbb{N}$, using item (a), there exists $A_n \in \mathcal{A}_\sigma$ with $E \subset A_n$ and $\mu^*(A_n) \leqslant \mu^*(E) + \frac{1}{n}$. Define $B = \bigcap\limits_{n=1}^{\infty} A_n$. Hence $B \in \mathcal{A}_{\delta\sigma}$, $E \subset B$ and $\mu^*(B) \leqslant \mu^*(A_n) \leqslant \mu^*(E) + \frac{1}{n}$ for each $n \in \mathbb{N}$. Making $n \to \infty$ we have $\mu^*(B) \leqslant \mu^*(E)$. Since $E \subset B$, we have $\mu^*(E) \leqslant \mu^*(B)$ and hence $\mu^*(B) = \mu^*(E)$ (until here the hypotheses that E is μ^*-measurable and $\mu^*(E) < \infty$ are not necessary).

Now since E and B are μ^*-measurable and μ^* is a measure when restricted to μ^*-measurable sets, we have

$$\mu^*(E) = \mu^*(B) = \mu^*(B \setminus E) + \mu^*(E),$$

and since $\mu^*(E) < \infty$, we obtain $\mu^*(B \setminus E) = 0$.

Solution to (c). If μ_0 is σfinite, there exists a pairwise disjoint sequence $\{A_j\}$ in $\mathcal{A}$ such that $X = \bigcup\limits_{j=1}^{\infty} A_j$ and $\mu_0(A_j) < \infty$ for each j.

Let E be a μ^*-measurable set. For each j, $E_j := E \cap A_j$ is μ^*-measurable and $\mu^*(E_j) \leqslant \mu^*(A_j) = \mu_0(A_j) < \infty$. From item (a), for each $n \in \mathbb{N}$ and j there exists a set $G_{j,n} \in \mathcal{A}_\sigma$ such that $E_j \subset G_{j,n}$ and $\mu^*(G_{j,n}) \leqslant \mu^*(E_j) + \frac{2^{-j}}{n}$. Since $\mu^*(E_j) < \infty$, we have $\mu^*(G_{j,n} \setminus E_j) \leqslant \frac{2^{-j}}{n}$.

Moreover, we can assume that $G_{j,n} \subset A_j$, for otherwise we could consider its intersection with A_j (which is still in $\mathcal{A}_\sigma$).

Now take $H_n = \bigcup_{j=1}^{\infty} G_{j,n}$, for each $n \in \mathbb{N}$. Hence $H_n \in \mathcal{A}_\sigma$ and $H_n \setminus E = \bigcup_{j=1}^{\infty} (G_{j,n} \setminus E_j)$, therefore we have

$$\mu^*(H_n \setminus E) = \mu^*\Big(\bigcup_{j=1}^{\infty} G_{j,n} \setminus E_j\Big) \leqslant \sum_{j=1}^{\infty} \mu^*(G_{j,n} \setminus E_j) \leqslant \sum_{j=1}^{\infty} \frac{2^{-j}}{n} = \frac{1}{n},$$

for each $n \in \mathbb{N}$.

Now take $B = \bigcap_{n=1}^{\infty} H_n$. We have $B \in \mathcal{A}_{\sigma\delta}$ and

$$\mu^*(B \setminus E) \subset \mu^*(H_n \setminus E) \leqslant \frac{1}{n},$$

for each $n \in \mathbb{N}$. Making $n \to \infty$, we obtain the result.

Exercise 19. Let μ^* be an outer measure on X induced from a finite premeasure μ_0. If $E \subset X$, define the **inner measure** of E to be $\mu_*(E) = \mu_0(X) - \mu^*(E^c)$. Then E is μ^*-measurable if and only if $\mu^*(E) = \mu_*(E)$.

Hint: Use Exercise 18.

Solution. Note that $\mu^*(E) \leqslant \mu^*(X) = \mu_0(X) < \infty$ for all $E \subset X$.

If $E \subset X$ is μ^*-measurable, then we have

$$\mu_0(X) = \mu^*(X) = \mu^*(X \cap E) + \mu^*(X \cap E^c) = \mu^*(E) + \mu^*(E^c),$$

and since μ^* is always finite, we have $\mu^*(E) = \mu_0(X) - \mu^*(E^c) = \mu_*(E)$.

Now, if $\mu_*(E) = \mu^*(E)$, we have $\mu_0(X) = \mu^*(E) + \mu^*(E^c)$. We will apply the proof of item (b) to both E and E^c to obtain $B_1, B_2 \in \mathcal{A}_{\sigma\delta}$ such that $E \subset B_1$, $E^c \subset B_2$ with $\mu^*(B_1) = \mu^*(E)$ and $\mu^*(B_2) = \mu^*(E^c)$. Thus we obtain

$$\mu_0(X) = \mu^*(E) + \mu^*(E^c) = \mu^*(B_1) + \mu^*(B_2).$$

Also, since B_2 is μ^*-measurable, we obtain

$$\mu_0(X) = \mu^*(B_2) + \mu^*(B_2^c),$$

and by the finitude of μ^* we obtain $\mu^*(B_1) = \mu^*(B_2^c)$. But $B_2^c \subset E \subset B_1$, and since B_1 and B_2 are both measurable, we obtain $\mu^*(B_1 \setminus B_2^c) = 0$.

But $B_1 \setminus E \subset B_1 \setminus B_2^c$ and hence $\mu^*(B_1 \setminus E) \leqslant \mu^*(B_1 \setminus B_2^c) = 0$. Hence, from item (b) of Exercise 18, E is μ^*-measurable.

Exercise 20. Let μ^* be an outer measure on X, $\mathcal{M}^*$ the σ-algebra of the μ^*-measurable sets, $\overline{\mu} = \mu^*|_{\mathcal{M}^*}$, and μ^+ the outer measure induce by $\overline{\mu}$ as in (2.3.1) (with $\overline{\mu}$ and $\mathcal{M}^*$ replacing μ_0 and $\mathcal{A}$).

(a) If $E \subset X$, we have $\mu^*(E) \leqslant \mu^+(E)$, with equality iff there exists $A \in \mathcal{M}^*$ with $E \subset A$ and $\mu^*(A) = \mu^*(E)$.

(b) If μ^* is induced from a premeasure, then $\mu^* = \mu^+$. (Use Exercise 18(a)).

(c) If $X = \{0, 1\}$, there exists an outer measure μ^* on X such that $\mu^* \neq \mu^+$.

Solution to (a). Let $E \subset X$ and $\epsilon > 0$. By definition of μ^+, there exists $B \in \mathcal{M}^*$ such that $E \subset B$ and $\overline{\mu}(B) \leqslant \mu^+(E) + \epsilon$. Since $B \in \mathcal{M}^*$, $\overline{\mu}(B) = \mu^*(B)$ and hence

$$\mu^*(E) \leqslant \mu^*(B) \leqslant \mu^+(E) + \epsilon,$$

and since $\epsilon > 0$ is arbitrary, we obtain $\mu^*(E) \leqslant \mu^+(E)$.

Now assume that there exists $A \in \mathcal{M}^*$ with $E \subset A$ and $\mu^*(A) = \mu^*(E)$, then $\overline{\mu}(A) = \mu^*(A) = \mu^*(E)$, and since $\mu^+(E) \leqslant \overline{\mu}(A)$, we obtain $\mu^+(E) \leqslant \mu^*(E)$, and the equality holds.

Conversely, if $\mu^*(E) = \mu^+(E)$ then given $n \in \mathbb{N}$ there exists $A_n \in \mathcal{M}^*$ such that $E \subset A_n$ and $\overline{\mu}(A) \leqslant \mu^*(E) + 1/n$. Setting $A = \bigcap_{n=1}^{\infty} A_n$ we have $A \in \mathcal{M}^*$, $E \subset A$ and

$$\mu^*(A) \leqslant \mu^+(A) \leqslant \overline{\mu}(A) \leqslant \mu^*(E) \leqslant \mu^*(A),$$

and thus $\mu^*(A) = \mu^*(E)$.

Solution to (b). Assume that μ^* is induced by a premeasure mu_0 on an algebra $\mathcal{A}$. From item (a), it suffices to show that for each $E \subset X$, there exists $B \in \mathcal{M}^*$ with $E \subset B$ and $\mu^*(B) = \mu^*(E)$.

From item (a) of Exercise 18, given $E \subset X$ and $n \in \mathbb{N}$, there exists $A_n \in \mathcal{A}_\sigma$ such that $\mu^*(E) \leqslant \mu^*(A) \leqslant \mu^*(E) + \frac{1}{n}$. Setting $B = \bigcap_{n=1}^{\infty} A_n$, we have $E \subset B$, $B \in \mathcal{M}^*$ (since $\mathcal{M}^*$ is a σ-algebra that contains $\mathcal{A}$), and we have $\mu^*(E) \leqslant \mu^*(B) \leqslant \mu^*(A_n) \leqslant \mu^*(E) + \frac{1}{n}$ for all $n \in \mathbb{N}$. Taking $n \to \infty$ concludes the result.

Solution to (c). Let μ^* defined in $\mathcal{P}(X)$ by $\mu^*(\varnothing) = 0$, $\mu^*(\{0\}) = \mu^*(\{1\}) = 1$ and

$\mu^*(X) = \frac{3}{2}$. Then μ^* is an outer measure on X. Now since

$$\mu^*(X) = \frac{3}{2} < 2 = \mu^*(\{0\}) + \mu^*(\{1\}),$$

we see that neither $\{0\}$ nor $\{1\}$ are μ^*-measurable sets and hence $\mathcal{M}^* = \{\varnothing, X\}$, $\overline{\mu}(\varnothing) = 0$ and $\overline{\mu}(X) = \frac{3}{2}$.

But then $\mu^+(\{0\}) = \mu^+(\{1\}) = \frac{3}{2}$, and therefore $\mu^* \neq \mu^+$.

Exercise 21. Let μ^* be an outer measure induced from a premeasure and $\overline{\mu}$ the restriction of μ^* to the μ^*-measurable sets. Then $\overline{\mu}$ is saturated.

Hint: Use Exercise 18.

Solution. Let $\mathcal{M}^*$ be the σ-algebra of the μ^*-measurable sets. We need to prove that a locally measurable set is also measurable. That is, if $E \cap A \in \mathcal{M}^*$ for all $A \in \mathcal{M}^*$ with $\overline{\mu}(A) < \infty$, then $E \in \mathcal{M}^*$.

Now let $F \subset X$ with $\mu^*(F) < \infty$ and $\epsilon > 0$. From Exercise 18 item (a), there exists $A \in \mathcal{A}_\sigma$ with $F \subset A$ and $\overline{\mu}(A) = \mu^*(A) \leqslant \mu^*(F) + \epsilon < \infty$. Hence, since E is locally measurable, $E \cap A \in \mathcal{M}^*$ and hence

$$\begin{aligned}\mu^*(A) &= \mu^*(A \cap (E \cap A)) + \mu^*(A \cap (E \cap A)^c) = \mu^*(E \cap A) + \mu^*(A \cap E^c) \\ &\geqslant \mu^*(F \cap E) + \mu^*(F \cap E^c),\end{aligned}$$

and thus

$$\mu^*(F) + \epsilon \geqslant \mu^*(A) \geqslant \mu^*(F \cap E) + \mu^*(F \cap E^c),$$

and since $\epsilon > 0$ is arbitrary, we obtain $\mu^*(F) \geqslant \mu^*(F \cap E) + \mu^*(F \cap E^c)$, which proves that E is μ^*-measurable and therefore $\overline{\mu}$ is saturated.

Exercise 22. Let $(X, \mathcal{M}, \mu)$ be a measure space, μ^* the outer measure induced by μ according to (2.3.1), $\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets, and $\overline{\mu} = \mu^*|_{\mathcal{M}^*}$.

(a) If μ is σ-finite, then $\overline{\mu}$ is the completion of μ (use Exercise 18).

(b) In general, $\overline{\mu}$ is the saturation of the completion of μ (see Exercises 16 and 21).

Solution to (a). Let $F \subset N$ where $N \in \mathcal{N}$ (see Theorem 2.1.9 for the notation). Since $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}^*$, by item (b) of Exercise 18, there exists $B \in \mathcal{M}$ with $N \subset B$ and $\mu^*(B \setminus N) = 0$. Therefore we have

$$\mu^*(B \setminus F) \leqslant \mu^*(B \setminus N) + \mu^*(B \cap (N \setminus F)) = 0,$$

since $\mu^*(B \setminus N) = 0$ and $\mu^*(B \cap (N \setminus F)) \leqslant \mu^*(N) = 0$. Thus by Exercise 18 (b), F is μ^*-measurable. This implies that $\overline{\mathcal{M}} \subset \mathcal{M}^*$ (since $\overline{\mathcal{M}} \subset \mathcal{M}^*$).

Conversely, if $E \in \mathcal{M}^*$, then also $E^c \in \mathcal{M}^*$ and from item (c) of Exercise 18, there exist $B_1, B_2 \in \mathcal{M}$ with $E \subset B_1$, $E^c \subset B_2$ such that $\mu^*(B_1 \setminus E) = \mu^*(B_2 \setminus E^c) = 0$. Now $B_2^c \subset E \subset B_1$ and

$$E = \underbrace{B_2^c}_{\in \mathcal{M}} \cup \underbrace{E \setminus B_2^c}_{\subset B_1 \setminus B_2^c}.$$

Since $B_1 \setminus B_2^c = (B_1 \setminus E) \cup (E \setminus B_2^c)$, we have

$$\mu^*(B_1 \setminus B_2^c) \leqslant \mu^*(B_1 \setminus E) + \underbrace{\mu^*(E \setminus B_2^c)}_{=\mu^*(B_2 \setminus E^c)} = 0,$$

and $B_1 \setminus B_2^c \in \mathcal{N}$. Thus $\mathcal{M}^* \subset \overline{\mathcal{M}}$, and proves the equality.

It remains to prove that $\overline{\mu}$ coincides with the completion of μ in $\mathcal{M}$, which we will call ν.

Let $E \in \mathcal{M}^*$, then $E \in \overline{\mathcal{M}}$ and thus $E = A \cup B$, where $A \in \mathcal{M}$ and $B \subset N$, where $N \in \mathcal{N}$, and without loss of generality, we can assume that $A \cap B = \varnothing$ (otherwise we could write $E = (A \setminus N) \cup (B \cup (A \cap N))$). By the previous computations we know that $B \in \mathcal{M}^*$ and thus

$$\overline{\mu}(E) = \mu^*(E) = \mu^*(A) + \mu^*(B) = \mu^*(A) = \mu(A) = \nu(E).$$

Thus $\overline{\mu} = \nu$ on $\mathcal{M}^*$, and concludes the result.

Solution to (b). We know that by Exercise 21, $\overline{\mu}$ is a saturated measure on $\mathcal{M}^*$. Now let $\overline{\mathcal{M}}$ be the completion of $\mathcal{M}$ and, as before, ν be the completion of μ. Let E be a locally measurable set for ν, we will show that $E \in \mathcal{M}^*$.

Let $F \subset X$ with $\mu^*(F) < \infty$ and $\epsilon > 0$. From Exercise 18 item (a), there exists $A \in \mathcal{M}$ with $F \subset A$ and $\mu(A) = \mu^*(A) \leqslant \mu^*(F) + \epsilon < \infty$. Since $A \in \mathcal{M} \subset \overline{\mathcal{M}}$, $\nu(A) = \mu(A) < \infty$ and E is locally measurable for ν, we have $E \cap A \in \overline{\mathcal{M}}$, and hence $E \cap A = B \cup C$, where $B \cap C = \varnothing$, $B \in \mathcal{M}$ and $C \subset N \in \mathcal{N}$.

We have

$$\mu^*(F \cap E) \overset{F \subseteq A}{=} \mu^*(F \cap (E \cap A)) \leqslant \mu^*(F \cap B) + \underbrace{\mu^*(F \cap C)}_{=0} = \mu^*(F \cap B)$$

and

$$\mu^*(F \cap E^c) = \mu^*(F \setminus E) = \mu^*(F \setminus (E \cap A)) \overset{B \subset E \cap A}{\leqslant} \mu^*(F \setminus B) = \mu^*(F \cap B^c),$$

Thus, using the fact that B is μ^* measurable and the two previous inequalities, we have

$$\mu^*(F) \geqslant \mu^*(F \cap B) + \mu^*(F \cap B^c) \geqslant \mu^*(F \cap E) + \mu^*(F \cap E^c),$$

which shows that $E \in \mathcal{M}^*$. Hence $\widetilde{\overline{\mathcal{M}}} \subset \mathcal{M}^*$.

Now let $E \in \mathcal{M}^*$. We want to show that E is locally measurable for ν. To that end, we will prove some claims first.

<u>Claim 1</u>: If $F \subset X$ is such that $\mu^*(F) = 0$, then there exists $N \in \mathcal{N}$ with $F \subset N$.

In fact, if $F \subset X$, then given $n \in \mathbb{N}$, using item (a) of Exercise 18, we have $N_n \in \mathbb{N}$ with $F \subset N_n$ and $\mu(N_n) \leqslant \frac{1}{n}$. Then the set $N = \bigcap\limits_{n=1}^{\infty} N_n$ satisfies the required conditions.

<u>Claim 2</u>: If $E \in \mathcal{M}^*$ is such that $\mu^*(E) < \infty$, then $E \in \overline{\mathcal{M}}$.

In fact, using item (b) of Exercise 18, there exists $B \in \mathcal{M}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$. But $B \setminus E \in \mathcal{M}^*$, and using again item (b) of Exercise 18, there exists $C \in \mathcal{M}$ with $B \setminus E \subset C$ and $\mu^*(C \setminus (B \setminus E)) = 0$.

Let $D = B \cap C$. Since $B \setminus E \subset C$, we have $B \setminus E \subset D$ and hence $B \setminus D \subset E$. We can write $E = (B \setminus D) \cup (E \setminus (B \setminus D))$, and $B \setminus D \in \mathcal{M}$, since B, C and D are in $\mathcal{M}$.

Now note that

$$E \setminus (B \setminus D) = E \cap (B \cap D^c)^c = E \cap (B^c \cup D) = E \cap D \subset E \cap C \subset C \setminus (B \setminus E),$$

and therefore $\mu^*(E \setminus (B \setminus D)) = 0$. By Claim 1, there exists $N \in \mathcal{N}$ with $E \setminus (B \setminus D) \subset N$, and concludes the proof of Claim 2.

<u>Claim 3</u>: If $A \in \overline{\mathcal{M}}$ then $A \in \mathcal{M}^*$ and $\mu^*(A) \leqslant \nu(A)$.

In fact, we proved in item (a) that $A \in \mathcal{M}$. We can write $A = G \cup H$ with $G \cap H = \varnothing$, $G \in \mathcal{M}$ and $H \subset N \in \mathcal{N}$. Hence

$$\mu^*(A) \leqslant \mu^*(G) + \mu^*(H) = \mu^*(G) = \mu(G) = \nu(A),$$

and concludes the proof of this claim.

Now we can prove that is $E \in \mathcal{M}^*$, then E is locally measurable for ν. To this end, let $A \in \overline{\mathcal{M}}$ with $\nu(A) < \infty$. By Claim 3, $\mu^*(A) \leqslant \nu(A) < \infty$ and $A \in \mathcal{M}^*$. Thus $E \cap A \in \mathcal{M}^*$ and $\mu^*(E \cap A) \leqslant \mu^*(A) < \infty$, we by Claim 2, we obtain that $E \cap A \in \overline{\mathcal{M}}$, which shows the local measurability of E.

Therefore $\mathcal{M}^* \subset \widetilde{\overline{\mathcal{M}}}$, and hence $\mathcal{M}^* = \widetilde{\overline{\mathcal{M}}}$.

Now we have to prove that $\overline{\mu} = \tilde{\nu}$ in $\mathcal{M}^*$. But if $E \in \mathcal{M}^*$ then we have:

<u>Case 1:</u> If $E \in \overline{\mathcal{M}}$, we have $E = A \cup B$ with $A \cap B = \varnothing$, and $A \in \mathcal{M}$ and $B \subset N \in \mathcal{N}$.

Thus
$$\tilde{\nu}(E) = \nu(E) = \mu(A) = \overline{\mu}(A) \leqslant \overline{\mu}(E),$$
and
$$\overline{\mu}(E) = \mu^*(E) = \mu^*(A \cup B) \leqslant \mu^*(A) = \mu(A) = \nu(E) = \tilde{\nu}(E),$$
and proves that $\overline{\mu}(E) = \tilde{\nu}(E)$.

Case 2: If $E \in \mathcal{M}^* \setminus \overline{M}$, then by Claim 2 we must have $\mu^*(E) = \infty$. Hence
$$\overline{\mu}(E) = \mu^*(E) = \infty = \tilde{\nu}(E),$$
since by definition $\tilde{\nu}(E) = \infty$ for $E \in \mathcal{M}^* \setminus \overline{\mathcal{M}}$.

Exercise 23. Let $\mathcal{A}$ be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$ where $-\infty \leqslant a \leqslant b \leqslant \infty$.

(a) $\mathcal{A}$ is an algebra on $\mathbb{Q}$ (Use Theorem 1.4.6).

(b) The σ-algebra generated by $\mathcal{A}$ is $\mathcal{P}(\mathcal{Q})$.

(c) Define μ_0 on $\mathcal{A}$ by $\mu_0(\varnothing) = 0$ and $\mu_0(A) = \infty$ for $A \neq \varnothing$. Then μ_0 is a premeasure on $\mathcal{A}$, and there is more then one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to $\mathcal{A}$ is μ_0.

Solution to (a). Let $\mathcal{E}$ be the collection of the sets of the form $(a, b] \cap \mathbb{Q}$, where $-\infty \leqslant a \leqslant b \leqslant \infty$, regarded as subsets of $\mathbb{Q}$.

Then taking $a = b = 0$, we have $\varnothing = (0, 0] \cap \mathbb{Q} \in \mathcal{E}$. If $A = (a, b] \cap \mathbb{Q}$ and $B = (c, d] \cap \mathbb{Q}$, then let $r = \max\{a, c\}$ and $s = \min\{b, d\}$. We have $A \cap B = \varnothing \in \mathcal{E}$ if $r > s$ and $A \cap B = (r, s] \cap \mathbb{Q} \in \mathcal{E}$ if $r \leqslant s$.

Now let $E \in \mathcal{E}$. If $E = \varnothing$, then $E^c = \mathbb{Q} = (-\infty, \infty] \cap \mathbb{Q} \in \mathcal{E}$. If $E \neq \varnothing$, then $E = (a, b] \cap \mathbb{Q}$ and hence $E^c = [(-\infty, a] \cap \mathbb{Q}] \cup [(b, \infty] \cap \mathbb{Q}]$, which is a finite union of disjoint elements of $\mathcal{E}$.

Therefore $\mathcal{E}$ is an elementary family, and the collection of finite disjoint unions of elements of $\mathcal{E}$, namely $\tilde{\mathcal{A}}$ is an algebra, by Theorem 1.4.6. It only remains to see that $\mathcal{A} = \tilde{\mathcal{A}}$. Clearly $\tilde{\mathcal{A}} \subset \mathcal{A}$. For the reverse inclusion, let $E_1 = (a_1, b_1] \cap \mathbb{Q}$ and $E_2 = (a_2, b_2] \cap \mathbb{Q}$ be two elements of $\mathcal{E}$. We can write $(a_1, b_1] \cup (a_2, b_2]$ as follows:

Case 1: $a_1 \leqslant a_2 \leqslant b_2 \leqslant b_1$.

In this case $(a_1, b_1] \cup (a_2, b_2] = (a_1, b_1]$.

Case 2: $a_1 \leqslant a_2 \leqslant b_1 \leqslant b_2$.

In this case $(a_1, b_1] \cup (a_2, b_2] = (a_1, a_2] \cup (a_2, b_1] \cup (b_1, b_2]$.

Case 3: $b_1 = a_2$.

In this case $(a_1, b_1] \cup (a_2, b_2] = (a_1, b_2]$.

<u>Case 4:</u> $b_1 < a_2$.

In this case we do nothing.

We have four more cases, replacing a_1 by a_2, a_2 by a_1, b_1 by b_2 and b_2 by b_1. But independently of which case we are, we can write $E_1 \cup E_2$ as a finite *disjoint* union of elements os $\mathcal{E}$, and hence $E \in \tilde{\mathcal{A}}$, which concludes the proof.

Solution to (b). Note that if $r \in \mathbb{Q}$ then

$$\{r\} = \bigcap_{n=1}^{\infty} (r - \tfrac{1}{n}, r] \cap \mathbb{Q},$$

hence the σ-algebra generated by $\mathcal{A}$ contains all singletons (unitary sets). If $Q \subset \mathbb{Q}$, Q is countable and hence Q can be written as the countable unions of its points, which show us that Q is in the σ-algebra generated by $\mathcal{A}$. Since Q is an arbitrary subset of $\mathbb{Q}$, we obtain that the σ-algebra generated by $\mathbb{Q}$ is $\mathcal{P}(\mathbb{Q})$.

Solution to (c). Clearly μ_0 is a premeasure on $\mathcal{A}$. Is associated outer measure is

$$\mu^*(E) = \begin{cases} \infty, & \text{if } E \neq \varnothing, \\ 0, & \text{if } E = \varnothing. \end{cases}$$

Thus the σ-algebra of the μ^* measurable sets is $\mathcal{P}(\mathcal{Q})$ and hence μ^* is a measure on $\mathcal{P}(Q)$ which extends μ_0.

Consider the counting measure ν on $\mathcal{P}(\mathbb{Q})$. Then if $E \in \mathcal{E}$ and $E \neq \varnothing$ then

$$\nu(E) = \infty = \mu_0(E),$$

hence $\nu = \mu_0$ on $\mathcal{A}$, and hence $\nu = \mu_0$ on $\mathcal{A}$, but $\nu \neq \mu^*$, since $\nu(\{1\}) = 1 \neq \infty = \mu^*(\{1\})$.

Exercise 24. Let μ be a finite measure on $(X, \mathcal{M})$ and μ^* be the outer measure induced by μ. Suppose that $E \subset X$ satisfies $\mu^*(E) = \mu^*(X)$ (but not that $E \in \mathcal{M}$).

(a) If $A, B \in \mathcal{M}$ and $A \cap E = B \cap E$ then $\mu(A) = \mu(B)$.

(b) Let $\mathcal{M}_E = \{A \cap E \colon A \in \mathcal{M}\}$, and define the function ν on $\mathcal{M}_E$ by $\nu(A \cap E) = \mu(A)$ (which makes sense by (a)). Them $\mathcal{M}_E$ is a σ-algebra on E and ν is a measure on $\mathcal{M}_E$.

Solution of (a). We have, since $B \in \mathcal{M}$, that

$$\mu(B) + \mu(B^c) = \mu(X) = \mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c),$$

and thus $\mu^*(E \cap B) = \mu(B) + \mu(B^c) - \mu^*(E \cap B^c)$, and since $\mu^*(E \cap B^c) \leqslant \mu^*(B^c) = \mu(B^c)$, we have $\mu^*(E \cap B) \geqslant \mu(B)$ and therefore, since $A \cap E \subset A$ we have

$$\mu(A) \geqslant \mu^*(A \cap E) = \mu^*(B \cap E) \geqslant \mu(B).$$

Interchanging A and B we obtain $\mu(A) \leqslant \mu(B)$, and thus we have the equality.

Solution to (b). We prove that $\mathcal{M}_E$ is a σ-algebra on E. Clearly $\varnothing \in \mathcal{M}_E$, since $\varnothing \in \mathcal{M}$. Now let $\{A_j \cap E\}$ be a sequence of elements in $\mathcal{M}_E$, that is $\{A_j\}$ is a sequence in $\mathcal{M}$. Thus

$$\bigcup_{j=1}^{\infty}(A_j \cap E) = \Big(\bigcup_{j=1}^{\infty} A_j\Big) \cap E \in \mathcal{M}_E,$$

since $\bigcup\limits_{j=1}^{\infty} A_j \in \mathcal{M}$.

If $A \in \mathcal{M}$, then $E \setminus (A \cap E) = A^c \cap E$, hence $\mathcal{M}_E$ is closed under complements in E. Thus $\mathcal{M}_E$ is a σ-algebra on E.

Now we have to prove that ν is a measure on $\mathcal{M}_E$. Clearly $\nu(\varnothing) = 0$. Now let $\{A_j \cap E\}$ be a pairwise disjoint sequence on $\mathcal{M}_E$ and consider the sequence $\{A_j\}$ in $\mathcal{M}$.

<u>Claim:</u> There exists a pairwise disjoint sequence $\{B_j\}$ on $\mathcal{M}$ such that $\mu(A_j) = \mu(B_j)$ for all j.

In fact, if $\{A_j\}$ is already a pairwise disjoint sequence on $\mathcal{M}$, we have nothing to do. If this is note the case, then we consider $B_1 = A_1$ and $B_j = A_j \setminus (\bigcup_{k=1}^{j-1} A_k)$. Then the sequence $\{B_j\}$ is a pairwise disjoint sequence in $\mathcal{M}$. Now we show that $B_j \cap E = A_j \cap E$. In fact, since $B_j \subset A_j$ we have $B_j \cap E \subset A_j \cap E$. Now if $x \in A_j \cap E$ then $x \in A_j$ and $x \in E$. But since $\{A_j \cap E\}$ is pairwise disjoint, we have $x \notin E \cap A_i$ for all $i < j$, and since $x \in E$ this implies that $x \notin A_i$ for $i < j$, this means that $x \in B_j$ and hence $A_j \cap E = B_j \cap E$. From item (a), $\mu(A_j) = \mu(B_j)$ and completes the proof of the claim.

To conclude, we have

$$\begin{aligned}\nu\Big(\bigcup_{j=1}^{\infty}(A_j \cap E)\Big) &= \nu\Big(\Big(\bigcup_{j=1}^{\infty} A_j\Big) \cap E\Big) = \mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \mu\Big(\bigcup_{j=1}^{\infty} B_j\Big) \\ &= \sum_{j=1}^{\infty} \mu(B_j) = \sum_{j=1}^{\infty} \mu(A_j) = \sum_{j=1}^{\infty} \nu(A_j \cap E),\end{aligned}$$

and hence ν is a measure on $\mathcal{M}_E$.

2.5| Borel measures on the real line

Definition 2.5.1. *A measure μ on a topological space (X, τ) is called a* **Borel measure** *on X if its σ-algebra of definition is the Borel σ-algebra $\mathcal{B}_X$.*

In this section, we will see how to construct Borel measures on $\mathbb{R}$. Let us see first some motivation: consider a finite Borel measure μ on $\mathbb{R}$ and define $F \colon \mathbb{R} \to \mathbb{R}$ given by

$$F(x) = \mu((-\infty, x]) \quad \text{for } x \in \mathbb{R}.$$

Such function F is called **distribution function** of μ. This F is increasing and right continuous, since $(-\infty, x] = \bigcap_{j=1}^{\infty}(-\infty, x_j]$ for a sequence $x_j \to x^+$. Moreover, if $a < b$ then $(-\infty, b] = (-\infty, a] \cup (a, b]$ and thus $\mu((a, b]) = F(b) - F(a)$.

The process we will present here does the opposite direction: from an increasing and right-continuous function, we will construct a Borel measure μ on $\mathbb{R}$. The particular case $F(x) = x$ will lead us to the usual definition of "lenght".

Definition 2.5.2. *Sets of the form $(a, b]$, (a, ∞) or $\varnothing$, for $-\infty \leqslant a < b < \infty$ will be called* **h-intervals**.

Proposition 2.5.3. *We have:*

(i) *the intersection of two h-intervals is an h-interval;*

(ii) *the complement of an h-interval is either an h-interval or the disjoint union of two h-intervals.*

Proof. Follow the steps of the proof of Exercise 23, item (a). ■

By Theorem 1.4.6, the collection $\mathcal{A}$ of finite disjoint unions of h-intervals is an algebra, and using Proposition 1.3.3, the σ-algebra generated by $\mathcal{A}$ is $\mathcal{B}_{\mathbb{R}}$.

Now we will construct a premeasure on the algebra $\mathcal{A}$.

Definition 2.5.4. *let $F \colon \mathbb{R} \to \mathbb{R}$ be an increasing and right continuous function. Set $F(\infty) = \lim\limits_{x \to \infty} F(x)$ and $F(-\infty) = \lim\limits_{x \to -\infty} F(x)$ (both exist since F is increasing). We define:*

1. *$\mu_0((a, b]) = F(b) - F(a)$, if $-\infty \leqslant a \leqslant b < \infty$, and $\mu_0((a, \infty)) = F(\infty) - F(a)$ if $-\infty \leqslant a$,*

2. *if $\{I_j\}_{j=1}^n$ is finite pairwise disjoint sequence of h-intervals, define*

$$\mu_0\Big(\bigcup_{j=1}^n I_j\Big) = \sum_{j=1}^n \mu_0(I_j).$$

We have two remarks to make before continuing. First, note that taking $a = b$ in (1) we obtain $\mu_0(\varnothing) = 0$. Also the difference $F(\infty) - F(-\infty)$ is always well defined, since $F(\infty)$ and $F(-\infty)$ cannot be simultaneously ∞ or $-\infty$ (recall that F is increasing).

Lemma 2.5.5. *The function μ_0 defined above is well defined in $\mathcal{A}$, that is, if $E = \bigcup_{j=1}^n I_j = \bigcup_{k=1}^m F_k$ with $\{I_j\}_{j=1}^n$ and $\{F_k\}_{k=1}^m$ are two finite disjoint sequences of h-intervals then*

$$\sum_{j=1}^n \mu_0(I_j) = \sum_{k=1}^m \mu_0(F_k).$$

Proof. We will prove first that if I is an h-interval with $I = \bigcup_{p=1}^r J_p$, where $\{J_p\}_{p=1}^r$ is a finite disjoint sequence of h-intervals, then $\mu_0(I) = \sum_{p=1}^r \mu_0(J_p)$. To that end, we have two cases to consider:

<u>Case 1:</u> $I = (a, b]$ with $-\infty \leqslant a < b < \infty$. In this case, each J_p must be of the form $J_p = (a_p, b_p]$ with $-\infty \leqslant a_p < b_p < \infty$. We can reorder the index p, if necessary, to obtain $a = a_1 < b_1 = a_2 < b_2 < \cdots < b_{r-1} = a_r < b_r = b$, and we have

$$\sum_{p=1}^r \mu_0(J_p) = \sum_{p=1}^r (F(b_p) - F(a_p)) = F(b_r) - F(a_1) = F(b) - F(a) = \mu_0(I).$$

<u>Case 2:</u> $I = (a, \infty)$ with $-\infty \leqslant a$. In this case, exactly one of the J_p's must be (a_p, ∞), and all the others are of the form $J_p = (a_p, b_p]$ with $-\infty \leqslant a_p < b_p < \infty$. We can reorder the index, if necessary, to obtain $a_1 = a < b_1 = a_2 < \cdots < b_{r-1} = a_r$, and $J_r = (a_r, \infty)$. Thus

$$\begin{aligned}\sum_{p=1}^\infty \mu_0(J_p) &= F(\infty) - F(a_r) + \sum_{p=1}^{r-1} (F(b_p) - F(a_p)) = F(\infty) - F(a_1) \\ &= F(\infty) - F(a) = \mu_0(I).\end{aligned}$$

With this result, consider the general case stated in the lemma. Using Proposition 2.5.3, item (i), for each $j = 1, \cdots, n$ and $k = 1, \cdots, m$, the set $I_j \cap F_k$ is an h-interval. Moreover, for each j we have $I_j = \bigcup_{k=1}^m (I_j \cap F_k)$, and for each k we have $F_k = \bigcup_{j=1}^n (I_j \cap F_k)$, hence from what we proved

$$\sum_{j=1}^{n} \mu_0(I_j) = \sum_{j=1}^{n}\sum_{k=1}^{m} \mu_0(I_j \cap F_k) = \sum_{k=1}^{m}\sum_{j=1}^{n} \mu_0(I_j \cap F_k) = \sum_{k=1}^{m} \mu_0(F_k),$$

which concludes the proof. ■

Proposition 2.5.6. *The function μ_0 defined in Definition 2.5.4 is a premeasure on $\mathcal{A}$.*

Proof. We have to prove that if $\{I_j\}$ is a pairwise sequence in $\mathcal{A}$ with $\bigcup_{j=1}^{\infty} I_j \in \mathcal{A}$ then $\mu_0(\bigcup_{j=1}^{\infty} I_j) = \sum_{j=1}^{\infty} \mu_0(I_j)$. Since each I_j is a finite union of h-intervals (which we can assume to be pairwise disjoint), we can assume, after relabelling the sequence, that each I_j is a single h-interval. Also, since their union is in $\mathcal{A}$, it consists of a finite union of pairwise disjoint h-intervals, and we can partition each I_j in a finite number of h-interval such that the union of the intervals in each subsequence of this partition is a single h-interval. Since μ_0 is finitely additive, we may assume that $\bigcup_{j=1}^{\infty} I_j$ is a single h-interval I. In this case, we have

$$\mu_0(I) = \mu_0\Big(\bigcup_{j=1}^{n} I_j\Big) + \mu_0\Big(I \setminus \bigcup_{j=1}^{n} I_j\Big) \geqslant \mu_0\Big(\bigcup_{j=1}^{n} I_j\Big).$$

Now we prove the reverse inequality, and we will brake it into some cases:

<u>Case 1</u>: $I = (a, b]$ with $-\infty < a < b < \infty$.

In this case consider $\epsilon > 0$. Since F is right continuous, there exists $\delta > 0$ such that $F(a+\delta) - F(a) < \epsilon$. Also, if $I_j = (a_j, b_j]$, for each j there exists $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) < \epsilon 2^{-j}$. The open intervals $(a_j, b_j + \delta_j)$ cover the compact set $[a+\delta, b]$, and we can extract a finite subcover. If we discard any interval in this finite subcover which is contained inside a larger interval and (possibly) relabelling the index j, we can assume that:

(i) the intervals $(a_1, b_1 + \delta_1), \cdots, (a_N, b_N + \delta_N)$ cover $[a+\delta, b]$,

(ii) $b_j + \delta_j \in (a_{j+1}, b_{j+1} + \delta_{j+1})$ for $j = 1, \cdots, N-1$.

Then, we have:

$$
\begin{aligned}
\mu_0(I) = F(b) - F(a) < F(b) - F(a+\delta) + \epsilon &\leqslant F(b_N + \delta_N) - F(a_1) + \epsilon \\
&= F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} [F(a_{j+1}) - F(a_j)] + \epsilon \\
&\leqslant F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} [F(b_j + \delta_j) - F(a_j)] + \epsilon \\
&< \sum_{j=1}^{N} [F(b_j) + \epsilon 2^{-j} - F(a_j)] + \epsilon \\
&\leqslant \sum_{j=1}^{\infty} \mu_0(I_j) + 2\epsilon,
\end{aligned}
$$

and since $\epsilon > 0$ is arbitrary, this conclude the proof for this case.

Case 2: $I = (-\infty, b]$, with $-\infty < b < \infty$.

Using the same notations as in Case 1, given $M > 0$, there is a finite subcover of $[-M, b]$, satisfying (i) (with $[-M, b]$ instead of $[a+\delta, b]$) and (ii). Then

$$
F(b) - F(-M) \leqslant F(b_N + \delta_N) - F(a_1) \leqslant \sum_{j=1}^{\infty} \mu_0(I_j) + \epsilon.
$$

Since $\epsilon > 0$ is arbitrary, we have $F(b) - F(-M) \leqslant \sum_{j=1}^{\infty} \mu_0(I_j)$, and the result follows by making $M \to \infty$.

Case 3: $I = (a, \infty)$ with $-\infty < a < \infty$.

Using the same argument as in Case 2, we obtain $F(M) - F(a) \leqslant \sum_{j=1}^{\infty} \mu_0(I_j)$, and the result again follows by making $M \to \infty$.

Case 4: $I = \mathbb{R}$.

In this case, we can find a finite subcover of $[-M, N]$ for any given $M, N > 0$, and we obtain $F(N) - F(-M) \leqslant \sum_{j=1}^{\infty} \mu_0(I_j)$, and the result is prove making $M, N \to \infty$ (in any order). ∎

With this premeasure we can construct a Borel measure, and we have our following result.

Theorem 2.5.7. *Let $F\colon \mathbb{R} \to \mathbb{R}$ be an increasing and right continuous function. Then there exists a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a, b]) = F(a) - F(b)$ for all $a, b \in \mathbb{R}$. If G is another such function, then we have $\mu_F = \mu_G$ if and only if $F - G$ is constant. Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets,* *and we define*

$$F(x) = \begin{cases} \mu((0,x]) & \text{if } x > 0; \\ 0 & \text{if } x = 0; \\ -\mu((x,0]) & \text{if } x < 0, \end{cases}$$

then F is increasing and right continuous, and $\mu = \mu_F$.

Proof. Using Proposition 2.5.6, we know that μ_0 given in Definition 2.5.4 is a premeasure on $\mathcal{A}$. Moreover, since $\mathbb{R} = \bigcup\limits_{j\in\mathbb{Z}} (j, j+1]$ and $\mu_0((j,j+1]) = F(j+1) - F(j) < \infty$ for all j, the premeasure μ_0 is σ-finite on $\mathbb{R}$. Hence, by Theorem 2.3.5, items (i) and (iii), there exists a unique extension μ_F of μ_0 to the σ-algebra generated by $\mathcal{A}$, which is $\mathcal{B}_{\mathbb{R}}$.

Now set $k = F(0) - G(0)$. If $x \geqslant 0$, then

$$F(x) - F(0) = \mu_F((0,x]) = \mu_G((0,x]) = G(x) - G(0),$$

and hence $F(x) - G(x) = k$. Now if $x < 0$ then

$$F(0) - F(x) = \mu_F((x,0]) = \mu_G((x,0]) = G(0) - G(x),$$

and again we obtain $F(x) - G(x) = k$. Hence $F - G$ is constant. For the converse, if $F - G$ is constant then μ_F and μ_G both coincide with μ_0 on $\mathcal{A}$ and hence, by the uniqueness of the extension, $\mu_F = \mu_G$.

For the last claim, if $0 \leqslant x \leqslant y$ then $(0,x] \subset (0,y]$ and $F(x) = \mu((0,x]) \leqslant \mu((0,y]) = F(y)$, by the monotonicity property of μ (see Theorem 2.1.3, item (a)). Now if $x \leqslant y \leqslant 0$ then $(y,0] \subset (x,0]$, and again by monotonicity we have $F(x) = -\mu((x,0]) \leqslant -\mu((y,0]) = F(y)$. For $x \leqslant 0 \leqslant y$, we have $F(x) \leqslant 0 \leqslant F(y)$, and therefore F is increasing.

Now let $x \geqslant 0$ and $x_n \searrow x$ as $n \to \infty$. Then $(0, x_{n+1}] \subset (0, x_n]$ for all n, $(0,x] = \bigcap\limits_{n=1}^{\infty} (0, x_n]$ and $\mu((0,x_1]) = F(x_1) < \infty$. By the continuity from above (see Theorem 2.1.8, item (b)), we have

$$F(x) = \mu((0,x]) = \mu\Big(\bigcap_{n=1}^{\infty} (0,x_n]\Big) = \lim_{n\to\infty} \mu((0,x_n]) = \lim_{n\to\infty} F(x_n),$$

and proves that F is right continuous at x. For $x < 0$ the proof is analogous, and F is right continuous.

Now if $a, b \in \mathbb{R}$ and $0 \leqslant a \leqslant b$, then $(0,b] = (0,a] \cup (a,b]$ and $\mu((a,b]) = \mu((0,b]) - \mu((0,a]) = F(b) - F(a)$. If $a \leqslant b \leqslant 0$ then $(a,0] = (a,b] \cup (b,0]$ and $\mu((a,b]) = \mu((a,0]) - \mu((b,0]) = -F(a) + F(b) = F(b) - F(a)$. Now if $a \leqslant 0 \leqslant b$, then $(a,b] = (a,0] \cup (0,b]$ and again $\mu((a,b]) = F(b) - F(a)$. Hence, it is clear that μ coincides with μ_0 in $\mathcal{A}$, and by the uniqueness of the extension, $\mu = \mu_F$. ■

It is worth to remark that this theory could be made with h-intervals of the form $[a, b)$ and left continuous functions. Also, if μ is a finite Borel measure on $\mathbb{R}$, then $\mu = \mu_F$ where $F(x) = \mu((-\infty, x])$ is the cumulative distribution of the measure μ, and this function differs from Theorem 2.5.7 by the constant $\mu((-\infty, 0])$.

The theory we developed before gives, for an increasing and right continuous function F, not only a Borel measure on $\mathbb{R}$, but a complete measure $\overline{\mu}_F$ on a σ-algebra that contains $\mathcal{B}_{\mathbb{R}}$ (see Theorem 2.1.9). We will see that $\overline{\mu}_F$ is just the completion of μ_F and its domain is always strictly larger than $\mathcal{B}_{\mathbb{R}}$. To this complete measure, which we again denote by μ_F, we give the name of **Lebesgue-Stieltjes measure** associated to F.

We will, from now on, explore further regularity properties of Lebesgue-Stieltjes measures. To this end, we will fix a complete Lebesgue-Stieltjes measure μ on $\mathbb{R}$ associated to the increasing and right continuous function F, and we denote by $\mathcal{M}_\mu$ the σ-algebra which is the domain of μ (which contains $\mathcal{B}_{\mathbb{R}}$). We know, from Theorem 2.3.5, that for each $E \in \mathcal{M}_\mu$ we have

$$\begin{aligned}\mu(E) &= \inf\left\{\sum_{j=1}^{\infty}[F(b_j) - F(a_j)] \colon E \subset \bigcup_{j=1}^{\infty}(a_j, b_j]\right\}\\ &= \inf\left\{\sum_{j=1}^{\infty}\mu((a_j, b_j]) \colon E \subset \bigcup_{j=1}^{\infty}(a_j, b_j]\right\},\end{aligned}$$

just noting that a set of the form (a, ∞) can be written as $(a, \infty) = \bigcup\limits_{n=1}^{\infty}(a+n-1, a+n]$, and we already know that $\mu_0((a, \infty)) = \sum\limits_{n=1}^{\infty}\mu_0((a+n-1, a+n])$, since μ_0 is a premeasure on $\mathcal{A}$.

First, we will see that we can compute the measure of E using open intervals, instead of h-intervals.

Lemma 2.5.8. *For any $E \in \mathcal{M}_\mu$, we have*

$$\mu(E) = \inf\left\{\sum_{j=1}^{\infty}\mu((a_j, b_j)) \colon E \subset \bigcup_{j=1}^{\infty}(a_j, b_j)\right\}.$$

Proof. We define

$$\nu(E) = \inf\left\{\sum_{j=1}^{\infty}\mu((a_j, b_j)) \colon E \subset \bigcup_{j=1}^{\infty}(a_j, b_j)\right\}.$$

First assume that $E \subset \bigcup_{j=1}^{\infty}(a_j, b_j)$. We can assume, without loss of generality that $b_j > a_j$ (otherwise (a_j, b_j) is empty and we can discard it). Consider an strictly increasing sequence $\{c_{j,k}\}_k$, with $c_{j,1} = a_j$ and $c_{j,k} \to b_j$ as $k \to \infty$, and define $I_{j,k} = (c_{j,k}, c_{j,k+1}]$. Hence

$(a_j, b_j) = \bigcup_{k=1}^{\infty} I_{j,k}$ and $E \subset \bigcup_{j,k=1}^{\infty} I_{j,k}$. Thus

$$\sum_{j=1}^{\infty} \mu((a_j, b_j)) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mu(I_{j,k}) = \sum_{j,k=1}^{\infty} \mu(I_{j,k}) \geqslant \mu(E).$$

Since this is true for any cover of E with open intervals, we have $\nu(E) \geqslant \mu(E)$.

Now for the converse, let $\epsilon > 0$. By definition of μ, there exists $\{(a_j, b_j]\}$ with $E \subset \bigcup_{j=1}^{\infty}(a_j, b_j]$ and $\sum_{j=1}^{\infty} \mu((a_j, b_j]) \leqslant \mu(E) + \epsilon$. For each j, we choose $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) < 2^{-j}\epsilon$ (this can be done since F is right continuous) and we have

$$\begin{aligned}\nu(E) &\leqslant \sum_{j=1}^{\infty} \mu((a_j, b_j + \delta_j)) \leqslant \sum_{j=1}^{\infty} \mu((a_j, b_j + \delta_j]) = \sum_{j=1}^{\infty} \mu((a_j, b_j]) + \sum_{j=1}^{\infty} \mu((b_j, b_j + \delta_j]) \\ &\leqslant \mu(E) + \epsilon + \sum_{j=1\infty} [F(b_j + \delta_j) - F(b_j)] \leqslant \mu(E) + \epsilon + \sum_{j=1}^{\infty} 2^{-j}\epsilon = \mu(E) + 2\epsilon,\end{aligned}$$

and since $\epsilon > 0$ is arbitrary, we obtain $\nu(E) \leqslant \mu(E)$. ■

Proposition 2.5.9. *For $E \in \mathcal{M}_\mu$ we have*

$$\mu(E) = \inf\{\mu(U) \colon E \subset U \text{ and } U \text{ is open}\}.$$

Proof. Let $\mu_{op}(E) = \inf\{\mu(U) \colon E \subset U \text{ and } U \text{ is open}\}$. Clearly if $E \subset U$, since $U \in \mathcal{B}_{\mathbb{R}} \subset \mathcal{M}_\mu$, we have by the monotonicity property of μ that $\mu(E) \leqslant \mu(U)$. Hence $\mu(E) \leqslant \mu_{op}(E)$.

For the converse inequality, let $\epsilon > 0$. By Lemma 2.5.8, there exists $\{(a_j, b_j)\}$ such that $E \subset \bigcup_{j=1}^{\infty}(a_j, b_j)$ and $\sum_{j=1}^{\infty} \mu((a_j, b_j)) \leqslant \mu(E) + \epsilon$. Since $\bigcup_{j=1}^{\infty}(a_j, b_j)$ is open, we obtain $\mu_{op}(E) \leqslant \mu(E) + \epsilon$, and since $\epsilon > 0$ is arbitrary, we have $\mu_{op}(E) \leqslant \mu(E)$, which conclude the proof. ■

Proposition 2.5.10. *For $E \in \mathcal{M}_\mu$ we have*

$$\mu(E) = \sup\{\mu(K) \colon K \subset E \text{ and } K \text{ is compact}\}.$$

Proof. Set $\mu_c(E) = \sup\{\mu(K) \colon K \subset E \text{ and } K \text{ is compact}\}$. Clearly if $K \subset E$ then $\mu(K) \leqslant \mu(E)$ ($\mu(K)$ is defined since K is closed, and hence $K \in \mathcal{B}_{\mathbb{R}} \subset \mathcal{M}_\mu$). Thus $\mu_c(E) \leqslant \mu(E)$, and we just have to prove the converse inequality.

Assume that E is bounded (hence $\mu(E) < \infty$). If E is closed then E is compact, and the equality follows taking $K = E$. If E is not closed, let $\epsilon > 0$ be fixed. By Proposition 2.5.9 we can choose an open set U with $\overline{E} \setminus E \subset U$ and $\mu(U) \leqslant \mu(\overline{E} \setminus E) + \epsilon$. Take $K = \overline{E} \setminus U$, which is compact and $K \subset E$. We have

$$\begin{aligned}
\mu(K) &= \mu(E) - \mu(E \cap U) = \mu(E) - [\mu(U) - \mu(U \setminus E)] \\
&= \mu(E) - \mu(U) + \mu \underbrace{(U \setminus E)}_{\supset \overline{E} \setminus E} \\
&\geqslant \mu(E) - \mu(\overline{E} \setminus E) - \epsilon + \mu(\overline{E} \setminus E) \\
&= \mu(E) - \epsilon,
\end{aligned}$$

and hence $\mu_c(E) \geqslant \mu(E) - \epsilon$, and since ϵ is arbitrary we obtain $\mu_c(E) \geqslant \mu(E)$, if E is bounded.

Now, if E is unbounded consider $E_j = E \cap (j, j+1]$ for each $j \in \mathbb{Z}$. Then E_j is bounded and by the previous computations, for each $j \in \mathbb{Z}$ there exists a compact set $K_j \subset E_j$ with $\mu(K_j) \geqslant \mu(E_j) - 2^{-|j|}$. Now, for each $n \in \mathbb{N}$, define $H_n = \bigcup_{j=-n}^{n} K_j$. Then H_n is compact, $H_n \subset E$ and

$$\begin{aligned}
\mu_c(E) \geqslant \mu(H_n) &= \mu\Big(\bigcup_{j=-n}^{n} K_j\Big) = \sum_{j=-n}^{n} \mu(K_j) \geqslant \sum_{j=-n}^{n} [\mu(E_j) - \epsilon 2^{-|j|}] \\
&= \sum_{j=-n}^{n} \mu(E_j) - 3\epsilon = \mu\Big(\bigcup_{j=-n}^{n} E_j\Big) - 3\epsilon.
\end{aligned}$$

Therefore $\mu_c(E) \geqslant \mu\Big(\bigcup_{j=-n}^{n} E_j\Big) - 3\epsilon$, and since $\mu(E) = \lim_{n\to\infty} \mu\Big(\bigcup_{j=-n}^{n} E_j\Big)$ (continuity from below), making $n \to \infty$ we obtain $\mu_c(E) \geqslant \mu(E) - 3\epsilon$, and since ϵ is arbitrary, we obtain the reverse inequality and conclude the result. ■

THEOREM 2.5.11. *If $E \subset \mathbb{R}$, the following conditions are equivalent:*

(a) $E \in \mathcal{M}_\mu$.

(b) $E = V \setminus N_1$ *where V is a G_δ set and* $\mu(N_1) = 0$.

(c) $E = H \cup N_2$ *where H is a F_σ set and* $\mu(N_2) = 0$.

Proof. Clearly, since μ is complete by hypotheses, N_1 and N_2 are in $\mathcal{M}_\mu$ and hence (b) and (c) clearly imply (a).

We will prove that (a) implies both (b) an (c). For (a) implies (b), for each $j \in \mathbb{Z}$, we set $E_j = E \cap (j, j+1]$ and so we have $\mu(E_j) < \infty$. If $j \in \mathbb{Z}$ is fixed, for each $k \in \mathbb{N}$ we have $E_j \subset U_{j,k}$, with $U_{j,k}$ open and $\mu(U_{j,k} \setminus E_j) \leqslant k^{-1} 2^{-|j|}$.

Now take $U_k = \bigcup_{j=-\infty}^{\infty} U_{j,k}$. We have $U_k \setminus E \subset \bigcup_{j=-\infty}^{\infty} (U_{j,k} \setminus E_j)$ and hence $\mu(U_k \setminus E) \leqslant \sum_{j=-\infty}^{\infty} \mu(U_{j,k} \setminus E_j) \leqslant \frac{3}{k}$. Clearly each U_k is open and taking $V = \bigcap_{k=1}^{\infty} U_k$ we have V a G_δ set and $V \setminus E \subset U_k \setminus E$ for all k, which implies that

$$\mu(V \setminus E) \leqslant \mu(U_k \setminus E) \leqslant \frac{3}{k} \text{ for all } k,$$

and hence $\mu(V \setminus E) = 0$. Taking $N_1 = V \setminus E$ we prove that (a) implies (b).

For (a) implies (c), note that since $E \in \mathcal{M}_\mu$ then $E^c \in \mathcal{M}_\mu$. Then using that (a) implies (b), we can write $E^c = V \setminus N_1$ where V is a G_δ set and $\mu(N_1) = 0$. Thus $E = (V \setminus N_1)^c = V^c \cup N_1$, where V^c is a F_σ set, and concludes the proof. ■

This theorem says roughly that all sets in $\mathcal{M}_\mu$ are reasonably simple (open or compact) modulo sets of measure zero. Another useful proposition that states that measurable sets with finite measure can be approximated by a finite union of open intervals is the following:

Proposition 2.5.12. *If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for every $\epsilon > 0$ there is a set A which is a finite union of open intervals such that $\mu(E \Delta A) < \epsilon$.*

Proof. In fact, for a given $\epsilon > 0$, there exists $K \subset E \subset U$ with K compact and U open, such that $\mu(U \setminus E) < \frac{\epsilon}{2}$ and $\mu(E \setminus K) < \frac{\epsilon}{2}$. Now, since U is an open set of $\mathbb{R}$, we can write $U = \bigcup_{j=1}^{\infty} (a_j, b_j)$, with $a_j < b_j$ for all j. Since $K \subset U$ is compact, the exists a finite subcover of $\{(a_j, b_j)\}$ that covers K, and after possible relabeling, we can assume that $K \subset \bigcup_{j=1}^{n} (a_j, b_j) := A$.

Thus we have $K \subset A \subset U$ and

$$\mu(A \setminus E) \leqslant \mu(U \setminus E) < \frac{\epsilon}{2} \text{ and } \mu(E \setminus A) \leqslant \mu(E \setminus K) < \frac{\epsilon}{2},$$

and hence $\mu(E \Delta A) = \mu(E \setminus A) + \mu(A \setminus E) < \epsilon$. ■

2.5.1 | The Lebesgue measure on the real line

Now we will take a look at the properties of the most important measure on $\mathbb{R}$, the **Lebesgue measure**, which is the complete Lesbesgue-Stieltjes measure μ_F associated with the function $F(x) = x$, and the measure of each interval is simply its length. The Lebesgue measure will be denoted by m, and the σ-algebra of the m-measurable sets is called the class of the **Lebesgue measurable sets**, and will be denoted by $\mathcal{L}$. The restriction of m to $\mathcal{B}_{\mathbb{R}}$ will be also called Lebesgue measure.

Among the most important properties of Lebesgue measure are its invariance under translations and simple behavior under dilations. For $E \subset \mathbb{R}$, $s, r \in \mathbb{R}$, we define

$$E + s = \{x + s \colon x \in E\} \quad \text{and} \quad rE = \{rx \colon x \in E\}.$$

Theorem 2.5.13. *Let $E \in \mathcal{L}$. Then if $s, r \in \mathbb{R}$, we have*

(a) *$E + s \in \mathcal{L}$ and $m(E + s) = m(E)$,*

(b) *$rE \in \mathcal{L}$ and $m(rE) = |r|m(E)$.*

Proof. First note that the collection of all open intervals of $\mathbb{R}$ is invariant by translations and dilations, and hence so is $\mathcal{B}_{\mathbb{R}}$; that is, the translation and dilation of Borel sets are still Borel sets. Hence, for $s, r \in \mathbb{R}$ and $E \in \mathcal{B}_{\mathbb{R}}$ we define

$$m_s(E) = m(E + s) \quad \text{and} \quad m^r(E) = m(rE).$$

Now if (a, b) is an open interval, we have

$$m_s((a, b)) = m((a, b) + s) = m((a + s, b + s)) = (b + s) - (a + s) = b - a = m((a, b)),$$

and

$$\begin{aligned} m^r((a, b)) = m(r(a, b)) &= \begin{cases} m((ra, rb)) = rb - ra = r(b - a) = rm((a, b)) & \text{if } r \geqslant 0, \\ m((rb, ra)) = ra - rb = -r(b - a) = -rm((a, b)) & \text{if } r < 0 \end{cases} \\ &= |r|m((a, b)), \end{aligned}$$

hence m_s and m^r agrees with m and $|r|m$, respectively. Therefore, they agree on finite unions of intervals, and by uniqueness (see Theorem 2.5.7) they must agree on $\mathcal{B}_{\mathbb{R}}$.

In particular, if $E \in \mathcal{B}_{\mathbb{R}}$ is such that $m(E) = 0$, then $m(E + s) = 0$ and $m(rE) = 0$, which shows that the class of Borel sets of zero measure is preserved by translations and dilations. If $E \in \mathcal{L}$ is such that $m(E) = 0$, we know that there exists a G_δ set $A \in \mathcal{B}_{\mathbb{R}}$ such that $E \subset A$ and $m(A) = 0$ (take U_n open with $E \subset U_n$ and $\mu(U_n) < 1/n$, and set $A = \bigcap_{n=1}^{\infty} U_n$). Hence $E + s \subset A + s$ and $rE \subset rA$, with $m(A + s) = m(A) = 0$ and $m(rA) = |r|m(A) = 0$, and since m is complete, $E + s$ and rE are in $\mathcal{L}$ and they both have zero Lebesgue measure.

Now, using item (c) of Theorem 2.5.11, each Lebesgue set E is the union of a Borel set and a set of Lebesgue measure zero. Thus its translation $E+s$ and dilation rE are also Lebesgue sets and $m(E+s)=m(E)$ and $m(rE)=|r|m(E)$. ■

An impressive remark is that measure and topological properties contain some surprises. In fact, since the Lebesgue measure of each point is zero, then the Lebesgue measure of each countable set is also zero. In particular $m(\mathbb{Q})=0$ and $\mathbb{Q}$ is a dense subset of $\mathbb{R}$. But we can make things more interesting still.

Consider an enumeration $\{r_j\}$ of $\mathbb{Q}$ in $[0,1]$. Fix $\epsilon>0$ and consider I_j the interval centered at r_j of length $2^{-j}\epsilon$. The set $U=(0,1)\cap\bigcup_{j=1}^{\infty} I_j$ is an open dense subset of $[0,1]$ (which is "large", topologically speaking) but $m(U)\leqslant\sum_{j=1}^{\infty}2^{-j}\epsilon=\epsilon$ (which is "small", measurably speaking). Furthermore, the set $K=[0,1]\setminus U$ is closed and nowhere dense (which is "small", topologically speaking) but $m(K)\geqslant m([0,1])-m(U)=1-\epsilon$ (which is "large", measurably speaking).

2.5.2 | The Cantor set

We will present an example of a Lebesgue null set, with the cardinality of the continuum, namely the **Cantor set**.

Consider the set $E_0=[0,1]$. Remove from E_0 the middle third $(\frac{1}{3},\frac{2}{3})$ e let $E_1=[0,\frac{1}{3}]\cup[\frac{2}{3},1]$. Remove the middle third from each remaining interval and let

$$E_2=[0,\tfrac{1}{9}]\cup[\tfrac{2}{9},\tfrac{3}{9}]\cup[\tfrac{6}{9},\tfrac{7}{9}]\cup[\tfrac{8}{9},1],$$

and proceed with this construction, obtaining a set E_j in each step. We have $E_1\supset E_2\supset E_3\supset\cdots$, and each E_j is the union of 2^j disjoint closed intervals, each one with lenght 3^{-j} (ans thus $m(E_j)=2^j3^{-j}$ for each j).

Define $C=\bigcap\limits_{j=1}^{\infty}E_j$. This set C is called the **Cantor set**, and it is clearly compact. Since it is an intersection of a decreasing sequence of compact sets, it is nonempty. Clearly $C\in\mathcal{B}_{\mathbb{R}}$, and hence C is Lebesgue measurable and moreover

$$m(C)\leqslant m(E_j)=\left(\frac{2}{3}\right)^j\to 0 \text{ as } j\to\infty,$$

and hence $m(C)=0$.

We will explore some topological properties of C.

Proposition 2.5.14. *If $0 \leqslant \alpha < \beta \leqslant 1$, then there exist $k, m \in \mathbb{N}_0$ such that the interval (α, β) contains an interval of the form*

$$\left(\tfrac{3k+1}{3^m}, \tfrac{3k+2}{3^m}\right). \tag{2.5.1}$$

Proof. First, choose $m \in \mathbb{N}_0$ such that $4 < 3^m(\beta - \alpha)$. We have

$$\frac{3^m\beta - 2}{3} - \frac{3^m\alpha - 1}{3} = \frac{3^m(\beta - \alpha) - 1}{3} > \frac{4-1}{3} = 1,$$

and hence there exists an integer $k \in (\frac{3^m\alpha-1}{3}, \frac{3^m\beta-2}{3})$, and therefore

$$\frac{3^m\alpha - 1}{3} < k < \frac{3^m\beta - 2}{3} \quad \Longrightarrow \quad \alpha < \frac{3k+1}{3^m} \quad \text{and} \quad \frac{3k+2}{3^m} < \beta,$$

and concludes the proof. ■

The intervals removed from $[0,1]$ to form the Cantor set are precisely the intervals of the form (2.5.1). Since each interval contains an interval of the form (2.5.1) (by this last proposition) the Cantor set C contains no interval. This implies that C is totally disconnected, that is, the only connected subsets of C are points.

Proposition 2.5.15. *Every point of C is a limit point of C. In other words, C has no isolated points.*

Proof. Let $x \in C$ and I an open interval containing x. Let I_j be the interval of E_j that contains x, and choose j large so that $I_j \subset I$. Choose x_j as the endpoint of I_j with $x_j \neq x$. By construction of C, all the endpoints of the intervals of E_j are in C, and hence $x_j \in C$, which proves that x is a limit point of C. ■

Let $x \in [0,1]$ and consider its expansion in base 3, that is, $x = \sum\limits_{j=1}^{\infty} 3^{-j} a_j$, where $a_j \in \{0, 1, 2\}$.

Lemma 2.5.16. *$x \in C$ if and only if its base 3 expansion $x = \sum\limits_{j=1}^{\infty} 3^{-j} a_j$ is such that $a_j \neq 1$ for all j.*

Proof. See [1, Page 38]. ■

Proposition 2.5.17. *We have* $\operatorname{card}(C) = \mathfrak{c}$.

Proof. Let $x \in C$ and $\sum_{j=1}^{\infty} 3^{-j} a_j$ its base 3 expansion. By the previous lemma, $a_j = 0$ or 2 for all j, and we can define $f\colon C \to [0,1]$ by $f(x) = \sum_{j=1}^{\infty} 2^{-j} b_j$, where $b_j = \frac{a_j}{2} \in \{0,1\}$ for all j. Since each real number in $[0,1]$ can be written in base 2, this function f is surjective, and hence $\operatorname{card}(C) \geqslant \operatorname{card}([0,1]) = \mathfrak{c}$. Since $C \subset [0,1]$, we have the equality. ■

One can see that if $x, y \in C$ and $x < y$, then $f(x) < f(y)$, unless x, y are the endpoints of one subinterval removed from $[0,1]$ to form C, since in this case we would have $f(x) = \frac{k}{2^m}$ for some $m \in \mathbb{N}_0$ and $k \in \{0, 1, \cdots, 2^m\}$, and $f(y)$ is the other base two expansion of $f(x)$. Hence we can extend this function, from C to $[0,1]$ by setting f constant (and equal to f of the endpoints) of each interval missing from C. This function f is still increasing, and since $f([0,1]) = [0,1]$, f can have no jump discontinuities, therefore f is continuous. This function f is called the **Cantor function** or the **Cantor-Lebesgue function**

2.5.3 | Cantor-type sets of positive measure

In this subsection we will generalize the construction of the Cantor set done above, to obtain sets which are compact, nowhere dense, totally disconnected, with no isolated points and with the cardinality of the continuum, but with positive measure.

Let $I = [a,b]$ be a bounded interval ($a, b \in \mathbb{R}$ and $a < b$) and $\alpha \in (0,1)$. Set $c = \frac{a+b}{2}$ and $r = \frac{\alpha(b-a)}{2}$. The interval $(c-r, c+r) \subset I$ is called the **open middle α^{th}** of I, and we have $m((c-r, c+r)) = 2r = \alpha(b-a) = \alpha m(I)$.

Now we make the construction as follows. Let $\{\alpha_j\}$ any sequence of numbers in $(0,1)$ and $K_0 = I = [a,b]$. We obtain K_1 be removing the open middle α_1^{th} of K_0. Next K_2 is obtained by removing the open middle α_2^{th} of each one of the two intervals that make K_1. Inductively, K_j is obtained by removing the open middle α_j^{th} of the 2^{j-1} intervals that make K_{j-1}.

Define $K = \bigcap_{j=1}^{\infty} K_j$. This set is called the **generalized Cantor set**, and is nonempty, compact, nowhere dense, totally disconnected, with no isolated points and has the cardinality of the continuum. When $K_0 = [0,1]$ and $\alpha_j = \frac{1}{3}$ for all j we obtain the Cantor set C.

Now, at each step we obtain $m(K_j) = (1-\alpha_j) m(K_{j-1})$ and hence, using this process j times, we have $m(K_j) = (b-a) \prod_{n=1}^{j} (1-\alpha_n)$. Using the continuity from above, we have

$$m(K) = \lim_{j \to \infty} m(K_j) = (b-a) \lim_{j \to \infty} \prod_{n=1}^{j} (1-\alpha_n).$$

If $\alpha_j = \alpha \in (0,1)$ for all j then $m(K) = (b-a) \lim\limits_{n \to \infty} (1-\alpha)^n = 0$. However, if $\alpha_j \to 0$ sufficiently fast, we have chances to obtain $m(K) > 0$. In fact, for each $\beta \in (0,1)$, we will see that we can choose a sequence $\alpha_j \to 0$ such that $\lim\limits_{j \to \infty} \prod\limits_{n=1}^{j} (1-\alpha_n) = \beta$, and hence $m(K) = (b-a)\beta$.

2.5.4 | A set not Lebesgue measurable in $\mathbb{R}$

One question that has to be answered is the following: are all subsets of $\mathbb{R}$ Lebesgue measurable? That is, $\mathcal{L} = \mathcal{P}(\mathbb{R})$?

In this subsection we will see that this is not the case, constructing a subset of $\mathbb{R}$ that is not Lebesgue measurable. But it comes as no surprise that such a set has to be something quite strange, that is, for a set to be not Lebesgue measurable, something has to be very wrong!

To begin, consider $E = [0,1)$, and define in E the following relation: $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Clearly $\sim$ is an equivalence relation and for each $x \in [0,1)$ we consider its equivalence class $[x]$. Define $\mathfrak{E} = \{[x] \colon x \in E\}$ and using the Axiom of Choice, consider $N \subset [0,1)$ be a set with exactly one element of each equivalence class in $\mathfrak{E}$.

Consider now $R = \mathbb{Q} \cap [0,1)$ and for each $r \in R$ define

$$N_r = \{x + r \colon x \in N \cap [0, 1-r)\} \cup \{x + r - 1 \colon x \in N \cap [1-r, 1)\},$$

that is, we shift N by r units to the right, but the part of this shift that sticks out of $[0,1)$ we bring back by one unit. With this construction $N_r \subset [0,1)$ for each $r \in R$.

We have the following two properties:

Lemma 2.5.18.

(a) $[0,1) = \bigcup_{r \in R} N_r$.

(b) $N_r \cap N_s = \varnothing$ *if* $r, s \in R$ *and* $r \neq s$.

Proof. **(a)** Let $x \in [0,1)$. By the construction of N, there exists a representative of the class $[x]$, we call it x_0, in N. If $x_0 \leqslant x$, set $r = x - x_0 \in R$. Then $x = x_0 + r \in N_r$, since $x_0 \in N \cap [0, 1 - x + x_0) = N \cap [0, 1-r)$. Now if $x_0 > x$, define $r = x - x_0 + 1 \in R$. In this case $x = x_0 + r - 1 \in N_r$, since $x_0 \in N \cap [x_0 - x, 1) = N \cap [1-r, 1)$. In any of the two cases, there exists $r \in R$ such that $x \in N_r$, and proves (a).

(b) Assume that $x \in N_r \cap N_s$, with $r, s \in R$ and $r \neq s$. Then we have $x = x_0 + r$ (or $x = x_0 + r - 1$) for some $x_0 \in N$ and $x = x_1 + s$ (or $x = x_1 + s - 1$) for some $x_1 \in N$. In any of the four possibilities, we obtain $x_0 - x_1 \in \mathbb{Q}$, and since both x_0, x_1 are in N, by construction of N, this implies that $x_0 = x_1$, which in turn implies that $r = s$ or $r = s + 1$ or $s = r + 1$. Since $r \neq s$ we have $r = s + 1$ or $s = r + 1$, but since $r, s \in R = \mathbb{Q} \cap [0,1)$, this gives us a contradiction. Therefore $N_r \cap N_s = \varnothing$. ∎

THEOREM 2.5.19. *N is not Lebesgue measurable.*

Proof. Assume that this is not the case, that is, assume that N is Lebesgue measurable. Then, for each $r \in R$ we have

$$\begin{aligned} m(N) &= m(N \cap [0, 1-r)) + m(N \cap [1-r, 1)) \\ &= m(N \cap [0, 1-r) + r) + m(N \cap [1-r, 1) + r - 1) \\ &= m(N_r), \end{aligned}$$

using the translation invariance property of the Lebesgue measure. Hence, since R is countable, using item (a) of the previous lemma we obtain

$$1 = m([0,1)) = m\Big(\bigcup_{r \in R} N_r\Big) = \sum_{r \in R} m(N_r) = \sum_{r \in R} m(N).$$

But thus last equality gives us a contradiction, since the last sum on the right can only be 0 (if $m(N) = 0$) of ∞ (if $m(N) > 0$). Therefore N cannot be Lebesgue measurable. ∎

2.6 | SOLVED EXERCISES FROM [1, PAGE 39]

EXERCISE 25. Complete the proof of Theorem 1.19.

Solution. See Theorem 2.5.11.

EXERCISE 26. Prove Proposition 1.20 (Use Theorem 1.18).

Solution. See Theorem 2.5.12.

EXERCISE 27. Prove Proposition 1.22a. (Show that is $x, y \in C$ and $x < y$, there exists $z \notin C$ such that $x < z < y$).

Solution. This result is proved in Subsection 2.5.2. To elaborate as the hint presented, consider $x = \alpha$, $y = \beta$. By Proposition 2.5.14, the interval (x, y) contains an interval of the form $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m})$ for some integers $m \in \mathbb{N}_0$ and $k \in \{0, 1, \cdots, 3^m\}$. Since this interval has empty intersection with C, any z in this interval is such that $z \notin C$ and $x < z < y$.

EXERCISE 28. Let $F\colon \mathbb{R} \to \mathbb{R}$ be an increasing and right continuous function, and let μ_F be the associated measure. Then $\mu_F(\{a\}) = F(a) - F(a-)$, $\mu_F([a,b)) = F(b-) - F(a-)$, $\mu_F([a,b]) = F(b) - F(a-)$ and $\mu_F((a,b)) = F(b-) - F(a)$.

Recall that $F(x_0-) = \lim_{x\to x_0^-} F(x)$.

Solution. Note that $\{a\} = \bigcap_{j=1}^{\infty} (a - \frac{1}{j}, a]$ and by the continuity from above, we have

$$\mu_F(\{a\}) = \lim_{j\to\infty} \mu_F((a - \tfrac{1}{j}, a]) = \lim_{j\to\infty} [F(a) - F(a - \tfrac{1}{j})] = F(a) - F(a-).$$

Now

$$\begin{aligned}
\mu_F([a,b]) &= \mu_F(\{a\}) + \mu_F((a,b]) = F(a) - F(a-) + F(b) - F(a) = F(b) - F(a-),\\
\mu_F((a,b)) &= \mu_F((a,b]) - \mu_F(\{b\}) = F(b) - F(a) - (F(b) - F(b-)) = F(b-) - F(a),\\
\mu_F([a,b)) &= \mu_F(\{a\}) + \mu_F((a,b)) = F(a) - F(a-) + F(b-) - F(a) = F(b-) - F(a-).
\end{aligned}$$

Exercise 29. Let E be a Lebesgue measurable set.

(a) If $E \subset N$, where N is the nonmeasurable set describe in Subsection 2.5.4, then $m(E) = 0$.

(b) If $m(E) > 0$, then E contais a nonmeasurable set. (If suffices to assume $E \subset [0,1]$)

Solution to (a). If $E \subset N$ is measurable, then with the notation of Subsection 2.5.4 let $F = \bigcup_{r\in R} E_r \subset \bigcup_{r\in R} N_r = [0,1)$, where

$$E_r = \{x + r : x \in E \cap [0, 1-r)\} \cup \{x + r - 1 : x \in E \cap [1-r, 1)\},$$

and each E_r is measurable (since E is measurable) and since $E \subset N$ the sequence $\{E_r\}$ is pairwise disjoint. Thus

$$m(F) = \sum_{r\in R} m(E_r) = \sum_{r\in R} m(E),$$

and since $F \subset [0,1)$, we have $m(F) \leqslant 1 < \infty$ and this implies that $m(E) = 0$, for if $m(E) > 0$, the previous equality would imply that $m(F) = \infty$.

P.S.: This item remains true if we replace N with N_r for some $r \in R$.

Solution to (b). Assume that $m(E) > 0$.

$\star$ Since m is semifinite, we can assume that $m(E) < \infty$, since if $m(E) = \infty$ there exists $F \subset E$ with $0 < m(F) < m(E)$.

$\star$ Also we can assume that E is bounded, since if E is unbounded, since m is σ-finite, we can write $E = \bigcup_{j\in\mathbb{Z}} E \cap (j, j+1]$, and since $0 < m(E) < \infty$, this implies that at least one of the $E \cap (j, j+1]$ has positive measure.

$\star$ Finally, we can assume that $E \subset [0,1]$, since if this is not the case we take $s = |\min\{\inf E, 0\}|$ and $E_s = E + s \subset [0,\infty)$ and if $r = \sup E_s + 1$ then $(1/r)E \subset [0,1)$, and if the result is proven for $(1/r)E$ with a nonmeasurable set N, then $rN - s \subset E$ is also nonmeasurable.

Thus we will prove the result for this case $E \subset [0,1)$ with $m(E) > 0$. Consider the nonmeasurable set N described in Subsection 2.5.4. Since $\bigcup_{r \in R} N_r = [0,1)$, we have $E = \bigcup_{r \in R} E \cap N_r$. We have N_r nonmeasurable for each $r \in R$. In fact, if for some $r \in R$, N_r is measurable then so is N, which is a contradiction. Assume now that each $E \cap N_r$ is measurable. Then by item (a), with N_r instead of N, we obtain $m(E \cap N_r) = 0$ and hence $m(E) = 0$, which is a contradiction. Hence $E \cap N_r$ is nonmeasurable for some $r \in R$, and this is a nonmeasurable set contained in E.

Exercise 30. If $E \in \mathcal{L}$ and $m(E) > 0$, for any $\alpha < 1$ the is an open interval I such that $m(E \cap I) > \alpha m(I)$.

Solution. For $\alpha \leqslant 0$, the result holds with $I = \mathbb{R}$. Now we prove the result for $0 < \alpha < 1$, and to that end we consider two cases.

Case 1: Assume that $m(E) < \infty$.

Assume to te contrary that there exists $0 < \alpha < 1$ such that $m(E \cap I) \leqslant \alpha m(I)$ for all open intervals I.

With this assumption, if $\{I_j\}$ is a pairwise disjoint sequence of open intervals such that $E \subset \bigcup_{j=1}^{\infty} I_j$ then

$$m(E) = m\Big(E \cap \bigcup_{j=1}^{\infty} I_j\Big) = \sum_{j=1}^{\infty} m(E \cap I_j) \leqslant \alpha \sum_{j=1}^{\infty} m(I_j). \tag{2.6.1}$$

Now, take $\epsilon > 0$. There exists an open set U with $E \subset U$ and $m(U) \leqslant m(E) + \epsilon$. Since U is open, there exists a pairwise disjoint sequence $\{I_j\}$ of open intervals such that $U = \bigcup_{j=1}^{\infty} I_j$. By (2.6.1) we have

$$m(E) \leqslant \alpha \sum_{j=1}^{\infty} m(I_j) = \alpha m(U) \leqslant \alpha m(E) + \alpha\epsilon,$$

which implies that $m(E) \leqslant \frac{\alpha}{1-\alpha}\epsilon$, for each $\epsilon > 0$, since $m(E) < \infty$. Thus $m(E) = 0$, and contradicts the assumption that $m(E) > 0$.

Case 2: $m(E) = \infty$. Since m is semifinite, take $F \subset E$ with $0 < m(F) < \infty$. For such F, by

Case 1, there exists an interval I such that $m(F \cap I) > \alpha m(I)$. Hence

$$m(E \cap I) \geqslant m(F \cap I) > \alpha m(I),$$

and concludes the results.

Exercise 31. If $E \in \mathcal{L}$ and $m(E) > 0$, the set $E - E = \{x - y \colon x, y \in E\}$ contains an interval centered at 0. (If I is an in Exercise 30, with $\alpha > \frac{3}{4}$, then $E - E$ contains $(-\frac{1}{2}m(I), \frac{1}{2}m(I))$).

Solution. Clearly $0 \in E - E$ and moreover, if $z = x - y \in E - E$ then $-z = y - x \in E - E$.

Let $\frac{3}{4} < \alpha < 1$. By Exercise 30, there exists an open interval I such that $m(E \cap I) > \frac{3}{4}m(I)$, and we can assume that $I = (x_0 - r, x_0 + r)$ (we can assume that I is bounded, since the previous inequality would not hold when I is not bounded). Note that $r = \frac{1}{2}m(I)$. By the considerations above, if we shows that $(0, r) \subset E - E$, the result is proven.

Let $z \in (0, r)$ and assume that $z \notin E - E$, hence $z \neq x - y$ for all $x, y \in E$. Define $E_1 = E \cap (x_0 - r, x_0]$ and $E_2 = E \cap (x_0, x_0 + r)$. If $x \in E_1$ then $x + z \in I$ and $x + z \notin E$ (since if $x + z = y \in E$ then $z = x - y \in E - E$). Hence $E_1 + z \subset I \setminus E$.

Analogously $E_2 - z \subset I \setminus E$. Therefore

$$\begin{aligned} m(E_1) &= m(E_1 + z) \leqslant m(I \setminus E) = m(I \setminus (E \cap I)) = m(I) - m(E \cap I) \\ m(E_2) &= m(E_2 - z) \leqslant m(I \setminus E) = m(I) - m(E \cap I), \end{aligned}$$

and we obtain

$$m(E \cap I) = m(E_1 \cup E_2) = m(E_1) + m(E_2) \leqslant 2[m(I) - m(E \cap I)],$$

which implies that $m(E \cap I) \leqslant \frac{2}{3}m(I)$. But then

$$\tfrac{3}{4}m(I) < m(E \cap I) \leqslant \tfrac{2}{3}m(I),$$

which gives us a contradiction. Hence $z \in E - E$, and thus $(0, \alpha) \subset E - E$, which concludes the proof.

Exercise 32. Suppose $\{\alpha_j\} \subset (0, 1)$.

(a) $\prod_{j=1}^{\infty}(1 - \alpha_j) > 0$ if and only if $\sum_{j=1}^{\infty} \alpha_j < \infty$ (compare $\sum_{j=1}^{\infty} \log(1 - \alpha_j)$ to $\sum_{j=1}^{\infty} \alpha_j$).

(b) Given $\beta \in (0,1)$, exhibit a sequence $\{\alpha_j\}$ such that $\prod_{j=1}^{\infty}(1-\alpha_j) = \beta$.

Solution to (a). Note first that

$$\prod_{j=1}^{\infty}(1-\alpha_j) = \lim_{n\to\infty}\prod_{j=1}^{n}(1-\alpha_j) = \lim_{n\to\infty}\prod_{j=1}^{n} e^{\log(1-\alpha_j)} = \lim_{n\to\infty} e^{\sum_{j=1}^{n}\log(1-\alpha_j)},$$

and since all terms $\log(1-\alpha_j)$ are negative, we have

$$\prod_{j=1}^{\infty}(1-\alpha_j) = e^{\sum_{j=1}^{\infty}\log(1-\alpha_j)} = e^{-\sum_{j=1}^{\infty}\log(1-\alpha_j)^{-1}}.$$

Now we compare $\sum_{j=1}^{\infty}\log(1-\alpha_j)^{-1}$ to $\sum_{j=1}^{\infty}\alpha_j$. To do that, consider the real function $f\colon [0,\infty) \to \mathbb{R}$ given by $f(x) = (1-x)e^x$. We have $f(0) = 1$ and $f'(x) = -xe^x < 0$ for all $x \in (0,1)$. Hence f is strictly decreasing and $f(x) \leqslant 1$ for all $x \in [0,1)$, which means that $(1-x)e^x \leqslant 1$ for all $x \in [0,1)$. Applying log on both sides, we obtain $x \leqslant \log(1-x)^{-1}$ for all $x \in [0,1)$. Thus taking $x = \alpha_j$, the Comparison Test gives us that if $\sum \log(1-\alpha_j)^{-1}$ converges then $\sum \alpha_j$ converges.

For the other side, consider the function $g\colon [0,1/2) \to \mathbb{R}$ given by $g(x) = (1-x)e^{2x}$. Thus $g(0) = 1$ and $g'(x) = e^{2x}(1-2x) > 0$ if $x \in [0,1/2)$. Hence g is strictly increasing and $1 \leqslant g(x)$ for all $x \in (0,1/2)$, which implies that $1 \leqslant (1-x)e^{2x}$ for all $x \in [0,1/2)$. Applying log on both sides, we obtain $\log(1-x)^{-1} \leqslant 2x$ for all $x \in [0,1/2)$. Thus taking $x = \alpha_j$, the Comparison Test gives us that if $\sum \alpha_j$ converges, then $\sum \log(1-\alpha_j)^{-1}$ converges (since $\sum \alpha_j$ converges, there exists $j_0 \in \mathbb{N}$ such that $\alpha_j \in (0,1/2)$ for all $j \geqslant j_0$, and we apply the Comparison Test for $j \geqslant j_0$).

This concludes the proof of (a), since $\prod_{j=1}^{\infty}(1-\alpha_j) > 0$ if and only if $\sum \log(1-\alpha_j)^{-1}$ converges.

Solution to (b). Let us construct first the following: given $\gamma < 0$, construct a sequence $\gamma_j \subset (0,1)$ such that $\sum \log \gamma_j = \gamma$.

Take $r = 1 - \frac{1}{\gamma}$. Then $r > 1$, since $\gamma < 0$, and for each j, define $\gamma_j = e^{-r^{-j}}$. Thus $\gamma_j \in (0,1)$ for all j. Now we have

$$\sum_{j=1}^{\infty}\log\gamma_j = -\sum_{j=1}^{\infty} r^{-j} = -\frac{1}{r-1} = \gamma.$$

Now for $\beta \in (0,1)$, set $\gamma = \log\beta$ and consider the sequence γ_j from the previous construction. Defining $\alpha_j = 1 - \gamma_j$ we have

$$\prod_{j=1}^{\infty}(1-\alpha_j) = e^{\sum\limits_{j=1}^{\infty}\log(1-\alpha_j)} = e^{\sum\limits_{j=1}^{\infty}\log\gamma_j} = e^{\gamma} = \beta,$$

and we conclude the construction.

Explicitly we have

$$\alpha_j = 1 - e^{r^{-j}} \text{ for all } j, \text{ where } r = 1 - \frac{1}{\log\beta}.$$

Exercise 33. There exists a Borel set $A \subset [0,1]$ such that $0 < m(A \cap I) < m(I)$ for every subinterval I of $[0,1]$. (Hint: every subinterval of $[0,1]$ contains Cantor-type sets of positive measure).

Solution. Consider $R = \mathbb{Q} \cap [0,1]$ and $\mathcal{I} = \{(a,b)\colon a, b \in R \text{ with } a < b\}$ the collection of all intervals with rational endpoints in $[0,1]$. We know that $\mathcal{I}$ is countable, thus we have $\mathcal{I} = \{I_j\}$.

There exists a Cantor-set type $K_1 \subset I_1$ with positive measure (K_1 is compact and totally disconnected). Since $I_1 \setminus K_1$ is open, there exists an open interval $I_1^* \subset I_1 \setminus K_1$, and a Cantor-type set $W_1 \subset I_1^*$ with positive measure (W_1 is also compact and totally disconnected). Clearly $K_1 \cap W_1 = \varnothing$, and since they are both compact and totally disconnected, so is $C_1 = K_1 \cup W_1$.

Now $I_2 \setminus C_1$ is open, hence there exists an interval $I_2^* \subset I_2 \setminus C_2$, and a Cantor-type set K_2 with positive measure. Also, there exists a Cantor-type set $W_2 \subset I_2^* \setminus K_2$ (since the latter is open) with positive measure. Hence $(K_1 \cup K_2) \cap (W_1 \cup W_2) = \varnothing$, and $C_2 = K_1 \cup K_2 \cup W_1 \cup W_2 \subset I_1 \cup I_2$ is compact and totally disconnected.

Assume we have constructed K_j, W_j with $(\bigcup_{k=1}^{j} K_k) \cap (\bigcup_{k=1}^{j} W_k) = \varnothing$ and $C_j = \bigcup_{k=1}^{j}(K_k \cup W_k) \subset \bigcup_{k=1}^{j} I_k$ is compact and totally disconnected. Thus $I_{j+1} \setminus C_j$ is open and contains an interval I_{j+1}^* which in turn contains a Cantor-type set K_{j+1} of positive measure. Also $I_{j+1}^* \setminus K_{j+1}$ contains a Cantor-type set of positive measure W_{j+1}.

Therefore $(\bigcup_{k=1}^{j+1} K_k) \cap (\bigcup_{k=1}^{j+1} W_k) = \varnothing$ and $C_{j+1} = \bigcup_{k=1}^{j+1}(K_k \cup W_k) \subset \bigcup_{k=1}^{j+1} I_k$ is compact and totally disconnected.

Now we define $A = \bigcup_{j=1}^{\infty} K_j$, which is a Borel set (countable union of compact sets). If I is a subinterval of $[0,1]$, then there exists j_0 such that $I_{j_0} \subset I$, and hence

$$m(A \cap I) \geqslant m(A \cap I_{j_0}) \geqslant m(K_{j_0} \cap I_{j_0}) \overset{Kj_0 \subset I_{j_0}}{=} m(K_{j_0}) > 0.$$

Also, since W_{j_0} is disjoint from K_j for all $j \in \mathbb{N}$ we have

$$m(A \cap I) < m(A \cap I) + m(W_j \cap I) \leqslant m(I),$$

and thus $0 < m(A \cap I) < m(I)$.

CHAPTER 3

INTEGRATION

3.1 | Measurable functions

Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be two measurable spaces and $f\colon X \to Y$ a function. We define $f^{-1}(\mathcal{N}) = \{f^{-1}(E)\colon E \in \mathcal{N}\}$.

Proposition 3.1.1. *The collection $f^{-1}(\mathcal{N})$ is a σ-algebra on X.*

Proof. This result follows since f^{-1} preserves unions, intersections and complements. ■

Definition 3.1.2. *$f\colon X \to Y$ is called* **$(\mathcal{M}, \mathcal{N})$-measurable** *(or simply* **measurable** *if $\mathcal{M}, \mathcal{N}$ are understood) if $f^{-1}(\mathcal{N}) \subset \mathcal{M}$, that is, if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.*

Proposition 3.1.3. *If $\mathcal{N}$ is generated by $\mathcal{E}$ then $f\colon X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.*

Proof. Clearly the only if part is trivial. The if part follows from the fact that $\{E \subset Y\colon f^{-1}(E) \in \mathcal{M}\}$ is a σ-algebra on X that contains $\mathcal{E}$, and hence it contains $\mathcal{N}$. ■

Corollary 3.1.4. *If X and Y are topological spaces, every continuous function $f\colon X \to Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.*

Proof. This result follows from the fact that f is continuous if and only if $f^{-1}(U)$ is open in X for every open subset U of Y. ■

If $(X, \mathcal{M})$ is a measurable space and $f\colon X \to \mathbb{R}$ or $f\colon X \to \mathbb{C}$ or $f\colon X \to \overline{\mathbb{R}}$ then it will be called **$\mathcal{M}$-measurable** (or simply **measurable**), if it is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ or $(\mathcal{M}, \mathbb{B}_{\mathbb{C}})$ or $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$ measurable. In particular $f\colon \mathbb{R} \to \mathbb{R}$ or $f\colon \mathbb{R} \to \mathbb{C}$ or $f\colon \mathbb{R} \to \overline{\mathbb{R}}$ is called **Lebesgue (Borel) measurable** if it is $(\mathcal{L}, \mathcal{B}_{\mathbb{R}})$ $((\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}}))$ or $(\mathcal{L}, \mathcal{B}_{\mathbb{C}})$ $((\mathcal{B}_{\mathcal{R}}, \mathcal{B}_{\mathbb{C}}))$ or $(\mathcal{L}, \mathcal{B}_{\overline{\mathbb{R}}})$ $((\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}}))$ measurable, respectively.

Remark 3.1.5. *If $f, g\colon \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable, it does not follow that $f \circ g$ is Lebesgue measurable, even if g is continuous. If $E \in \mathcal{B}_{\mathbb{R}}$ we have $f^{-1}(E) \in \mathcal{L}$, but unless $f^{-1}(E) \in \mathcal{B}_{\mathbb{R}}$ there is no guarantee that $g^{-1}(f^{-1}(E))$ will be in $\mathcal{L}$ (we will see the existence of a nonborelian Lebesgue measurable set in a following exercise).*

However if f is Borel measurable then $f \circ g$ is Lebesgue or Borel measurable whenever g is.

Proposition 3.1.6. *If $(X, \mathcal{M})$ is a measurable space and $f\colon X \to \mathbb{R}$, the following are equivalent:*

(a) *f is $\mathcal{M}$-measurable.*

(b) *$f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.*

(c) *$f^{-1}([a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.*

(d) *$f^{-1}((-\infty, a)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.*

(e) *$f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$.*

Proof. This follows from Propositions 3.1.3 and 1.3.3. ■

We often need to consider the measurability of a function f on subsets of X. In this case, if $(X, \mathcal{M})$ is a measurable space, f is a (real or complex) function and $E \in \mathcal{M}$, we say that f is **measurable on E** if $f^{-1}(B) \cap E \in \mathcal{M}$ for all Borel sets B; or equivalently, if $f|_E$ is $\mathcal{M}_E$ measurable, where $\mathcal{M}_E = \{F \cap E\colon F \in \mathcal{M}\}$.

Given a nonempty set X, a family $\{(Y_\alpha, \mathcal{N}_\alpha)\}_\alpha \in \Lambda$ a collection of measurable spaces and $f_\alpha\colon X \to Y_\alpha$ is a map for each $\alpha \in \Lambda$, there is a unique smallest σ-algebra on X such that all f_α are measurable, namely the σ-algebra $\mathcal{M}$ generated by the sets $f_\alpha^{-1}(E_\alpha)$ with $E_\alpha \in \mathcal{N}_\alpha$ and $\alpha \in \Lambda$. This σ-algebra $\mathcal{M}$ is called the **σ-algebra generated** by $\{f_\alpha\}_{\alpha\in\Lambda}$.

If $X = \prod\limits_{\alpha\in\Lambda} Y_\alpha$, we see that the product σ-algebra on X (see Section 1.2) is the σ-algebra generated by the coordinate maps $\pi_\alpha\colon X \to Y_\alpha$, $\alpha \in \Lambda$.

Proposition 3.1.7. *Let $(X, \mathcal{M})$ and $\{(Y_\alpha, \mathcal{N}_\alpha)\}_{\alpha\in\Lambda}$ be measurable spaces, $Y = \prod\limits_{\alpha\in\Lambda} Y_\alpha$, $\mathcal{N} = \otimes_{\alpha\in\Lambda}\mathcal{N}_\alpha$, and $\pi_\alpha\colon Y \to Y_\alpha$ the coordinate maps for each $\alpha \in \Lambda$. Then $f\colon X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable if and only if $f_\alpha = \pi_\alpha \circ f$ is $(\mathcal{M}, \mathcal{N}_\alpha)$-measurable for all $\alpha \in \Lambda$.*

Proof. Assume that f is $(\mathcal{M}, \mathcal{N})$-measurable and fix $\alpha \in \Lambda$. If $E_\alpha \in \mathcal{N}_\alpha$, then by definition of $\mathcal{N}$, we have $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{N}$ and hence $f_\alpha^{-1}(E_\alpha) = (\pi_\alpha \circ f)^{-1}(E_\alpha) = f^{-1}(\pi_\alpha^{-1}(E_\alpha)) \in \mathcal{M}$, and f_α if $(\mathcal{M}, \mathcal{N}_\alpha)$-measurable.

For the converse, since $\mathcal{N}$ is generated by the family $\mathcal{E} = \bigcup_{\alpha\in\alpha} \bigcup_{E_\alpha\in\mathcal{N}_\alpha} \pi_\alpha^{-1}(E_\alpha)$ and f_α is $(\mathcal{M}, \mathcal{N}_\alpha)$-measurable for all $\alpha \in \Lambda$, we have

$$f^{-1}(\pi_\alpha^{-1}(E_\alpha)) = (\pi_\alpha \circ f)^{-1}(E_\alpha) = f_\alpha^{-1}(E_\alpha) \in \mathcal{M}.$$

Hence, by Proposition 3.1.3, it follows that f is $(\mathcal{M}, \mathcal{N})$-measurable. ■

Corollary 3.1.8. *A function $f\colon X \to \mathbb{C}$ is $\mathcal{M}$-measurable if and only if $\operatorname{Re} f\colon X \to \mathbb{R}$ and $\operatorname{Im} f\colon X \to \mathbb{R}$ are $\mathcal{M}$-measurable.*

Proof. This follows from Proposition 3.1.7 since $\mathcal{B}_{\mathbb{C}} = \mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ by (1.3.1). ■

Proposition 3.1.9. *Let $f, g\colon X \to \mathbb{K}$ be $\mathcal{M}$-measurable functions, where $\mathbb{K}$ can be $\mathbb{C}$ or $\mathbb{R}$. Then $f + g$ and fg are also $\mathcal{M}$-measurable.*

Proof. Define $F\colon X \to \mathbb{K}\times\mathbb{K}$, $\phi\colon \mathbb{K}\times\mathbb{K} \to \mathbb{K}$ and $\psi\colon \mathbb{K}\times\mathbb{K} \to \mathbb{K}$ by $F(x) = (f(x), g(x))$ for $x \in X$, $\phi(x, y) = x+y$ and $\psi(x, y) = xy$ for $x, y \in \mathbb{K}$. By Proposition 1.3.2, $\mathcal{B}_{\mathbb{K}\times\mathbb{K}} = \mathcal{B}_{\mathbb{K}} \otimes \mathcal{B}_{\mathbb{K}}$, and by Proposition 3.1.7, F is $(\mathcal{M}, \mathcal{B}_{\mathbb{K}\times\mathbb{K}})$-measurable. By Corollary 3.1.4, ϕ and ψ are $(\mathcal{B}_{\mathbb{K}\times\mathbb{K}}, \mathcal{B}_{\mathbb{K}})$-measurable. Thus, $f + g = \phi \circ F$ and $fg = \psi \circ F$ are $\mathcal{M}$-measurable. ■

This result holds for functions $f, g\colon X \to \overline{\mathbb{R}}$, if we take care with the indetermination $\infty - \infty$ and define by convention that $0 \cdot \infty = 0 \cdot (-\infty) = 0$ (see Exercise 2 ahead).

Now we will see how measurable functions behave under limits.

Proposition 3.1.10. *Let $\{f_j\}$ be a sequence of $\overline{\mathbb{R}}$ valued measurable functions on $(X, \mathcal{M})$. Then the functions:*

(a) $g_1(x) = \sup_j f_j(x)$,

(b) $g_2(x) = \limsup_{j\to\infty} f_j(x)$,

(c) $h_1(x) = \inf_j f_j(x)$,

(d) $h_2(x) = \liminf_{j\to\infty} f_j(x)$

are all $\mathcal{M}$-measurable. Also, if $f(x) = \lim_{j\to\infty} f(x)$ exists for every $x \in X$, then f is also $\mathcal{M}$-measurable.

Proof. We prove that

$$g_1^{-1}((a, \infty]) = \bigcup_{j=1}^{\infty} f_j^{-1}((a, \infty]).$$

In fact, let $x \in g_1^{-1}((a, \infty])$, hence $g_1(x) = \sup_j f_j(x) > a$. Thus there exists j_0 such that $f_{j_0}(x) > a$, for otherwise we would have $g_1(x) \leqslant a$ which is a contradiction, hence the $\subset$ inclusion holds. The converse inclusion $\supset$ is trivial. Thus g_1 is $\mathcal{M}$-measurable.

Analogously, we can show that $h_1^{-1}([-\infty, a)) = \bigcup_{j=1}^{\infty} f_j^{-1}([-\infty, a))$, and thus h_1 is also $\mathcal{M}$-measurable.

Now, for each $k \in \mathbb{N}$, the function $r_k(x) = \sup\limits_{j>k} f_j(x)$ is $\mathcal{M}$-measurable by (a), and hence $g_2(x) = \inf\limits_{k} r_k(x)$ is $\mathcal{M}$-measurable. Analogously h_2 is $\mathcal{M}$-measurable. When the limit exists, we have $f = g_2 = h_2$, and f is also $\mathcal{M}$-measurable. ■

Corollary 3.1.11. *If $f, g\colon X \to \overline{\mathbb{R}}$ are $\mathcal{M}$-measurable, then so are $\max(f, g)$ and $\min(f, g)$.*

Proof. Use the previous result to the sequence $f_1 = f$, $f_2 = g$ and $f_n = g$ for $n \geqslant 3$. ■

Corollary 3.1.12. *If $\{f_j\}$ is a sequence of complex-valued functions and $f(x) = \lim\limits_{j\to\infty} f_j(x)$ exists for all x, then f is measurable.*

Proof. Just apply Corollary 3.1.8. ■

3.1.1 | Decompositions of functions

For future use, we will present two useful decompositions of functions.

Definition 3.1.13. *Let $f\colon X \to \overline{\mathbb{R}}$. Then we define*

$$f^+(x) = \max(f(x), 0) \qquad \text{and} \qquad f^-(x) = \max(-f(x), 0) = -\min(f(x), 0),$$

the **positive** *and* **negative parts** *of f, respectively.*

Clearly we have

$$f = f^+ - f^- \qquad \text{and} \qquad |f| = f^+ + f^-,$$

and more specifically we have $f(x) = f^+(x)$ iff $f(x) \geqslant 0$ and $f(x) = -f^-(x)$ iff $f(x) < 0$. Also, if f is measurable, then both f^+ and f^- are, by Corollary 3.1.11. Also, this implies that if f is measurable, then $|f|$ is also measurable (the converse is not true in general).

Before presenting the other decomposition, we recall the **sign function** in $\mathbb{C}$, given by

$$\operatorname{sgn}(z) = \begin{cases} \dfrac{z}{|z|} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases} \tag{3.1.1}$$

Thus we have $z = |z|\operatorname{sgn}(z)$ for all $z \in \mathbb{C}$.

Definition 3.1.14. *Let $f: X \to \mathbb{C}$. We define the* **polar decomposition** *of f as*

$$f(x) = |f(x)|\mathrm{sgn}(f(x)) \quad \text{for all } x \in X.$$

Proposition 3.1.15. *If f is measurable, then so are $|f|$ and $\mathrm{sgn}(f)$.*

Proof. The function $z \mapsto |z|$ is continuous on $\mathbb{C}$, hence $|f|$ is measurable if f is. Now, the function $z \mapsto \mathrm{sgn}(z)$ is continuous on $\mathbb{C} \setminus \{0\}$. Hence, if $U \subset \mathbb{C}$ is open, then $\mathrm{sgn}^{-1}(U)$ is either open (when $0 \notin U$), or of the form $V \cup \{0\}$ where V is open (when $0 \in U$). In either one of these cases, $\mathrm{sgn}^{-1}(U)$ is a Borel set, hence sgn is Borel measurable. Hence $\mathrm{sgn}(f) = \mathrm{sgn} \circ f$ is measurable. ∎

3.1.2 | Simple functions

We will now discuss the concept of *simple functions*, which are the building block for the theory of integration. To begin, we set $(X, \mathcal{M})$ a measurable space, and we need first the following definition:

Definition 3.1.16. *Let $E \subset X$. The function defined by*

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E, \end{cases} \tag{3.1.2}$$

is called the **characteristic function** *of E (also known as* **indicator function** *of E, and also denoted by 1_E).*

Proposition 3.1.17. *$\chi_E: X \to \mathbb{R}$ is $\mathcal{M}$-measurable if and only if $E \in \mathcal{M}$.*

Proof. If χ_E is $\mathcal{M}$-measurable, then $E = \chi_E^{-1}([1, \infty)) \in \mathcal{M}$. Now if $E \in \mathcal{M}$, then

$$\chi_E^{-1}([a, \infty)) = \begin{cases} X & \text{if } a \leqslant 0, \\ E & \text{if } 0 < a \leqslant 1, \\ \varnothing & \text{if } a > 1, \end{cases}$$

and $\chi_E^{-1}([a, \infty)) \in \mathcal{M}$ in any case, hence χ_E is $\mathcal{M}$-measurable. ∎

Definition 3.1.18. *A function $s: X \to \mathbb{C}$ is said to be a* **simple function** *if there exists $c_1, \cdots, c_n \in \mathbb{C}$ and $E_1, \cdots, E_n \in \mathcal{M}$ such that*

$$s(x) = \sum_{j=1}^{n} c_j \chi_{E_j}(x) \quad \text{for all } x \in X.$$

The following proposition is straightforward.

Proposition 3.1.19. *A function $s\colon X \to \mathbb{C}$ is simple if and only if s is $\mathcal{M}$-measurable and $s(X)$ is finite.*

In fact, if s is simple, we can write

$$s(x) = \sum_{j=1}^{n} c_j \chi_{E_j}(x) \quad \text{for each } x \in X, \tag{3.1.3}$$

where $\{c_j\}_{j=1}^n$ is a finite sequence of distinct elements and $E_j = s^{-1}(\{c_j\})$ for each $j = 1, \cdots, n$. The decomposition (3.1.3) is called the **standard representation** of s, and writes s as a finite linear combination with distinct coefficients, of characteristic functions of disjoint measurable sets whose union if the whole space X.

On note to remember that even if one of the c_j is zero (that can happen), we still need to envision $c_j\chi_{E_j}$ as a part of the standard representation, since the set E_j may have a role to play when f interacts with other functions.

Clearly, when s and r are simple functions, then so are $s + r$ and sr. We will see that we can approximate any measurable function by a sequence of simple functions, in a very well behaved way.

Theorem 3.1.20. *If $f\colon X \to \overline{\mathbb{R}}$ is a $\mathcal{M}$-measurable function with $f(x) \geqslant 0$ for all $x \in X$, then there exists a sequence $\{s_n\}$ of real simple functions such that $0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant f$ such that $s_n \to f$ pointwise and $s_n \to f$ uniformly on any set on which f is bounded.*

Proof. Fix $n \in \mathbb{N}$ and we split $[0, \infty]$ in the following way:

$$J_n^0 = [0, 2^{-n}] \quad \text{and} \quad J_n^k = (k2^{-n}, (k+1)2^{-n}],$$

for $k = 1, \cdots, 2^{2n}-1$. Hence $\bigcup_{k=0}^{2^{2n}-1} J_n^k = [0, 2^n]$, and we set $I_n = (2^n, \infty]$. Thus $\{J_n^k\}_{k=1}^{2^{2n}-1} \cup \{I_n\}$ is a finite sequence of disjoint sets whose union is $[0, \infty]$, for each n.

Now we define $E_n^k = f^{-1}(J_n^k)$ and $F_n = f^{-1}(I_n)$, then $\{E_n^k\}_{k=1}^{2^{2n}-1} \cup \{F_n\}$ is a sequence of pairwise disjoint sets whose union is X, and since f is measurable, each one of these sets is measurable. Also $0 \leqslant f(x) \leqslant 2^{-n}$ for $x \in E_n^0$, $k2^{-n} < f(x) \leqslant (k+1)2^{-n}$ for $x \in E_n^k$ $(k = 1, \cdots, 2^{2n} - 1)$ and $f(x) > 2^n$ for $x \in F_n$.

Hence, for each n, define

$$s_n(x) = \sum_{k=0}^{2^{2n}-1} k2^{-n}\chi_{E_n^k}(x) + 2^n\chi_{F_n}(x) \quad \text{for each } x \in X.$$

Therefore, by construction, $0 \leqslant s_n \leqslant s_{n+1} \leqslant f$ for each n, and also $0 \leqslant f(x) - s_n(x) \leqslant 2^{-n}$ for each $x \in \bigcup_{k=0}^{2^{2n}-1} E_n^k$, and the result follows. ■

One important thing to notice is that we are splitting the image of f in intervals, and using these intervals to split the domain of f. This process is different from what we do in the classical theory of Riemann integration, where we split the domain in intervals.

Corollary 3.1.21. *If $f\colon X \to \overline{\mathbb{R}}$ is measurable, there exists a sequence $\{s_n\}$ of real simple functions such that $0 \leqslant |s_1| \leqslant |s_2| \leqslant \cdots \leqslant |f|$, $s_n \to f$ pointwise and $s_n \to f$ uniformly on any set on which f is bounded.*

Proof. Using Theorem 3.1.20 for f^+ and f^- we obtain sequences $\{q_n\}$ and $\{r_n\}$ of real simple functions with $0 \leqslant q_1 \leqslant q_2 \leqslant \cdots \leqslant f^+$, $0 \leqslant r_1 \leqslant r_2 \leqslant \cdots \leqslant f^-$, with $q_n \to f^+$ and $r_n \to f^-$ pointwise, and uniformly for sets on which f^+ and f^- are bounded, respectively. Setting $A = f^{-1}([0, \infty])$ and $B = A^c$, we know that $A, B \in \mathcal{M}$, $A \cup B = X$, $q_n|_B = 0$ and $r_n|_A = 0$ for each n (since $0 \leqslant s_n \leqslant f^+$ and $0 \leqslant r_n \leqslant f^-$ for all n).

Set $s_n = q_n - r_n$ for each n. Then, for each n, we have $0 \leqslant |s_n| \leqslant q_n + r_n \leqslant f^+ + f^- = |f|$, and also $f - s_n = (f^+ - q_n) - (f^- - r_n)$, hence $s_n \to f$ pointwise, and $s_n \to f$ uniformly on set which f is bounded.

Now if remains to prove that $|s_n| \leqslant |s_{n+1}|$ for each n. If $x \in A$ then $|s_n(x)| = q_n(x) \leqslant q_{n+1}(x) = s_{n+1}(x) = |s_{n+1}(x)|$ (since $r_n(x) = 0$ for $x \in A$ for all n). Analogously, if $x \in B$, then $|s_n(x)| = r_n(x) \leqslant r_{n+1}(x) = |s_{n+1}(x)|$ (since $s_n(x) = 0$ for $x \in B$ for all n), and we conclude the proof. ■

Proposition 3.1.22. *If $f\colon X \to \mathbb{C}$ is measurable, there exists a sequence $\{s_n\}$ of simple functions such that $0 \leqslant |s_1| \leqslant |s_2| \leqslant \cdots \leqslant |f|$, $s_n \to f$ pointwise and $s_n \to f$ uniformly on any set on which f is bounded.*

Proof. By Corollary 3.1.8, $\mathrm{Re} f\colon X \to \mathbb{R}$ and $\mathrm{Im} f\colon X \to \mathbb{R}$ are measurable, and hence, by Corollary 3.1.21, there are sequences of simple functions $\{s_n^1\}$ and $\{s_n^2\}$ such that $0 \leqslant |s_n^1| \leqslant |s_2^1| \leqslant \cdots \leqslant |\mathrm{Re}(f)|$, $0 \leqslant |s_1^2| \leqslant |s_2^2| \leqslant \cdots \leqslant |\mathrm{Im}(f)|$, $s_n^1 \to \mathrm{Re} f$ and $s_n^2 \to \mathrm{Im} f$ pointwise, and uniformly on sets which $\mathrm{Re} f$ and $\mathrm{Im} f$ are bounded respectively.

Thus the sequence $s_n = s_n^1 + i s_n^2$ for each n has the desired properties. ■

3.1.3 | Measurability of functions on complete measure spaces

We fix $(X, \mathcal{M}, \mu)$ a measure space. When we want study measurable functions, it is advantageous to exclude the behavior of measurable functions on μ-null sets. On this note, this study is much simpler when μ is complete.

Proposition 3.1.23. *The following implications are true if and only if μ is complete.*

(a) *If f is measurable and $f = g$ μ-a.e., then g is measurable.*

(b) *If f_n is measurable for $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e., then f is measurable.*

Proof. Assume that μ is a complete measure. We prove (a) and (b).

(a) Let $N = \{x \in X\colon f(x) \neq g(x)\}$. Then by hypothesis, $N \in \mathcal{M}$ and $\mu(N) = 0$. Thus

$$g^{-1}((a,\infty)) = \underbrace{(g^{-1}((a,\infty)) \cap N)}_{\subset N} \cup \underbrace{(f^{-1}((a,\infty)) \cap N^c)}_{\in \mathcal{M}} \in \mathcal{M},$$

since μ is complete (and thus $g^{-1}((a,\infty)) \cap N \in \mathcal{M}$). Therefore g is measurable.

(b) Let $g = \limsup\limits_{n\to\infty} f_n$. Then g is measurable by Proposition 3.1.10, and $g = f$ μ-a.e.. By item (a), f is measurable.

Now we prove that if these implications are true, then μ is a complete measure. In fact, assume that μ is not complete. Then there exists a μ-null set N (that is, $N \in \mathcal{M}$ and $\mu(N) = 0$) and a subset F of N which is not measurable.

Define $f = \chi_N$ and $g = \chi_F$. Then if $x \notin N$ we have $f(x) = g(x)$, thus $f = g$ μ-a.e., but f is measurable and g is not, thus (a) does not hold. For (b), define $f_n = (1 + \frac{1}{1+n})\chi_N$ for $n \in \mathbb{N}$ and $g = \chi_F$. Hence $f_n \to f$ uniformly on X and thus $f_n \to g$ μ-a.e., and (b) also does not hold. ∎

However, there is no much harm if we forget about the completeness of the measure.

Proposition 3.1.24. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. If f is an $\overline{\mathcal{M}}$-measurable function on X, there is an $\mathcal{M}$-measurable function g such that $f = g$ $\overline{\mu}$-almost everywhere.*

Proof. First, assume that $f = \chi_E$ for some $E \in \overline{\mathcal{M}}$. Thus $E = G \cup F$, where $G \in \mathcal{M}$, $F \subset N \in \mathcal{N}$ (see the notation in Theorem 2.1.9), and we can assume that $G \cap F = \varnothing$. Set $g = \chi_G$, which is $\mathcal{M}$-measurable. Then $\{x \in X\colon f(x) \neq g(x)\} = F$ which is a $\overline{\mu}$-null set. Thus the result is true if f is a $\overline{\mathcal{M}}$-measurable simple function. For the general case, let $\{\phi_n\}$ be a sequence of $\overline{\mathcal{M}}$-measurable simple functions such that $\phi_n \to f$, and for each n, choose a

$\mathcal{M}$-measurable simple function ψ_n, such that $\psi_n = \phi_n$ outside a set $E_n \in \overline{\mathcal{M}}$ with $\overline{\mu}(E_n) = 0$. For each n, there exists a μ-null set N_n such that $E_n \subset N_n$. Hence, setting $N = \bigcup_{n=1}^{\infty} N_n$, we have $N \in \mathcal{M}$, $\mu(N) = 0$ and $\bigcup_{n=1}^{\infty} E_n \subset N$. Now define $g = \lim_{n\to\infty} \chi_{X\setminus N}\psi_n$. Then g is the limit of a sequence of $\mathcal{M}$-measurable functions, hence g is $\mathcal{M}$-measurable and

$$g(x) = \begin{cases} \lim_{n\to\infty} \psi_n(x) = \lim_{n\to\infty} \phi_n(x) = f(x) & \text{if } x \in N^c, \\ 0 & \text{if } x \in N, \end{cases}$$

since if $x \in N^c$ then $x \notin \bigcup_{n=1}^{\infty} E_n$ and hence $\psi_n = \phi_n$ for all n. Therefore $f = g$ except possibly in N, which is a μ-null set (and thus a $\overline{\mu}$-null set). ■

We end this section with a final result regarding properties that hold almost everywhere and completion of measure.

Proposition 3.1.25. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. A property P holds μ-a.e. if and only if it hold $\overline{\mu}$-a.e.*

Proof. Assume that P holds μ-a.e. Since $\mathcal{M} \subset \overline{\mathcal{M}}$ and $\overline{\mu} = \mu$ on $\mathcal{M}$, P holds $\overline{\mu}$-a.e.

Now assume that P holds $\overline{\mu}$-a.e. Thus there exists $E \in \overline{\mathcal{M}}$ with $\overline{\mu}(E) = 0$ and such that P holds in $X \setminus E$. By Proposition 2.1.10, there exists $N \in \mathcal{M}$ with $\mu(N) = 0$ and $E \subset N$. Hence $X \setminus N \subset X \setminus E$, and thus P holds in $X \setminus N$. Therefore P holds μ-a.e. ■

3.2| Solved exercises from [1, Page 48]

In Exercises 1-7, $(X, \mathcal{M})$ is a measurable space.

Exercise 1. Let $f\colon X \to \overline{\mathbb{R}}$ and $Y = f^{-1}(\mathbb{R})$. Then f is measurable iff $f^{-1}(\{-\infty\}) \in \mathcal{M}$, $f^{-1}(\{\infty\}) \in \mathcal{M}$ and $f|_Y\colon Y \to \mathbb{R}$ is measurable.

Solution. Assume that f is measurable. Firstly note that, since $\mathbb{R} \in \mathcal{B}_{\overline{\mathbb{R}}}$, we have $Y \in \mathcal{M}$. In Y we consider the σ-algebra $\mathcal{M}_Y = \{E \cap Y\colon E \in \mathcal{M}\} \subset \mathcal{M}$, and we have

$$(f|_Y)^{-1}((a,\infty)) = \underbrace{f^{-1}((a,\infty))}_{\in \mathcal{M}} \cap Y \in \mathcal{M}_Y,$$

since $(a,\infty) \in \mathcal{B}_{\overline{\mathbb{R}}}$ for each $a \in \mathbb{R}$, and this implies that if B is a real Borel set, then $(f|_Y)^{-1}(B) \in \mathcal{M}$. Now $\{\infty\} = \bigcap_{n=1}^{\infty}(n,\infty] \in \mathcal{B}_{\overline{\mathbb{R}}}$ and hence $f^{-1}(\{\infty\}) \in \mathcal{M}$, since f is measurable. Analogously for $\{-\infty\}$.

Conversely, note that $Y = X \setminus f^{-1}(\{\pm\infty\})$, and since both X and $\{\pm\infty\}$ are in $\mathcal{M}$ (the latter by hypothesis), we have $Y \in \mathcal{M}$ and thus $\mathcal{M}_Y \subset \mathcal{M}$. Let $B \in \mathcal{B}_{\overline{\mathbb{R}}}$. Hence

$$f^{-1}(B) = f^{-1}(B \cap \mathbb{R}) \cup f^{-1}(B \cap \{\pm\infty\}) = \underbrace{(f|_Y)^{-1}(B \cap \mathbb{R})}_{\in \mathcal{M}_Y \subset \mathcal{M}} \cup \underbrace{f^{-1}(B \cap \{\pm\infty\})}_{\in \mathcal{M}} \in \mathcal{M},$$

since $B \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$ by Theorem 1.3.4. Thus f is measurable.

Exercise 2. Suppose $f, g \colon X \to \overline{\mathbb{R}}$ are measurable.

(a) fg is measurable (where $0 \cdot (\pm\infty) = 0$).

(b) Fix $a \in \overline{\mathbb{R}}$ and define

$$h(x) = \begin{cases} a & \text{if } f(x) = -g(x) = \pm\infty, \\ f(x) + g(x) & \text{otherwise.} \end{cases}$$

Then h is measurable.

Solution to (b). We will firstly prove item (b). To that end, note that the set

$$\begin{aligned} Y_\infty &= \{x \in X \colon f(x) = -g(x) = \pm\infty\} \\ &= f^{-1}(\{\infty\}) \cap g^{-1}(\{-\infty\}) \cup f^{-1}(\{-\infty\}) \cap g^{-1}(\{\infty\}) \end{aligned}$$

is measurable, since f and g are measurable functions.

Now we have

$$h^{-1}(\{\infty\}) = f^{-1}(\infty) \cap g^{-1}((-\infty, \infty]) \cup f^{-1}((-\infty, \infty]) \cap g^{-1}(\{\infty\})$$

and again, since both f and g are measurable, $h^{-1}(\{\infty\})$ is measurable. Analogously for $h^{-1}(\{-\infty\})$. Now, let $b \in \mathbb{R}$. We have

$$h^{-1}((b, \infty]) = h^{-1}((b, \infty)) \cup h^{-1}(\{\infty\}),$$

and

$$h^{-1}((b, \infty)) = \begin{cases} (f+g)^{-1}((b, \infty)) & \text{if } a \leqslant b \\ (f+g)^{-1}((b, \infty)) \cup Y_\infty & \text{if } b < a, \end{cases}$$

and since $f|_{f^{-1}(\mathbb{R})}$ and $g|_{g^{-1}(\mathbb{R})}$ are measurable, and Y_∞ is a measurable set, we have $h^{-1}((b, \infty))$ measurable. Hence $h^{-1}((b, \infty])$ is measurable, and therefore h is measurable.

Solution to (a). Let $\mathbb{Q}^+ = \{r \in \mathbb{Q} \colon r > 0\}$ and $\mathbb{Q}^- = \{r \in \mathbb{Q} \colon r < 0\}$, which are countable sets.

Before continuing, we will prove the following claim: let $y_1, y_2 > 0$ be such that $y_1 y_2 > b \geqslant 0$. Then there exists $r \in \mathbb{Q}^+$ such that $y_1 > r$ and $y_2 > \frac{b}{r}$. Indeed, if $b = 0$, then choose any $r \in \mathbb{Q}^+$ such that $r < y_1$. Now we assume that $b > 0$. Then we choose $r \in \mathbb{Q}^+$ such that $r < y_1$ and $ry_2 > b$ (such r exists, for otherwise $ry_2 \leqslant b$ for all $r < y_1$, and by density of $\mathbb{Q}$, we obtain $y_1 y_2 \leqslant b$, which is a contradiction).

Now assume that $f, g \geqslant 0$ and $b \geqslant 0$. Using our claim, we can write

$$(fg)^{-1}((b,\infty]) = \{x \in X \colon f(x)g(x) > b\} = \bigcup_{r \in \mathbb{Q}^+} f^{-1}((r,\infty]) \cap g^{-1}((b/r,\infty]),$$

and hence $(fg)^{-1}((b,\infty])$ is measurable. If $b < 0$ then $(fg)^{-1}((b,\infty]) = X$, also measurable. Therefore fg is measurable.

Now for the general case, consider the measurable functions $f^+, f^-, g^+, g^- \geqslant 0$, which are all measurable, such that $f = f^+ - f^-$ and $g = g^+ - g^-$. We have

$$fg = (f^+ - f^-)(g^+ - g^-) = \underbrace{f^+g^+ + f^-g^-}_{:=F} \underbrace{-(f^+g^- + f^-g^+)}_{:=G},$$

where $F \geqslant 0$ and $-G \geqslant 0$ are measurable by our previous computations (and hence G is also measurable). Now we prove that $\{x \in X \colon F(x) = -G(x) = \infty\} = \varnothing$.

If $F(x) = \infty$ we have $f^+(x)g^+(x) + f^-(x)g^-(x) = \infty$. We will brake this situation into two cases:

Case 1: Either $f^+(x) = \infty$ and $g^+(x) > 0$ or $f^+(x) > 0$ and $g^+(x) = \infty$. In this case $f^-(x) = g^-(x) = 0$ and we have $G(x) = 0$.

Case 2: Either $f^-(x) = \infty$ and $g^-(x) > 0$ or $f^-(x) > 0$ and $g^-(x) = \infty$. In this case $f^+(x) = g^+(x) = 0$ and $G(x) = 0$.

Hence $F(x) = \infty$ implies $G(x) = 0$ and thus $\{x \in X \colon F(x) = -G(x) = \infty\} = \varnothing$. Thus by item (b), $fg = F + G$ is measurable.

EXERCISE 3. If $\{f_n\}$ is a sequence of measurable functions on X, then $\{x \in X \colon \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is a measurable set.

Solution. As in Proposition 3.1.10, set

$$g_2(x) = \limsup_{n\to\infty} f_n(x) \quad \text{and} \quad h_2(x) = \liminf_{n\to\infty} f_n(x),$$

which are measurable functions. Hence, by the previous exercise, the function

$$F(x) = \begin{cases} 1 & \text{if } g_2(x) = h_2(x) = \pm\infty \\ g_2(x) - h_2(x) & \text{otherwise,} \end{cases}$$

is measurable. Furthermore $\{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\} = F^{-1}(0)$, and thus it is a measurable set.

Exercise 4. If $f\colon X \to \overline{\mathbb{R}}$ and $f^{-1}((r,\infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$, then f is measurable.

Solution. Let $a \in \mathbb{R}$ and $\{r_n\}$ a decreasing sequence in $\mathbb{Q}$ such that $a = \lim_{n\to\infty} r_n$. Then

$$f^{-1}((a,\infty]) = \bigcup_{n=1}^{\infty} f^{-1}((r_n,\infty]) \in \mathcal{M},$$

hence f is measurable.

Exercise 5. If $X = A \cup B$ where $A, B \in \mathcal{M}$, a function on X is measurable iff f is measurable on A and B.

Solution. Let $f_A = f|_A$ and $f_B = f|_B$. If f is measurable, then for each $C \in \mathcal{B}_{\overline{\mathbb{R}}}$ we have

$$(f_J)^{-1}(C) = f^{-1}(C) \cap J \in \mathcal{M},$$

for $J = A, B$. Hence f_A and f_B are measurable. Now for the converse, note that

$$f^{-1}(C) = (f^{-1}(C) \cap A) \cup (f^{-1}(C) \cap B) = (f_A)^{-1}(C) \cup (f_B)^{-1}(C) \in \mathcal{M},$$

and thus f is measurable.

Exercise 6. The supremum of an uncountable family of measurable $\overline{\mathbb{R}}$-valued functions on X can fail to be measurable (unless the σ-algebra $\mathcal{M}$ is very special).

Solution. Assume that X is uncountable and $\mathcal{M}$ is σ-algebra such that $\{x\} \in \mathcal{M}$ (and therefore each countable set is measurable). Assume that there exists a nonmeasurable set F in X. Define, for each $x \in X$, the function $f_x\colon X \to \overline{\mathbb{R}}$ by

$$f_x(y) = \chi_{\{x\}}(y) \quad \text{for each } y \in X.$$

Hence $\{f_x\}_{x\in F}$ is an uncountable family of measurable functions and

$$\sup_{x\in N} f_x(y) = \chi_F(y) \quad \text{for all } y \in X,$$

which is not a measurable function.

Exercise 7. Suppose that for each $\alpha \in \mathbb{R}$ we are given a set $E_\alpha \in \mathcal{M}$ such that $E_\alpha \subset E_\beta$ whenever $\alpha < \beta$, $\bigcup_{\alpha\in\mathbb{R}} E_\alpha = X$ and $\bigcap_{\alpha\in\mathbb{R}} E_\alpha = \varnothing$. Then there is a measurable function $f\colon X \to \mathbb{R}$ such that $f(x) \leqslant \alpha$ on E_α and $f(x) \geqslant \alpha$ on E_α^c for every α (Use Exercise 4).

Solution. We define

$$f(x) = \inf\{\alpha \in \mathbb{R} \colon x \in E_\alpha\} \quad \text{for each } x \in \mathbb{R}.$$

We claim that this function satisfies all the required conditions. First, note that by construction, we have $f(x) \leqslant \alpha$ if $x \in E_\alpha$. Also, if $x \in E_\alpha^c$ then $x \notin E_\alpha$ and hence $x \notin E_\beta$ for all $\beta \leqslant \alpha$, hence $f(x) \geqslant \alpha$.

Now, since $\bigcup_{\alpha\in\mathbb{R}} E_\alpha = X = \bigcup_{\alpha\in\mathbb{R}} E_\alpha^c$ (since $\bigcap_{\alpha\in\mathbb{R}} E_\alpha = \varnothing$), for any given $x \in X$ there exist $\alpha, \beta \in \mathbb{R}$ such that $x \in E_\alpha \cap E_\beta^c$, this implies that $\alpha \leqslant \beta$ and

$$-\infty < \beta \leqslant f(x) \leqslant \alpha < \infty,$$

and thus we have shown that $f(X) \subset \mathbb{R}$.

It remains to prove the measurability of f. To that end, note that if $f(x) < r$ then there exists $\alpha \in \mathbb{R}$ such that $f(x) < \alpha < r$ and $x \in E_\alpha$. Since $E_\beta \subset E_\alpha$ for $\beta < \alpha$, then for any $\alpha < q < r$ with $q \in \mathbb{Q}$ we have $x \in E_q$. Conversely if $q \in \mathbb{Q}$ is such that $q < r$ and $x \in E_q$ then $f(x) \leqslant q < r$. We have just proved that

$$f^{-1}((-\infty, r)) = \bigcup_{q<r,\ q\in\mathbb{Q}} E_q \in \mathcal{M}.$$

Thus

$$f^{-1}([r, \infty)) = \bigcap_{q<r,\ q\in\mathbb{Q}} E_q^c \in \mathcal{M},$$

for each $r \in \mathbb{Q}$. Therefore, by Exercise 4, f is measurable.

Exercise 8. If $f\colon \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

Solution. Since f is measurable iff $-f$ is measurable, we can assume without loss of generality that f is monotonically increasing. Now let $a \in \mathbb{R}$ and $x \in f^{-1}([a,\infty))$, that is, $f(x) \geqslant a$. If $y \geqslant x$, then $a \leqslant f(x) \leqslant f(y)$, which implies that $y \in f^{-1}([a,\infty))$.

In other words, we have proven that if $x \in f^{-1}([a,\infty))$ then the ray $[x,\infty) \subset f^{-1}([a,\infty))$, and thus $f^{-1}([a,\infty))$ is an interval. Therefore f is Borel measurable.

Exercise 9. Let $f\colon [0,1] \to [0,1]$ be the Cantor function and let $g(x) = f(x) + x$.

(a) g is a bijection from $[0,1]$ to $[0,2]$ and $h = g^{-1}$ is continuous from $[0,2]$ to $[0,1]$.

(b) If C is the Cantor set $m(g(C)) = 1$.

(c) Be Exercise 29 of Chapter 1, $g(C)$ contains a Lebesgue nonmeasurable set A. Let $B = g^{-1}(A)$. Then B is Lebesgue measurable but no Borel.

(d) There exist a Lebesgue measurable function F and a continuous function G on $\mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.

Solution to (a). g is a continuous (sum of two continuous functions) and increasing (sum of two increasing functions) such that $g(0) = 0$ and $g(1) = f(1) + 1 = 2$. If we show that g is strictly increasing, the g will be a bijection. Let $0 \leqslant x < y \leqslant 1$. Then $f(x) \leqslant f(x)$ and $x < y$, then $g(x) < g(y)$, and g is strictly increasing. Since $[0,1]$ is compact and $\mathbb{R}$ is a Hausdorff space, its inverse $h = g^{-1}$ is continuous (see [2, Theorem 26.6])
Solution to (b). Using that g is a bijection, we can write

$$[0,2] = g([0,1]) = g(([0,1] \setminus C) \cup C) = g([0,1] \setminus C) \cup g(C),$$

and hence

$$m(g(C)) = 2 - m(g([0,1] \setminus C)).$$

Now C is closed, and hence $[0,1] \setminus C$ is open, and can be written as a countable union of disjoint open intervals, namely $[0,1] \setminus C = \bigcup\limits_{n=1}^{\infty} I_n$, where $\{I_n = (a_n, b_n)\}$ is a pairwise disjoint family of open intervals. Hence

$$\begin{aligned} m(g([0,1] \setminus C)) &= m\Big(g\Big(\bigcup_{n=1}^{\infty} I_n\Big)\Big) = m\Big(\bigcup_{n=1}^{\infty} g(I_n)\Big) \\ &= \sum_{n=1}^{\infty} m(g(I_n)) = \sum_{n=1}^{\infty} m((f(a_n) + a_n, f(b_n) + b_n)) \\ &= \sum_{n=1}^{\infty} [f(b_n) - f(a_n) + b_n - a_n], \end{aligned}$$

and recalling that f is constant in each interval outside C, hence is constant on each I_n, we have $f(b_n) = f(a_n)$ for all n and thus

$$m(g([0,1] \setminus C)) = \sum_{n=1}^{\infty} [b_n - a_n] = m\Big(\bigcup_{n=1}^{\infty} I_n\Big) = m([0,1] \setminus C) = m([0,1]) - m(C) = 1,$$

since $m(C) = 0$. Therefore $m(g(C)) = 1$.

Solution to (c). Note that since $A \subset g(C)$, then $B = g^{-1}(A) \subset C$. But $m(C) = 0$ and m is a complete measure, thus B is Lebesgue measurable. Now assume that B is Borel measurable. Since g^{-1} is continuous, $A = g(B) = (g^{-1})^{-1}(B)$ is a Borel set, which is a contradiction, since A is not Lebesgue measurable.

Solution to (d). Define $F = \chi_B$ and $G = g^{-1}$. Thus F is Lebesgue measurable (since B is a Lebesgue measurable set), and G is continuous. But

$$(F \circ G)^{-1}((1/2, \infty)) = g(F^{-1}((1/2, \infty))) = g(B) = A,$$

which is not Lebesgue measurable. Hence $F \circ G$ is not Lebesgue measurable.

Exercise 10. Prove Proposition 3.1.23.

Exercise 11. Suppose that f is a function defined on $\mathbb{R} \times \mathbb{R}^k$ such that $f(x, \cdot)$ is Borel measurable for each $x \in \mathbb{R}$ and $f(\cdot, y)$ is continuous for each $y \in \mathbb{R}^k$. For $n \in \mathbb{N}$, define f_n as follows. For $i \in \mathbb{Z}$ let $a_i = i/n$, and for $a_i \leqslant x \leqslant a_{i+1}$ let

$$f_n(x, y) = \frac{f(a_{i+1}, y)(x - a_i) - f(a_i, y)(x - a_{i+1})}{a_{i+1} - a_i}.$$

Then f_n is Borel measurable on $\mathbb{R} \times \mathbb{R}^k$ and $f_n \to f$ pointwise; hence f is Borel measurable on $\mathbb{R} \times \mathbb{R}^k$. Conclude by induction that every function of $\mathbb{R}^n$ that is continuous in each variable separately is Borel measurable.

Solution. Note first that if $(x, y) \in \mathbb{R} \times \mathbb{R}^k$ we have

$$f_n(x, y) - f(x, y) = \frac{(f(a_{i+1}, y) - f(x, y))(x - a_i) - (f(a_i, y) - f(x, y))(x - a_{i+1})}{a_{i+1} - a_i}.$$

But $a_{i+1} - a_i = 1/n$, $|x - a_i| \leqslant 1/n$ and $|x - a_{i+1}| \leqslant 1/n$. Therefore

$$|f_n(x, y) - f(x, y)| \leqslant |f(a_{i+1}, y) - f(x, y)| + |f(a_i, y) - f(x, y)|,$$

and by the continuity of $f(\cdot, y)$, we have $f_n(x,y) \to f(x,y)$ as $n \to \infty$.

Now for the Borel measurability of f_n we proceed as follows: since $n \geqslant 1$ is fixed, we write $\mathbb{R} = \bigcup_{i\in\mathbb{Z}} [a_i, a_{i+1}]$. In $A_i = [a_i, a_{i+1}] \times \mathbb{R}^k$, $f_n|_{A_i}$ is the sum of product of Borel measurable functions with continuous functions, and hence, it is Borel measurable. Using a simple generalization of Exercise 5 (writing $\mathbb{R} \times \mathbb{R}^k = \bigcup_{i\in\mathbb{Z}} A_i$), we have f_n Borel measurable on $\mathbb{R} \times \mathbb{R}^k$.

Now we prove the last argument by induction. For $n = 2$ the claim follows from what we just proved, since if $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous in each variable separately then $f(x, \cdot)$ is continuous (hence Borel measurable) and $f(\cdot, y)$ is continuous. Assume that the claim is true for n and assume that $f\colon \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a function which is continuous in each variable separately. Using the induction, $f_{\mathbb{R}^n}$ is a function which is continuous in each variable separately, hence it is Borel measurable, that is, $f(x, \cdot)$ is Borel measurable for each $x \in \mathbb{R}^n$. Also by assumption $f(\cdot, y)$ is continuous for each $y \in \mathbb{R}$, and therefore it is Borel measurable.

3.3 | INTEGRATION OF NONNEGATIVE FUNCTIONS

From now on, we consider fixed a measure space $(X, \mathcal{M}, \mu)$ and we define

$$L^+ = L^+(X, \mathcal{M}) = \{f\colon X \to [0,\infty] \colon f \text{ is measurable}\}.$$

LEMMA 3.3.1. *Assume that $\phi \in L^+$ is a simple function with $\phi = \sum_{j=1}^n a_j \xi_{E_j} = \sum_{k=1}^m b_k \chi_{F_k}$ with $\{E_j\}_{j=1}^n$ and $\{F_k\}_{k=1}^m$ finite sequences of disjoint measurable sets. Then*

$$\sum_{j=1}^n a_j \mu(E_j) = \sum_{k=1}^m b_k \mu(F_k).$$

Proof. For each $j = 1, \cdots, n$ and $k = 1, \cdots, m$ we define $G_{j,k} = E_j \cap F_k$. We have $E_j = \bigcup_{k=1}^m G_{j,k}$ and $F_k = \bigcup_{j=1}^n G_{j,k}$ for each $j = 1, \cdots, n$ and $k = 1, \cdots, m$. Moreover if $G_{j,k} \neq \varnothing$, letting $x \in G_{j,k}$ we have $a_j = \phi(x) = b_k$. Hence we define

$$c_{j,k} = \begin{cases} a_j & \text{if } G_{j,k} \neq \varnothing, \\ 0 & \text{if } G_{j,k} = \varnothing. \end{cases}$$

Thus we have

$$\sum_{j=1}^{n} a_j\mu(E_j) = \sum_{j=1}^{n}\sum_{k=1}^{m} c_{j,k}\mu(G_{j,k}) = \sum_{k=1}^{m}\sum_{j=1}^{n} c_{j,k}\mu(G_{j,k}) = \sum_{k=1}^{m} b_k\mu(F_k),$$

and the proof is complete. ■

Definition 3.3.2. *If $\phi \in L^+$ is a simple function with standard representation $\phi = \sum_{j=1}^{n} a_j\chi_{E_j}$, we define the* **integral** *of ϕ with respect to μ by*

$$\int \phi d\mu = \sum_{j=1}^{n} a_j\mu(E_j),$$

with the convention (as always) that $0 \cdot \infty = 0$.

We note that $\int \phi d\mu$ may be ∞, if $\mu(E_j) = \infty$ for some j on which $a_j > 0$. When there is no confusion on which is the measure μ, we will write $\int \phi$ instead of $\int \phi d\mu$. Also, sometimes it is convenient to display the argument of ϕ explicitly, and we can also use the notation $\int \phi(x)d\mu(x)$ (or $\int \phi\mu(dx)$). This integral is well defined, by Lemma 3.3.1.

Proposition 3.3.3. *Let $\phi \in L^+$ be a simple function and $A \in \mathcal{M}$. Then $\phi\chi_A$ is also a simple function in L^+ and*

$$\int \phi\chi_A d\mu = \sum_{j=1}^{n} a_j\mu(A \cap E_j).$$

Proof. Just note that $\phi\chi_A = \sum_{j=1}^{n} a_j\chi_{E_j\cap A}$ is the standard representation for $\phi\chi_A$, if $\sum_{j=1}^{n} a_j\chi_{E_j}$ is the standard representation for ϕ. ■

Definition 3.3.4. *If $\phi \in L^+$ is a simple function and $A \in \mathcal{M}$, we define*

$$\int_A \phi d\mu = \int \phi\chi_A d\mu = \sum_{j=1}^{n} a_j\mu(A \cap E_j).$$

The same remarks for $\int \phi d\mu$ also apply to $\int_A \phi d\mu$. Note also that $\int_X = \int$.

Proposition 3.3.5. *Let $\phi, \psi \in L^+$ be simple functions and $c \geqslant 0$. Then we have the following properties of the integral:*

(a) $\int c\phi = c\int \phi$;

(b) $\int (\phi + \psi) = \int \phi + \int \psi$;

(c) *if* $\phi \leqslant \psi$ *then* $\displaystyle\int \phi \leqslant \int \psi$*;*

(d) *the map* $\mathcal{M} \ni A \mapsto \int_A \phi d\mu$ *is a measure on* $\mathcal{M}$*.*

Proof. First of all we write $\phi = \sum_{j=1}^{n} a_j \chi_{E_j}$ and $\psi = \sum_{k=1}^{m} b_k \chi_{F_k}$ as their standard decompositions.

(a). Note that $c\phi = \sum_{j=1}^{n} c a_j \chi_{E_j}$ is the standard decomposition of the simple function $c\phi \in L^+$. Hence

$$\int c\phi = \sum_{j=1}^{n} c a_j \mu(E_j) = c \sum_{j=1}^{n} a_j \mu(E_j) = c \int \phi.$$

(b). We note that, for each j and k we have

$$E_j = \bigcup_{k=1}^{m} (E_j \cap F_k) \qquad \text{and} \qquad F_k = \bigcup_{j=1}^{n} (E_j \cap F_k),$$

since $X = \bigcup_{j=1}^{n} E_j = \bigcup_{k=1}^{m} F_k$. Hence $\phi + \psi$ is a L^+ simple function, with standard representation

$$\phi + \psi = \sum_{j=1}^{n} \sum_{k=1}^{m} (a_j + b_k) \chi_{E_j \cap F_k},$$

and hence

$$\begin{aligned}
\int (\phi + \psi) &= \sum_{j=1}^{n} \sum_{k=1}^{m} (a_j + b_k) \mu(E_j \cap F_k) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{m} a_j \mu(E_j \cap F_k) + \sum_{j=1}^{n} \sum_{k=1}^{m} b_k \mu(E_j \cap F_k) \\
&= \sum_{j=1}^{n} a_j \mu\Big(\bigcup_{k=1}^{m} (E_j \cap F_k)\Big) + \sum_{k=1}^{m} b_k \mu\Big(\bigcup_{j=1}^{n} (E_j \cap F_k)\Big) \\
&= \sum_{j=1}^{n} a_j \mu(E_j) + \sum_{k=1}^{m} b_k \mu(F_k) = \int \phi + \int \psi.
\end{aligned}$$

(c). Note that, by the decomposition made on item (b), we can write $\phi = \sum_{j=1}^{n} \sum_{k=1}^{m} a_j \chi_{E_j \cap F_k}$ and $\psi = \sum_{j=1}^{n} \sum_{k=1}^{m} b_k \chi_{E_j \cap F_k}$. If $x \in E_j \cap F_k$, since $\phi \leqslant \psi$, we must have $a_j \leqslant b_k$, and hence

$$\int \phi = \sum_{j=1}^{n} \sum_{k=1}^{m} a_j \mu(E_j \cap F_k) \leqslant \sum_{j=1}^{n} \sum_{k=1}^{m} b_k \mu(E_j \cap F_k) = \int \psi.$$

(d). It is clear that $\int_\varnothing \phi = 0$. Now assume that $\{A_i\}$ is a pairwise disjoint sequences of sets in $\mathcal{M}$ and $A = \bigcup\limits_{i=1}^{\infty} A_i$. Then

$$\begin{aligned}\int_A \phi &= \sum_{j=1}^{n} a_j \mu(E_j \cap A) = \sum_{j=1}^{n} a_j \mu\Big(\bigcup_{i=1}^{\infty} (E_j \cap A_i)\Big) = \sum_{j=1}^{n} a_j \sum_{i=1}^{\infty} \mu(E_j \cap A_i)\\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{n} a_j \mu(E_j \cap A_i) = \sum_{i=1}^{\infty} \int_{A_i} \phi,\end{aligned}$$

and proves that $\mathcal{M} \ni A \mapsto \int_A \phi d\mu$ is a measure on $\mathcal{M}$. ■

Definition 3.3.6. *If $f \in L^+$ we define the* **integral** *of f with respect to μ as*

$$\int f d\mu = \sup \left\{ \int \phi d\mu \colon 0 \leqslant \phi \leqslant f,\ \phi \in L^+ \textit{ is simple} \right\}.$$

Using item (c) of this last proposition, if $f = \psi$ is an L^+ simple function, then this definition coincides with the first one.

Proposition 3.3.7. *Let $f, g \in L^+$ and $c \geqslant 0$. Then:*

(a) $\displaystyle\int cf = c \int f;$

(b) $\displaystyle\int f \leqslant \int g$ *if* $f \leqslant g$.

Proof. **(a).** This follows from Proposition 3.3.5 item (a) and the fact that $\sup(cE) = c \sup E$ for $E \subset \overline{\mathbb{R}}$ and $c \geqslant 0$.
(b). If $0 \leqslant \phi \leqslant f$ and $\phi \in L^+$ is simple, then $0 \leqslant \phi \leqslant g$, and the result follows. ■

The same remark applies here, that is, if $A \in \mathcal{M}$, then we define $\int_A f = \int f \chi_A$. This definition also coincides with

$$\int_A f = \sup \left\{ \int_A \phi d\mu \colon 0 \leqslant \phi \leqslant f \text{ in } A,\ \phi \in L^+ \text{ is simple} \right\}.$$

We will now begin to state and prove the fundamental theorems in the theory of integration.

Theorem 3.3.8 (Monotone Convergence Theorem (MCT))**.** *If $\{f_n\}$ is a sequence in L^+ such that $f_n \leqslant f_{n+1}$ for all n and $f = \lim\limits_{n\to\infty} f_n (= \sup\limits_n f_n)$, then*

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof. Since $f_n \leqslant f_{n+1}$ for all n, Proposition 3.3.7 item (b), the sequence $\{\int f_n\}$ is increasing, hence its limit exists in $\overline{\mathbb{R}}$. Moreover, since $f_n \leqslant f$ for all n, we also have $\int f_n \leqslant \int f$ for all n and therefore

$$\lim_{n\to\infty} \int f_n \leqslant \int f.$$

To achieve the other inequality, we fix $\alpha \in (0,1)$ and let $\phi \in L^+$ be a simple function with $0 \leqslant \phi \leqslant f$. Define $E_n = \{x \in X \colon f_n(x) \geqslant \alpha\phi(x)\}$.

Claim 1: each E_n is a measurable set.

In fact, since both f_n and ϕ are measurable, the function $g_n = f_n - \alpha\phi$ is measurable. Also $E_n = g_n^{-1}([0,\infty])$, and thus E_n is a measurable set.

Claim 2: $E_n \subset E_{n+1}$ for all n.

In fact if $x \in E_n$ then $f_n(x) \geqslant \alpha\phi(x)$. But $f_n(x) \leqslant f_{n+1}(x)$ and hence $f_{n+1}(x) \geqslant f_n(x) \geqslant \alpha\phi(x)$, that is $x \in E_{n+1}$, and prove the claim.

Claim 3: $X = \bigcup_{n=1}^{\infty} E_n$.

In fact, fix $x \in X$. If $f_n(x) < \alpha\phi(x)$ for all n, we would have $f(x) \leqslant \alpha\phi(x)$, which is a contradiction, since $\phi \leqslant f$ and $\alpha \in (0,1)$, and hence x must belong to some E_n.

Hence, since $\alpha\phi\chi_{E_n} \leqslant f_n\chi_{E_n} \leqslant f_n$ we have

$$\int f_n \geqslant \int_{E_n} f_n \geqslant \int_{E_n} \alpha\phi = \alpha \int_{E_n} \phi.$$

Thus, using the fact that $\mathcal{M} \ni A \mapsto \int_A \phi$ is a measure on $\mathcal{M}$, the property of continuity from below of measures and the fact that $X = \bigcup_{n=1}^{\infty} E_n$, we have $\lim_{n\to\infty} \int f_n \geqslant \alpha \int \phi$. Taking the supremum over all $\phi \in L^+$ which are simple and such that $0 \leqslant \phi \leqslant f$, we obtain $\lim_{n\to\infty} \int f_n \geqslant \alpha \int f$, and since this is true for any $\alpha \in (0,1)$, taking the limit when $\alpha \to 1^-$, we obtain

$$\lim_{n\to\infty} \int f_n \geqslant \int f,$$

which concludes the proof. ■

The monotone convergence is essential. It states that to compute $\int f$, we only need to compute $\int \phi_n$ where $\{\phi_n\}$ is an increasing sequences of simple functions in L^+ that converge pointwise to f, which exists from Theorem 3.1.20. With this theorem, we can also prove the additivity of the integral.

Theorem 3.3.9. *If $\{f_n\}$ is a finite of infinite sequence in L^+ and $f = \sum_n f_n$, then*

$$\int f = \sum_n \int f_n.$$

Proof. Assume that f_1 and f_2 are L^+ functions. Using Theorem 3.1.20 we can find increasing sequences $\{\phi_j\}$ and $\{\psi_j\}$ of simple functions in L^+ such that $\lim_{j\to\infty} \phi_j = f_1$ and $\lim_{j\to\infty} \psi_j = f_2$. Using the MCT and the properties of the integral for simple functions we have

$$\int (f_1 + f_2) = \lim_{j\to\infty} \int (\phi_j + \psi_j) = \lim_{j\to\infty} \int \phi_j + \lim_{j\to\infty} \int \psi_j = \int f_1 + \int f_2.$$

An induction argument concludes the case for a finite number of functions. Now assume that we have an infinite sequence $\{f_n\}$. For each n, set $g_n = \sum_{k=1}^n f_k$. Then $\{g_n\}$ is an increasing sequence in L^+ that converges to f, and by the MCT and the previous case of finite sequences we have

$$\int f = \lim_{n\to\infty} \int g_n = \lim_{n\to\infty} \int \sum_{k=1}^{n} f_k = \lim_{n\to\infty} \sum_{k=1}^{n} \int f_k = \sum_{k=1}^{\infty} \int f_k.$$

■

Proposition 3.3.10. *If $f \in L^+$ then $\int f = 0$ if and only if $f = 0$ a.e.*

Proof. Assume first that f is simple, that is, $f = \sum_{j=1}^{n} a_j \chi_{E_j}$ and hence $\int f = \sum_{j=1}^{n} a_j \mu(E_j)$. If $f = 0$ a.e. then the sets on which $a_j > 0$ we must have $\mu(E_j) = 0$, hence $\int f = 0$. Reciprocally, if $\int f = 0$, then either $a_j = 0$ or $\mu(E_j) = 0$, and hence $f = 0$ a.e..

If f is not simple, let ϕ be a simple L^+ function with $0 \leqslant \phi \leqslant f$. If $f = 0$ a.e. then $\phi = 0$ a.e. and $\int f = \sup_\phi \int \phi = 0$. For the converse we will write

$$\{x \in X : f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n \quad \text{where} \quad E_n = \{x \in X : f(x) > 1/n\}.$$

So if it is false that $f = 0$ a.e., then we must have $\mu(E_n) > 0$ for some n. But then if $\phi = 1/n\chi_{E_n}$ and $f \geqslant f\chi_{E_n} > \phi \geqslant 0$ and $\phi \in L^+$ is a simple function, hence

$$\int f \geqslant \int \phi = \tfrac{1}{n}\mu(E_n) > 0,$$

and contradicts the fact that $\int f = 0$. ■

Corollary 3.3.11 (Improved MCT)**.** *If $\{f_n\} \subset L^+$, $f \in L^+$, $f_n \leqslant f_{n+1}$ for all n and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e., then*

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof. Assume that $\lim_{n\to\infty} f_x(x) = f(x)$ for all $x \in E$, where $\mu(E^c) = 0$. Then we have the following:

(i) $\{f_n\chi_E\}$ is an increasing sequence in L^+ that converges to $f\chi_E \in L^+$ (for all $x \in X$);

(ii) $f - f\chi_E = 0$ and $f_n - f_n\chi_E = 0$ a.e. (that is, they are equal for all x in E).

From (i), (ii), the previous proposition and the MCT we have

$$\int f = \int f\chi_E = \lim_{n\to\infty} \int f_n\chi_E = \lim_{n\to\infty} \int f_n.$$

■

They hypothesis that the sequence $\{f_n\}$ is increasing is fundamental for the MCT. Consider, for instance, $X = \mathbb{R}$ and μ the Lebesgue measure. Then

$$\chi_{(n,n+1)} \to 0 \qquad \text{and} \qquad n\chi_{(0,1/n)} \to 0$$

pointwise, but $\int \chi_{(n,n+1)} = \int n\chi_{(0,1/n)} = 1$ for all n.

However, if we remove this hypothesis, one inequality still holds.

LEMMA 3.3.12 (Fatou's Lemma). *If $\{f_n\}$ is any sequence in L^+ then*

$$\int \liminf_{n\to\infty} f_n \leqslant \liminf_{n\to\infty} \int f_n.$$

Proof. For each $k \geqslant 1$ we have $\inf\limits_{n\geqslant k} f_n \leqslant f_j$ for all $j \geqslant k$, hence $\int \inf_{n\geqslant k} f_n \leqslant \int f_j$ for all $j \geqslant k$, and thus

$$\int \inf_{n\geqslant k} f_n \leqslant \inf_{j\geqslant k} \int f_j.$$

But $\{\inf\limits_{n\geqslant k} f_n\}_k$ is an increasing sequence of L^+ functions that converges to $\liminf\limits_{k\to\infty} f_k$, and by the MCT we have

$$\int \liminf_{k\to\infty} f_k = \lim_{k\to\infty} \int \inf_{n\geqslant k} f_n \leqslant \lim_{k\to\infty} \inf_{j\geqslant k} \int f_j = \liminf_{k\to\infty} \int f_k.$$

■

COROLLARY 3.3.13. *If $\{f_n\} \subset L^+$, $f \in L^+$ and $f_n \to f$ a.e., then*

$$\int f \leqslant \liminf_{n\to\infty} \int f_n.$$

Proof. If the convergence is everywhere, this is a direct consequence of Fatou's Lemma, since $f = \liminf\limits_{n\to\infty} f_n$. If the convergence is only a.e., we modify f and f_n on a null set, as done in the Improved MCT. ■

Proposition 3.3.14. *If $f \in L^+$ and $\int f < \infty$ then $\{x \in X : f(x) = \infty\}$ is a null set and $\{x \in X : f(x) > 0\}$ is a σ-finite set.*

Proof. Set $E_\infty = \{x \in X : f(x) = \infty\}$. Since f is measurable, E_∞ is a measurable set. If $\mu(E_\infty) > 0$, then $f \geqslant f\chi_{E_\infty} \geqslant n\chi_{E_\infty}$ for all n, hence

$$\int f \geqslant \int n\chi_{E_\infty} = n\mu(E_\infty),$$

and making $n \to \infty$ we obtain that $\int f = \infty$, which is a contradiction. Therefore $\mu(E_\infty) = 0$.

For the second part, let $E_n = \{x \in X : f(x) > 1/n\}$ for each n and $E = \{x \in X : f(x) > 0\}$. Then $E = \bigcup_{n=1}^{\infty} E_n$ and each E_n is a measurable set. We will show that $\mu(E_n) < \infty$ for each n. Assume by absurd that $\mu(E_n) = \infty$ for some n. Then $f \geqslant f\chi_{E_n} > (1/n)\chi_{E_n}$ and hence

$$\int f \geqslant \int (1/n)\chi_{E_n} = (1/n)\mu(E_n) = \infty,$$

which is a contradiction and proves that E is σ-finite. ■

3.4 | Solved exercises from [1, Page 52]

Exercise 12. Prove Proposition 3.3.14.

Solution. It is already proven in the text.

Exercise 13. Suppose $\{f_n\} \subset L^+$, $f_n \to f$ pointwise and $\int f = \lim \int f_n < \infty$. Then $\int_E f = \lim \int_E f_n$ for all $E \in \mathcal{M}$. However, this need not to be true if $\int f = \lim \int f_n = \infty$.

Solution. By Fatou's Lemma, since $f_n\chi_E \to f\chi_E$, we have

$$\int_E f \leqslant \liminf \int_E f_n.$$

If we can prove that $\limsup \int_E f_n \leqslant \int_E f$, then we are done. To that end, note that if $E \in \mathcal{M}$ we have $f\chi_E \leqslant f$ and hence

$$\int_E f = \int f\chi_E \leqslant \int f < \infty,$$

and thus $\int_E f < \infty$ for all $E \in \mathcal{M}$. Since $f = f\chi_E + f\chi_{E^c}$ and $f_n = f_n\chi_E + f_n\chi_{E^c}$ we have

$$\int f = \int_E f + \int_{E^c} f \qquad \text{and} \qquad \int f_n = \int_E f_n + \int_{E^c} f_n \quad \text{for all } n.$$

But then Fatou's Lemma we have

$$\int f - \int_E f = \int_{E^c} f \leqslant \liminf \left(\int f_n - \int_E f_n \right) = \int f - \limsup \int_E f_n,$$

and since everything is finite, we have

$$\limsup \int_E f_n \leqslant \int_E f.$$

Now we show that this result can fail if $\int f = \lim\limits_{n\to\infty} \int f_n = \infty$. Let $X = \mathbb{R}$ and μ the Lebesgue measure. Consider $f = \chi_{[2,\infty)}$, $f_n = \chi_{[2,\infty)} + n\chi_{(0,1/n]}$ and $E = (0, 1]$. Then $f_n \to f$ pointwise, $\int f = \int f_n = \infty$ for all n and

$$\int_E f_n = \int n\chi_{(0,1/n]} = n\mu((0, 1/n]) = 1 \quad \text{for all } n,$$

but $\int_E f = 0$.

Exercise 14. If $f \in L^+$, let $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{M}$. Then λ is a measure on $\mathcal{M}$ and for any $g \in L^+$, $\int g d\lambda = \int g f d\mu$ (first suppose that g is simple).

Solution. Let $\{A_j\}$ be a pairwise disjoint sequence in $\mathcal{M}$ and $A = \bigcup\limits_{j=1}^{\infty} A_j$. Then, since $\chi_A = \sum\limits_{n=1}^{\infty} \chi_{A_n}$ we have

$$\lambda(A) = \int_A f d\mu = \int f\chi_A d\mu = \sum_{j=1}^{\infty} \int f\chi_{A_j} d\mu = \sum_{j=1}^{\infty} \int_{A_j} f d\mu = \sum_{j=1}^{\infty} \lambda(A_j).$$

Hence λ is a measure on $\mathcal{M}$.

Now for the other claim, assume that $g \in L^+$ is a simple function, with $g = \sum\limits_{j=1}^{n} a_j \chi_{E_j}$, with $X = \bigcup\limits_{j=1}^{n} E_j$ and $\{E_j\}_{j=1}^{n}$ is a pairwise disjoint finite sequence. Then

$$\int g d\lambda = \sum_{j=1}^{n} a_j \lambda(E_j) = \sum_{j=1}^{n} a_j \int_{E_j} f d\mu = \int \sum_{j=1}^{n} a_j \chi_{E_j} f d\mu = \int f \sum_{j=1}^{n} a_j \chi_{E_j} d\mu = \int f g d\mu.$$

Now if $g \in L^+$ let $\{\phi_j\}$ be an increasing sequence in L^+ of simple functions that converges to g. Then, using the result for simple function and the MCT we have

$$\int g d\lambda = \lim_{j\to\infty} \int \phi_j d\lambda = \lim_{j\to\infty} \int \phi_j f d\mu = \int g f d\mu,$$

and the last equality follows also from the MCT, since $\phi_j f$ increases to fg.

EXERCISE 15. If $\{f_n\} \subset L^+$, f_n decreases pointwise to f, and $\int f_1 < \infty$, then $\int f = \lim \int f_n$.

Solution. Before we begin, we note that since $f \leqslant f_n \leqslant f_1$ for all n, we have

$$\int f \leqslant \int f_n \leqslant \int f_1 < \infty,$$

and hence all the integrals are finite.

By Fatou's Lemma, we have $\int f \leqslant \liminf \int f_n$. The proof is complete if we can prove that $\limsup \int f_n \leqslant \int f$. To that end define $g_n = f_1 - f_n$. Then $\{g_n\} \subset L^+$ and g_n increases pointwise to $f_1 - f$. Hence by the MCT we have

$$\int (f_1 - f) = \lim \left(\int f_1 - f_n \right) = \liminf \left(\int f_1 - f_n \right) = \int f_1 - \limsup \int f_n.$$

Now note that of $g = f_1 - f$, then $f_1 = g + f$ and $\int f_1 = \int g + \int f$, and since all the integrals are finite, we have $\int g = \int f_1 - \int f$, that is $\int (f_1 - f) = \int f_1 - \int f$, thus we obtain

$$\int f_1 - \int f = \int (f_1 - f) = \int f_1 - \limsup \int f_n,$$

and hence $\int f = \limsup \int f_n$, and the proof is complete.

EXERCISE 16. If $f \in L^+$ and $\int f < \infty$, for every $\epsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f > (\int f) - \epsilon$.

Solution. By Exercise 12, the set $A = \{x \in X : f(x) > 0\}$ is σ-finite, and we can write $A = \bigcup_{j=1}^{\infty} A_j$ with $\mu(A_j) < \infty$ for all j. Without loss of generality, we can assume that $A_j \subset A_{j+1}$ for all j (taking $B_j = \bigcup_{k=1}^{j} A_j$ if necessary).

Since $\mathcal{M} \ni B \mapsto \lambda(B) = \int_B f d\mu$ is a measure and $f = f\chi_A$, from the continuity from below for the measure λ, we have

$$\int f = \int_A f = \lambda(A) = \lim_{j\to\infty} \lambda(A_j) = \lim_{j\to\infty} \int_{A_j} f.$$

Also $f\chi_{A_j} \leqslant f$, and $\int_{A_j} f \leqslant \int f$. Thus, since $\int f < \infty$, given $\epsilon > 0$ we can choose j sufficiently large so that

$$0 \leqslant \int f - \int_{A_j} f < \epsilon.$$

Taking $E = A_j$ concludes the result.

EXERCISE 17. Assume Fatou's Lemma and deduce the MCT from it.

Solution. Let $\{f_n\} \subset L^+$ a sequence that increases to f. Then since $f_n \leqslant f$ for all n, we have $\int f_n \leqslant \int f$ for all n and hence $\limsup \int f_n \leqslant \int f$.

By Fatou's Lemma, $\int f \leqslant \liminf \int f_n$, but then

$$\int f \leqslant \liminf \int f_n \limsup \int f_n \leqslant \int f,$$

so all inequalities are equalities and $\lim \int f_n = \int f$, which is the MCT.

3.5 | INTEGRATION OF COMPLEX FUNCTIONS

We continue our work with a fixed measure space $(X, \mathcal{M}, \mu)$.

DEFINITION 3.5.1 (Integral for extended-real valued functions). *If $f\colon X \to \overline{\mathbb{R}}$ is a measurable function, then both f^+ and f^- are in L^+ (see Definition 3.1.13). If at least one of the integrals $\int f^+$ and $\int f^-$ is finite, we define*

$$\int f = \int f^+ - \int f^-.$$

When both integrals are finite, we say that f is **integrable**.

PROPOSITION 3.5.2. *A measurable function $f\colon X \to \overline{\mathbb{R}}$ is integrable iff $\int |f| < \infty$.*

Proof. This follows directly from the fact that $|f| = f^+ + f^-$. ■

PROPOSITION 3.5.3. *The set of integrable functions $f\colon X \to \overline{\mathbb{R}}$ is a real vector space, and the integral is a linear functional on it.*

Proof. Assume that $f, g\colon X \to \overline{\mathbb{R}}$ are integrable and $a, b \in \mathbb{R}$. Then, since $|af + bg| \leqslant |a||f| + |b||g|$, it follows that $af + bg$ is integrable, hence it is a real vector space.

Now we show that the integral is a linear functional on it. If $a \geqslant 0$ then $(af)^+ = af^+$ and $(af)^- = af^-$ then $\int af = a \int f$. Now $(-f)^+ = f^-$ and $(-f)^- = f^+$ hence

$$\int(-f) = \int(-f)^+ - \int(-f)^- = \int f^- - \int f^+ = -\int f,$$

and the result for $a < 0$ follows from the previous two cases, therefore $\int af = a \int f$ for any $a \in \mathbb{R}$.

Now take $h = f + g$. Then $h = h^+ - h^-$, but $h = f + g = f^+ - f^- + g^+ - g^-$ and hence $h^+ - h^- = f^+ - f^- + g^+ - g^-$, therefore $h^+ + f^- + g^- = h^- + f^+ + g^+$. But the additivity of the integral in L^+ we have

$$\int h^+ + \int f^- + \int g^- = \int h^- + \int f^+ + \int g^+,$$

and hence

$$\int (f + g) = \int f + \int g,$$

which concludes the result. ■

Definition 3.5.4 (Integral for complex functions)**.** *We say that a complex valued function* $f\colon X \to \mathbb{C}$ *is* **integrable** *if* $\int |f| < \infty$*. More generally, if* $E \in \mathcal{M}$*, we say that* f *is* **integrable on** E *if* $\int_E |f| < \infty$*.*

Since $|f| \leqslant |\mathrm{Re} f| + |\mathrm{Im} f| \leqslant 2|f|$*, we see that* f *is integrable iff both* $\mathrm{Re} f$ *and* $\mathrm{Im} f$ *are integrable, and in this case, we define*

$$\int f = \int \mathrm{Re} f + i \int \mathrm{Im} f.$$

It follows as in Proposition 3.5.3 that the space of complex-valued integrable functions is a complex vector space, and the integral is a complex-linear functional on it.

Exercise 3.5.5. *Prove these last claims.*

We will denote this space by $L^1(\mu)$ (or $L^1(X, \mu)$, or $L^1(X)$, or simply L^1, depending on the context).

Proposition 3.5.6. *If* $f \in L^1$*, then*

$$\left| \int f \right| \leqslant \int |f|.$$

Proof. This is trivial if $\int f = 0$, since $\int |f| \geqslant 0$. If f is real, we have

$$\left| \int f \right| = \left| \int f^+ - \int f^- \right| \leqslant \int f^+ + \int f^- = \int |f|.$$

Now assume that f is complex and $\int f \neq 0$. Let $\alpha = \overline{\operatorname{sgn}(\int f)}$, then

$$\alpha \int f = \alpha\overline{\alpha}\left|\int f\right| = \underbrace{|\alpha|^2}_{=1}\left|\int f\right| = \left|\int f\right|,$$

and in particular $\int \alpha f$ is real. Hence

$$\left|\int f\right| = \operatorname{Re}\int \alpha f = \int \operatorname{Re}(\alpha f) \leqslant \int |\operatorname{Re}(\alpha f)| \leqslant \int |\alpha f| = \int |f|.$$

■

Proposition 3.5.7. *We have the following:*

(a) *if $f \in L^1$, then $\{x \in X : f(x) \neq 0\}$ is σ-finite.*

(b) *if $f, g \in L^1$, then $\int_E f = \int_E g$ for all $E \in \mathcal{M}$ iff $\int |f-g| = 0$ iff $f = g$ a.e.*

Proof. **(a).** We have

$$\{x \in X : f(x) \neq 0\} = \{x \in X : f^+(x) > 0\} \cup \{x \in X : f^-(x) > 0\},$$

and the result follows since each one of the sets on the right side is σ-finite (by Proposition 3.3.14).

(b). The equivalence that $\int |f-g| = 0$ iff $f = g$ a.e. follows by Proposition 3.3.10, since $|f-g| \in L^+$ and $|f-g| = 0$ a.e. iff $f = g$ a.e.

If $\int |f-g| = 0$ then if $E \in \mathcal{M}$, we have

$$\left|\int_E f - \int_E g\right| \leqslant \int_E |f-g| = \int \chi_E |f-g| \leqslant \int |f-g| = 0,$$

and hence $\int_E f = \int_E g$.

Now we prove that if $\int_E f = \int_E g$ then $f = g$ a.e., which completes the proof. Assume that it is false that $f = g$ a.e. and consider $h = f - g$, then it is false that $h = 0$ a.e. Writing $u = \operatorname{Re} h$ and $v = \operatorname{Im} h$ and considering

$$E_{r,s} = \{x \in X : r^s(x) > 0\} \quad \text{for } r = u, v \text{ and } s = \pm,$$

then at least one of the $E_{r,s}$ must have positive measure. Assume, for instance that $E_{u,+}$ has positive measure. Then

$$\mathrm{Re}\Big(\int_{E_{u,+}} f - \int_{E_{u,+}} g\Big) = \mathrm{Re}\int_{E_{u,+}} h = \int u^+ > 0,$$

since $u^- = 0$ in $E_{u,+}$. This implies that $\int_{E_{u,+}} h \neq 0$ and gives us a contradiction. Analogously for the other three cases. ■

With this proposition, we can make some additional remarks, that are significantly important for the theory of integration.

First: this proposition shows us that when we are integrating a function, its definition on any null set is irrelevant. That is, we can change the definition of f anyway we want in any null set and we obtain the same result when integrating. Hence, if $f\colon X \to \overline{\mathbb{R}}$ is integrable, we have already seen that $\{x \in X\colon f(x) = \infty\}$ is a null set, and hence $\{x \in X\colon f(x) = \pm\infty\}$ is also a null set. Redefining f to be, for instance, 0 in this set, then we can look at f as a real valued function. This means that, under integration, one does not need to consider integrable functions taking values in the extended real line, but only real values.

Hence, we will redefine $L^1(\mu)$ as follows:

Definition 3.5.8. *Consider f, g complex-valued integrable functions. We say that $f \sim g$ iff $f = g$ a.e.*

This relation $\sim$ is an equivalence relation in the set of complex-valued functions, and we can define $L^1(\mu)$ as the set of all such equivalence classes. This new $L^1(\mu)$ is still a complex vector space. But although now $L^1(\mu)$ is a space of equivalence classes, we will still write $f \in L^1(\mu)$ with the meaning that "f is an a.e.-defined integrable function". This is an abuse of notation, but it does not cause major confusions.

Proposition 3.5.9. *If $\overline{\mu}$ is the completion of μ, then there exists a one-to-one correspondence between $L^1(\mu)$ and $L^1(\overline{\mu})$.*

Proof. Assume that $f \in L^1(\overline{\mu})$, that is $f\colon X \to \mathbb{C}$ is an $\overline{\mathcal{M}}$-measurable function. Then, by Proposition 3.1.24, there exists a $\mathcal{M}$-measurable function $g\colon X \to \mathbb{C}$ such that $f = g$ $\overline{\mu}$-a.e. Thus to each f we associate $\Psi(f) = g$. If $\Psi(f_1) = \Psi(f_2)$ then $f_1 = f_2$ $\overline{\mu}$-a.e., and hence $f_1 = f_2$ in $L^1(\overline{\mu})$, so this association is injective.

If $g\colon X \to \mathbb{C}$ is an $\mathcal{M}$-measurable function, then g is also an $\overline{\mathcal{M}}$-measurable function, since $\mathcal{M} \subset \overline{\mathcal{M}}$, hence Ψ is surjective, since $\Psi(g) = g$.

It remains to show that if $f \in L^1(\overline{\mu})$ and $\Psi(f) = g$, then $g \in L^1(\mu)$. Since $f \in L^1(\overline{\mu})$ and

$f = g$ $\overline{\mu}$-a.e., we have $|f| = |g|$ $\overline{\mu}$-a.e. and

$$\int |g| d\mu = \int |g| d\overline{\mu} = \int |f| d\overline{\mu} < \infty,$$

and the first equality follows from the fact that g is $\mathcal{M}$-measurable and $\overline{\mu}$ is the completion of μ, therefore $g \in L^1(\mu)$. ■

Hence we can (and shall) identify the spaces $L^1(\mu)$ and $L^1(\overline{\mu})$.

DEFINITION 3.5.10. *For $f, g \in L^1(\mu)$ we define*

$$\rho(f, g) = \int |f - g|.$$

PROPOSITION 3.5.11. *The function ρ is a metric on L^1.*

Proof. It is clear that $0 \leqslant \rho(f, g) < \infty$ for all $f, g \in L^1$. Also $\rho(f, g) = 0$ iff $f = g$ a.e., that is, iff $f = g$ in L^1. It is trivial that $\rho(f, g) = \rho(g, f)$, and finally, since $|f - g| \leqslant |f - h| + |h - g|$, we have the triangle inequality. ■

DEFINITION 3.5.12 (Convergence in L^1). *Consider a sequence $\{f_n\} \subset L^1$. We say that* $f_n \to f$ **in** L^1 *if $f \in L^1$ and $\rho(f_n, f) \to 0$ as $n \to \infty$, that is, if $\int |f_n - f| \to 0$ as $n \to \infty$.*

Now, together with the MCT and Fatou's Lemma, the next theorem form the three fundamental convergence theorems of the theory of integration.

THEOREM 3.5.13 (The Dominated Convergence Theorem (DCT)). *Let $\{f_n\}$ be a sequence in L^1 such that*

(a) *$f_n \to f$ a.e.*

(b) *there exists a nonnegative $g \in L^1$ such that $|f_n| \leqslant g$ a.e. for all n.*

Then $f \in L^1$ and $\displaystyle\int f = \lim_{n\to\infty} \int f_n$.

Proof. Using Propositions 3.1.23 and 3.1.24, after perhaps a redefinition on a null set, f is a measurable function. Since $|f| \leqslant g$ a.e., $f \in L^1$. Taking real and imaginary parts, we can assume that f_n, f are real-valued, and hence $g + f_n \geqslant 0$ and $g - f_n \geqslant 0$ for all n. Thus, using Fatou's Lemma, we have

$$\int g + \int f \leqslant \liminf \int (g + f_n) = \int g + \liminf \int f_n$$

and

$$\int g - \int f \leqslant \liminf \int (g - f_n) = \int g - \limsup \int f_n,$$

therefore $\limsup \int f_n \leqslant \int f \leqslant \liminf \int f_n$, which concludes the result. ■

THEOREM 3.5.14. *Suppose that $\{f_j\}$ is a sequence in L^1 such that $\sum_{j=1}^{\infty} \int |f_j| < \infty$. Then $\sum_{j=1}^{\infty} f_j$ converges a.e. to a function in L^1 and*

$$\int \sum_{j=1}^{\infty} f_j = \sum_{j=1}^{\infty} \int f_j.$$

Proof. By Theorem 3.3.9, $\int \sum_{j=1}^{\infty} |f_j| = \sum_{j=1}^{\infty} \int |f_j| < \infty$, so the function $g = \sum_{j=1}^{\infty} |f_j|$ is in L^1. By Proposition 3.3.14, $\sum_{j=1}^{\infty} |f_j(x)|$ is finite for a.e. x, and for such x, the series $\sum_{j=1}^{\infty} f_j(x)$ converges. Thus $\left| \sum_{j=1}^{n} f_j \right| \leqslant g$ a.e for all n and we can apply the DCT to the sequence of partial sums to obtain

$$\int \sum_{j=1}^{\infty} f_j = \sum_{j=1}^{\infty} \int f_j.$$

■

THEOREM 3.5.15. *If $f \in L^1(\mu)$ and $\epsilon > 0$, there exists an integrable simple function $\phi = \sum a_j \chi_{E_j}$ such that $\int |f - \phi| d\mu < \epsilon$; that is, the integrable simple functions are dense in L^1 in the L^1 metric.*

If μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$, then the sets E_j in the definition of ϕ can be taken to be finite unions of bounded open intervals; moreover, there is a continuous function g that vanishes outside a bounded interval such that $\int |f - g| d\mu < \epsilon$.

Proof. Since f is measurable, using Proposition 3.1.22, there exists a sequence $\{\phi_n\}$ of simple functions such that $0 \leqslant |\phi_1| \leqslant |\phi_2| \leqslant \cdots \leqslant |f|$ and $\phi_n \to f$ pointwise. Clearly, since $f \in L^1$, each ϕ_n is in L^1 and $\phi_n - f \to 0$ pointwise. Since $|\phi_n - f| \leqslant |\phi_n| + |f| \leqslant 2|f| \in L^1$, using the DCT, we obtain

$$\int |\phi_n - f| \to 0,$$

and given $\epsilon > 0$ we can choose n such that $\int |\phi_n - f| < \epsilon$.

For the second part, we assume that μ is a Lebesgue-Stieltjes measure. We write $\phi = \phi_n$ and consider its standard decomposition $\phi = \sum_{j=1}^{m} a_j \chi_{E_j}$. We can assume that the a_j are all nonzero and the E_j are disjoint, discarding the sets on which ϕ is zero from the decomposition, and hence

$$\mu(E_j) = \int \chi_{E_j} = |a_j|^{-1} \int_{E_j} |\phi| \leqslant |a_j|^{-1} \int |f| < \infty.$$

Before continuing, note that for measurable sets E and F, we have $\mu(E \Delta F) = \rho(\chi_E, \chi_F) = \int |\chi_E - \chi_F|$, since $\chi_{E\Delta F} = |\chi_E - \chi_F|$. Hence, for each j, given $\epsilon > 0$, using Proposition 2.5.12, there exists a set A_j which is a finite union of open bounded intervals such that $\mu(E_j \Delta A_j) < \frac{\epsilon}{m|a_j|}$, and considering $\tilde{\phi} = \sum a_j \chi_{A_j}$ we have

$$\int |\tilde{\phi} - \phi| = \sum |a_j| \int |\chi_{E_j} - \chi_{A_j}| = \sum |a_j| \mu(E_j \Delta A_j) < \epsilon,$$

and hence $\int |\tilde{\phi} - f| < 2\epsilon$.

Hence, we can write $\tilde{\phi} = \sum_{k=1}^{p} b_k \chi_{I_k}$, where $I_k = (c_k, d_k]$ with $c_k, d_k \in \mathbb{R}$ for each k, and we can again assume that all b_k are nonzero and that the finite family $\{I_k\}$ of intervals is pairwise disjoint (here we use h-intervals and approximate then from the inside with open intervals). Now for each k, we will construct a continuous function $g_k \colon \mathbb{R} \to \mathbb{R}$ as follows:

(i) choose $\delta_k > 0$ such that

$$\mu((c_k, c_k + \delta_k]) + \mu((d_k, d_k + \delta_k]) < \frac{\epsilon}{p|b_k|},$$

and define $g_k = 1$ on $[c_k + \delta_k, d_k]$;

(ii) $g_k = 0$ on $(-\infty, c_k] \cup [d_k + \delta_k, \infty)$;

(iii) define g_k linear from 0 to 1 in $[c_k, c_k + \delta]$ and linear from 1 to 0 on $[d_k, d_k + \delta_k]$.

Thus $\int |b_k \chi_{I_k} - b_k g_k| \leqslant |b_k| \Big(\mu((c_k, c_k + \delta]) + \mu((d_k - \delta, d_k]) \Big) < \epsilon / p$. Therefore, $g = \sum b_k g_k$ is continuous, vanishes outside $\bigcup_{k=1}^{p} (c_k, d_k + \delta_k]$ and

$$\int |g - \tilde{\phi}| \leqslant \sum_{k=1}^{p} \int |b_k||\chi_{I_k} - g_k| < \epsilon,$$

and then

$$\int |f - g| < 3\epsilon.$$

■

The next theorem gives us a criterion for the validity of the interchange limits and derivatives with integrals.

Theorem 3.5.16. *Suppose that $f\colon X \times [a,b] \to \mathbb{C}$ with $-\infty < a < b < \infty$ and that $f(\cdot,t)\colon X \to \mathbb{C}$ is integrable for each $t \in [a,b]$. Define*

$$F(t) = \int_X f(x,t)d\mu(x) \qquad \textit{for each } t \in [a,b].$$

Then

(a) *if there exists $g \in L^1(\mu)$ such that $|f(x,t)| \leqslant g(x)$ for all x,t and $\lim_{t\to t_0} f(x,t) = f(x,t_0)$ for every x then $\lim_{t\to t_0} F(t) = F(t_0)$. In particular, if $f(x,\cdot)$ is continuous for every x, then F is continuous.*

(b) *if $\partial f/\partial t$ exists and there exists $h \in L^1(\mu)$ such that $|(\partial f/\partial t)(x,t)| \leqslant h(x)$ for all x,t then F is differentiable and*

$$F'(t) = \int_X \frac{\partial f}{\partial t}(x,t)d\mu(x) \qquad \textit{for all } t \in [a,b].$$

Proof. **(a).** Let $\{t_n\}$ be any sequence in $[a,b]$ such that $t_n \to t_0$ and define $f_n(x) = f(x,t_n)$ and $f_0(x) = f(x,t_0)$ for every $x \in X$. Hence $f_n \to f_0$ pointwise and $|f_n| \leqslant g$ for all n. Thus, applying the DCT we have $\int f_0 = \lim \int f_n$, that is,

$$F(t_0) = \int_X f(x,t_0)d\mu(x) = \int f_0 = \lim \int f_n = \lim \int_X f(x,t_n)d\mu(x) = \lim F(t_n).$$

Since this is true for every sequence $\{t_n\}$ in $[a,b]$ converging to t_0, the result follows.

(b). Take again any sequence $\{t_n\}$ in $[a,b]$ with $t_n \to t_0$, such that $t_n \neq t_0$ for all n, and define

$$h_n(x) = \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0} \qquad \text{for all } n \text{ and } x \in X.$$

Define also $h_0(x) = (\partial f/\partial t)(x,t_0)$ for all $x \in X$. Hence $h_n \to h_0$, and since each h_n is measurable, it follows that h_0 is measurable. By the Mean Value Theorem we have

$$|h_n(x)| = \frac{|f(x,t_n) - f(x,t_0)|}{|t_n - t_0|} \leqslant \sup_{t\in[a,b]} \left|\frac{\partial f}{\partial t}(x,t)\right| \leqslant h(x) \qquad \text{for all } x \in X,$$

and we can again apply the DCT to obtain $\int h_0 = \lim \int h_n$, that is,

$$F'(t_0) = \lim \frac{F(t_n) - F(t_0)}{t_n - t_0} = \lim \int h_n(x)d\mu(x) = \int h_0(x)d\mu(x) = \int_X \frac{\partial f}{\partial t}(x,t_0)d\mu(x).$$

■

The use of sequential limits is fundamental to treat continuous limits, since the DCT only deal with sequences. However, in similar situations in the future, we will just say $t \to t_0$, with the understanding that we are taking sequential limits.

In the particular case when $\mu = m$ is the Lebesgue measure in $\mathbb{R}$, this integral we have just developed is called the **Lebesgue integral**.

3.5.1 | Comparison between the Riemann and Lebesgue integrals

We will use Darboux's characterization of the Riemann integral, in terms of upper and lower sums, to compare it with the Lebesgue integral.

Let $[a, b]$ be a compact interval. By a **partition** of $[a, b]$, we mean a finite sequence $P = \{t_j\}_{j=0}^n$ such that $a = t_0 < t_1 < \cdots < t_n = b$.

Let f be an arbitrary bounded real-valued function defined on $[a, b]$. For each partition P, we define

$$S_P f = \sum_{j=1}^{n} M_j (t_j - t_{j-1}) \quad \text{and} \quad s_P f = \sum_{j=1}^{n} m_j (t_j - t_{j-1}),$$

where $M_j = \sup\limits_{x \in [t_{j-1}, t_j]} f(x)$ and $m_j = \inf\limits_{x \in [t_{j-1}, t_j]} f(x)$. The sums $S_P f$ and $s_P f$ are called **upper** and **lower sums** of f on P, respectively.

Then we define

$$\overline{I}_a^b(f) = \inf_P S_P f \quad \text{and} \quad \underline{I}_a^b(f) = \sup_P s_P f,$$

where the infimum are taken over all partitions P of $[a, b]$. They're called respectively the **upper** and **lower integrals** of f in $[a, b]$.

When $\overline{I}_a^b(f) = \underline{I}_a^b(f)$, their common value is the **Riemann integral** $\int_a^b f(x)dx$ and f is called **Riemann integrable**.

Theorem 3.5.17. *Let f be a bounded real-valued function on $[a, b]$. If f is Riemann integrable then f is Lebesgue measurable (and hence Lebesgue integrable on $[a, b]$, since it is bounded), and*

$$\int_a^b f(x)dx = \int_{[a,b]} f dm.$$

Proof. For each partition P of $[a, b]$ define

$$G_P = \sum_{j=1}^{n} M_j \chi_{(t_{j-1}, t_j]} \quad \text{and} \quad g_P = \sum_{j=1}^{n} m_j \chi_{(t_{j-1}, t_j]},$$

with the same notation as above. Thus $S_P f = \int G_P dm$ and $s_P f = \int g_P dm$.

We can choose a sequence $\{P_k\}$ of partitions of $[a,b]$ such that $P_k \subset P_{k+1}$ for all k, whose **mesh** ($= \max\limits_{j=1,\cdots,n}(t_j - t_{j-1})$) converges to zero and such that $S_{P_k} f$ and $s_{P_k} f$ both converge to $\int_a^b f(x)dx$.

Since $P_k \subset P_{k+1}$ for all k, $\{G_{P_k}\}$ is a decreasing sequence and $\{g_{P_k}\}$ is an increasing sequence. We can thus define $G = \lim G_{P_k}$ and $g = \lim g_{P_k}$, which are measurable functions, since they are limits of sequences of simple measurable functions. Since $g_{P_k} \leqslant f \leqslant G_{P_k}$ for all k, where $M = \sup f$, we have $g \leqslant f \leqslant G$ and using twice the DCT, we have

$$\int Gdm = \lim \int G_{P_k} dm = \lim S_{P_k} f = \int_a^b f(x)dx = \lim s_{P_k} f = \lim \int g_{P_k} dm = \int gdm.$$

Therefore $\int (G-g)dm = 0$, which implies that $G = g$ a.e. on $[a,b]$ by Proposition 3.3.10. Hence, since $g \leqslant f \leqslant G$, we have $f = g = G$ a.e. on $[a,b]$, thus f is measurable (since it is equal a.e. to a measurable function and m is complete) and

$$\int fdm = \int Gdm = \int gdm = \int_a^b f(x)dx.$$

■

Now we will characterize the set of Riemann integrable function on an interval $[a,b]$. To that end, let $f\colon [a.b] \to \mathbb{R}$ be a bounded function and define

$$H(x) = \lim_{\delta \to 0^+} \sup_{|y-x| \leqslant \delta} f(y) \quad \text{and} \quad h(x) = \lim_{\delta \to 0^+} \inf_{|y-x| \leqslant \delta} f(y).$$

Lemma 3.5.18. *f is continuous at $x \in [a,b]$ iff $H(x) = h(x)$.*

Proof. Assume that f is continuous at x. Then given $\epsilon > 0$, there exists $\delta_0 > 0$ such that if $0 < \delta < \delta_0$ and $|y-x| < \delta$ we have $|f(y) - f(x)| < \epsilon$, thus $f(y) - f(x) < \epsilon$ and $f(x) - f(y) < \epsilon$ for all y such that $|y-x| < \delta$. Hence

$$\sup_{|y-x|<\delta} f(y) - f(x) \leqslant \epsilon \quad \text{and} \quad f(x) - \inf_{|y-x|<\delta} f(y) \leqslant \epsilon,$$

and then

$$\sup_{|y-x|<\delta} f(y) - \inf_{|y-x|<\delta} f(y) \leqslant \sup_{|y-x|<\delta} f(y) - f(x) + f(x) - \inf_{|y-x|<\delta} f(y) \leqslant 2\epsilon.$$

Taking the limit when $\delta \to 0^+$ we have $H(x) - h(x) \leqslant 2\epsilon$, and since $\epsilon > 0$ is arbitrary, we obtain $H(x) - h(x) \leqslant 0$. Since $H(x) \geqslant h(x)$ for each x, we have $H(x) = h(x)$.

For the converse, assume that $H(x) = h(x) = \alpha$. Hence given $\eta > 0$, there exists $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ we have

$$|\alpha - \sup_{|y-x|<\delta} f(y)| < \eta/2 \quad \text{and} \quad |\alpha - \inf_{|y-x|<\delta} f(y)| < \eta/2,$$

which implies that $\sup_{|y-x|<\delta} f(y) - \inf_{|y-x|<\delta} f(y) < \eta$. But then for $|y - x| < \delta$ we have

$$|f(y) - f(x)| \leqslant \sup_{|y-x|<\delta} f(y) - \inf_{|y-x|<\delta} f(y) < \eta,$$

and proves that f is continuous at x. ■

Lemma 3.5.19. *In the notation of the proof of Theorem 3.5.17 we have $H = G$ and $h = g$ a.e.*

Proof. Consider the sequence of partitions $\{P_k\}$ used in the proof of Theorem 3.5.17 and set $E = \{\text{points of } P_k \text{ for all } k\}$. Since each P_k has a finite number of points, E is countable and hence has zero Lebesgue measure.

We will show that $H = G$ in $[a, b] \setminus E$. If $x \in [a, b] \setminus E$, then $G_{P_k}(x) \geqslant H(x)$, since if $x \in (t_{j-1}, t_j)$ we have $G_{P_k}(x) = \sup_{y\in(t_{j-1},t_j]} f(y) \geqslant H(x)$. Hence $G(x) \geqslant H(x)$.

If $H(x) < G(x)$, choose $a \in \mathbb{R}$ such that $H(x) < a < G(x)$. By definition of H, there exists $\delta_0 > 0$ such that if $0 < \delta < \delta_0$ we have $f(y) < a$ if $|y - x| < \delta$. But since the mesh of the partitions P_k tends to zero, for large k, $x \in (t_{j-1}, t_j)$ and $t_j - t_{j-1} < \delta$, hence

$$G_{P_k}(x) = M_j = \sup_{y\in(t_{j-1},t_j]} f(y) \leqslant a.$$

Since the sequence $\{G_{P_k}\}$ is decreasing, we have $G(x) \leqslant G_{P_k}(x) \leqslant a < G(x)$, which gives us a contradiction, hence $H = G$ in $[a, b] \setminus E$, therefore $H = G$ a.e.

Analogously we show that $h = g$ a.e. ■

Corollary 3.5.20. *H and h are measurable, $\int_{[a,b]} H dm = \overline{I}_a^b(f)$ and $\int_{[a,b]} h dm = \underline{I}_a^b(f)$.*

Proof. Since G and g are measurable, $H = G$, $h = g$ a.e. and m is a complete measure, H and h are measurable. Moreover

$$\int_{[a,b]} H dm = \int_{[a,b]} G dm = \lim \int_{[a,b]} G_{P_k} dm = \lim S_{P_k} f = \overline{I}_a^b(f),$$

and

$$\int_{[a,b]} h dm = \int_{[a,b]} g dm = \lim \int_{[a,b]} g_{P_k} dm = \lim s_{P_k} f = \underline{I}_a^b(f),$$

■

Theorem 3.5.21. *Let f be a bounded real-valued function. Then f is Riemann integrable iff $D(f) = \{x \in [a,b] : f \text{ is discontinuous at } x\}$ has zero Lebesgue measure.*

Proof. If f is Riemann integrable, using the notation of the previous results, by Corollary 3.5.20, we have

$$\int_{[a,b]} H dm = \int_a^b f(x)dx = \int_{[a,b]} h dm,$$

hence $H = h$ a.e. by Proposition 3.5.7. Thus $D(f)$ has zero Lebesgue measure by Lemma 3.5.18.

Conversely, if $D(f)$ has zero Lebesgue measure, $H = h$ a.e. by Lemma 3.5.18 and hence by Proposition 3.5.7 and Corollary 3.5.20 we obtain

$$\overline{I}_a^b(f) = \int_{[a,b]} H dm = \int_{[a,b]} h dm = \underline{I}_a^b(f),$$

hence f is Riemann integrable. ■

These results show that the proper Riemann integral is contained in a particular case of the Lebesgue integral. Some improper Riemann integrals can be interpreted as Lebesgue integrals immediately, but others still require a limiting procedure. Consider the follow example.

Example 3.5.22. *If f is a Riemann integrable function in $[0,b]$ for all $b > 0$ and Lebesgue integrable on $[0,\infty)$, then*

$$\int_{[0,\infty)} f dm = \lim_{b\to\infty} \int_0^b f(x)dx.$$

In fact consider the "sequence" $f_b = f\chi_{[0,b]}$. Then $f_b \to f$ pointwise as $b \to \infty$ and $|f_b| \leqslant |f|$, hence by the DCT we have

$$\int_{[0,\infty)} f dm = \lim_{b\to\infty} \int_{[0,\infty)} f\chi_{[0,b]} dm = \lim_{b\to\infty} \int_{[0,b]} f dm = \lim_{b\to\infty} \int_0^b f(x)dx.$$

However, the limit on the right side may exist even when f is not Lebesgue integrable.

For instance, consider $f = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \chi_{(n,n+1]}$. *Thus*

$$\int_{[0,\infty)} |f| dm = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

and hence f *is not Lebesgue integrable in* $[0,\infty)$, *but*

$$\lim_{b\to\infty} \int_0^b f(x)dx = \lim_{b\to\infty} \left(\sum_{n=1}^{\lfloor b \rfloor} \frac{(-1)^n}{n} + \frac{(-1)^{\lfloor b \rfloor+1}}{\lfloor b \rfloor + 1}(b - \lfloor b \rfloor) \right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which is convergent.

The Lebesgue theory offers two real and useful advantages over the Riemann theory. First, we have more powerful convergence theorem. Such results are not true in general for Riemann integrals. Also, there are much more Lebesgue integrable functions the Riemann ones. One simple example is the function $\chi_{\mathbb{Q}}$. Since is everywhere discontinuous, it is not Riemann integrable on any closed interval, however, it is Lebesgue integrable and $\in_{[a,b]} \chi_{\mathbb{Q}} dm = 0$.

Also, metric spaces on which the metric is defined with Lebesgue integrals are complete, but not when defined with the Riemann integral.

From now on, for real-valued functions, we will use the notation $\int_a^b f(x)dx$ for **Lebesgue integrals**.

3.5.2 | The gamma function

In this subsection we discuss the **gamma function**. To begin, let $z \in \mathbb{C}$ with $\text{Re}z > 0$ and define $f_z \colon (0,\infty) \to \mathbb{C}$ by $f_z(t) = t^{z-1}e^{-t}$, where $t^{z-1} = \exp[(z-1)\ln t]$.

Proposition 3.5.23. *If* $z \in \mathbb{C}$ *is such that* $\text{Re}z > 0$ *then* $f_z \in L^1((0,\infty))$.

Proof. We must show that $\int_0^\infty |f_z(t)|dt < \infty$. To that end, first note that, since $|t^{z-1}| = t^{\text{Re}z-1}$, for $0 < t < 1$ we have $|f_z(t)| \leqslant t^{\text{Re}z-1}$ and thus

$$\int_0^1 |f_z(t)|dt \leqslant \int_0^1 t^{\text{Re}z-1}dt = \frac{t^{\text{Re}z}}{\text{Re}z}\Big|_0^1 = \frac{1}{\text{Re}z} < \infty \quad \text{since } \text{Re}z > 0.$$

Now, for $t \geqslant 1$, set $\alpha = \text{Re}z - 1$ and $g(t) = t^\alpha e^{-t/2}$. Hence $|f_z(t)| = g(t)e^{-t/2}$ and since $\lim_{t\to\infty} g(t) = 0$, g is a bounded function for $t \geqslant 1$, which ensures us that there exists a constant

$C > 0$ such that $|f_z(t)| \leqslant Ce^{-t/2}$. Thus

$$\int_1^\infty |f_z(t)|dt \leqslant C\int_1^\infty e^{-t/2}dt = -2Ce^{-t/2}\Big|_1^\infty = 2Ce^{-1/2} < \infty,$$

and joining the two estimates, we have $f_z \in L^1((0,\infty))$. ■

Using this proposition we can make the following definition.

Definition 3.5.24 (Gamma function). *We define for* $\mathrm{Re} z > 0$ *the* **gamma function** *of* z *by*

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$

Proposition 3.5.25. *For* $\mathrm{Re} z > 0$ *we have* $\Gamma(z+1) = z\Gamma(z)$.

Proof. Let $a, b > 0$. Using the comparison between the Lebesgue and Riemann integrals, we can use integration by parts to obtain

$$\int_a^b t^z e^{-t}dt = -t^z e^{-t}\Big|_a^b + z\int_a^b t^{z-1}e^{-t}dt,$$

and letting $a \to 0^+$ and $b \to \infty$ (and recalling that $f_x \in L^1((0,\infty))$), we obtain the result. ■

Thus, if $-1 < \mathrm{Re} z < 0$, we can **define** the gamma function for z by the formula

$$\Gamma(z) = \frac{\Gamma(z+1)}{z},$$

since $\mathrm{Re} z + 1 > 0$. Inductively, we can use this procedure to define Γ for the entire complex plane, except for $\mathrm{Re} z = m$, where m is a nonpositive integer.

Proposition 3.5.26. *For every nonnegative integer* n*, we have* $\Gamma(n+1) = n!$.

Proof. We have $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1)\cdots n!\Gamma(1)$, for each nonnegative integer n. It remains to show that $\Gamma(1) = 1$, but this follows easily since

$$\Gamma(1) = \int_0^\infty e^{-t}dt = -e^{-t}\Big|_0^\infty = 1.$$

■

Together with the gamma function, we also have the **beta function**. It is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad \text{for } x, y > 0.$$

Proposition 3.5.27. *For $x, y > 0$ we have*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. This proof we be done later, as it requires some further definitions and results (See Exercise 60). ∎

3.6 | Solved exercises from [1, Page 59]

Exercise 18. Fatou's Lemma remains valid if the hypothesis that $f_n \in L^+$ is replaced by the hypothesis that $f_n \geqslant -g$ where $g \in L^+ \cap L^1$. What is the analogue of Fatou's lemma for nonpositive functions?

Solution. Fatou's Lemma. Assume that $\{f_n\}$ is a sequence of measurable functions such that $f_n \geqslant -g$ for some $g \in L^+ \cap L^1$. Thus

$$\int \liminf f_n \leqslant \liminf \int f_n.$$

Before proving the result, we will prove the following lemma:

Lemma. If $f \geqslant -g$, f is measurable and $g \in L^+ \cap L^1$, then $\int (f+g) = \int f + \int g$.

Proof. Indeed, if $f \in L^1$ the result is given in Theorem 3.5.14. Assume now that f is not integrable and write $f = f^+ - f^-$. Let $E^- = \{x \in X : f(x) < 0\}$. Since $f \geqslant -g$ we have $f(x) = -f^-(x)$ for $x \in E^-$ and hence $-f^-(x) \geqslant -g(x)$, thus $f^-(x) \leqslant g(x)$. Hence $\int f^- = \int_{E^-} f^- \leqslant \int_{E^-} g \leqslant \int g < \infty$. Thus, if f is not integrable, this means that $\int f^+ = \infty$ and hence $\int f = \infty$. It remains to prove that $\int (f+g) = \infty$. If $h = f + g$ (which is in L^+) we have $h = f^+ - f^- + g$ and hence $h + f^- = f^+ + g$, hence

$$\int h + \int f^- = \int (h + f^-) = \int f^+ + \int g,$$

and since $\int g$ and $\int f^-$ are finite, this follows that $\int h = \infty$. Hence, in any case $\int (f+g) = \int f + \int g$. ♠

Now we can prove this version of Fatou's Lemma. Since $f_n \geqslant -g$ we have $f_n + g \geqslant 0$ for all n, and we can apply Fatou's Lemma together with our previous lemma to obtain

$$\int \liminf f_n + \int g = \int \liminf (f_n + g) \leqslant \liminf \int (f_n + g) = \liminf \int f_n + \int g,$$

and since $g \in L^1$, the result holds.

Fatou's Lemma for nonpositive functions. If $\{f_n\}$ is a sequence of measurable functions with $f_n \leqslant 0$ for all n then

$$\limsup \int f_n \leqslant \int \limsup f_n.$$

Proof: Apply Fatou's Lemma for $\{-f_n\}$.

Exercise 19. Suppose $\{f_n\} \subset L^1(\mu)$ and $f_n \to f$ uniformly.

(a) If $\mu(X) < \infty$ then $f \in L^1(\mu)$ and $\int f_n \to \int f$.

(b) If $\mu(X) = \infty$, the conclusion of (a) can fail. (Find examples on $\mathbb{R}$ with the Lebesgue measure).

Solution to (a). Since $f_n \to f$ uniformly, there exists n_0 such that $|f_n - f| \leqslant 1$ for $n \geqslant n_0$. Hence

$$|f| \leqslant |f - f_{n_0}| + |f_{n_0}| \leqslant 1 + |f_{n_0}|,$$

and since $\mu(X) < \infty$, the constant function 1 is in $L^1(\mu)$, hence $f \in L^1(\mu)$.

Now

$$\left|\int f_n - \int f\right| \leqslant \int |f_n - f| \leqslant \sup_{x\in X} |f_n(x) - f(x)|\mu(X) \to 0$$

as $n \to \infty$, since $\mu(X) < \infty$, hence $\lim \int f_n = \int f$.

Solution to (b). Consider $f_n = \frac{1}{n}\chi_{[0,n]}$. Hence $f_n \to f \equiv 0$ uniformly on $\mathbb{R}$ but $\int f_n = 1$ for all n and $\int f = 0$.

Exercise 20. (A generalized Dominated Convergence Theorem) If $f_n, g_n, f, g \in L^1$, $f_n \to f$ and $g_n \to g$ a.e., $|f_n| \leqslant g_n$ and $\int g_n \to \int g$ then $\int f_n \to \int f$. (Rework the proof of the dominated convergence theorem).

Solution. We have $g_n + f_n \geqslant 0$ and $g_n - f_n \geqslant 0$ for all n. Using Fatou's Lemma, we have

$$\int g + \int f \leqslant \liminf \int (g_n + f_n) = \int g + \liminf \int f_n$$

and

$$\int g - \int f \leqslant \liminf \int (g_n - f_n) = \int g - \limsup \int f_n,$$

and we obtain the desired result.

Exercise 21. Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$. (Use Exercise 20).

Solution.

Assume that $\int |f_n| \to \int |f|$. We have $|f_n - f| \leqslant |f_n| + |f|$ for all n, $|f_n| + |f| \to 2|f|$ a.e. and $\int |f_n| + \int |f| \to 2\int |f|$. Thus, since since $f_n - f \to 0$ a.e., we can use Exercise 20 to conclude that $\int |f_n - f| \to 0$.

For the converse, assume that $\int |f_n - f| \to 0$. Since $||f_n| - |f|| \leqslant |f_n - f|$ for all n and $|f_n| - |f| \to 0$ a.e., we can use Exercise 20 to conclude that $\int (|f_n| - |f|) \to 0$, hence $\int |f_n| \to \int |f|$.

Exercise 22. Let μ be the counting measure on $\mathbb{N}$. Interpret Fatou's lemma, the MCT and the DCT as statements about infinite series.

Solution. Let $f\colon \mathbb{N} \to \mathbb{R}$ any function. Then since $\mathcal{M} = \mathcal{P}(\mathbb{N})$, f is automatically measurable. We write $f(n) = a_n$ for all n. If $f \geqslant 0$ then

$$\int f(n) d\mu(n) = \sum_{n=1}^{\infty} f(n)\mu(\{n\}) = \sum_{n=1}^{\infty} a_n.$$

If f is any function, the f is integrable iff $\int |f| < \infty$ iff $\sum\limits_{n=1}^{\infty} |a_n|$ is convergent.

Fatou's Lemma. If $\{a_{n,k}\}$ is a doubly indexed real sequence with $a_{n,k} \geqslant 0$ for all n, k then

$$\sum_{k=1}^{\infty} \liminf_{n\to\infty} a_{n,k} \leqslant \liminf_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}.$$

MCT. If $\{a_{n,k}\}$ is a doubly indexed real sequence and $\{b_k\}$ is a real sequence with $0 \leqslant a_{1,k} \leqslant a_{2,k} \leqslant \cdots \leqslant b_k$ and $\lim\limits_{n\to\infty} a_{n,k} = b_k$ If $\{a_{n,k}\}$ is a doubly indexed real sequence with $a_{n,k} \geqslant 0$ for all n, k, then

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = \sum_{k=1}^{\infty} b_k.$$

DCT. Assume that $\{a_{n,k}\}$ is a doubly indexed sequence in $\mathbb{R}$ with $\sum\limits_{k=1}^{\infty} a_{n,k}$ absolutely convergent for all n, $\lim\limits_{n\to\infty} a_{n,k} = b_k$ for all k, and there exists a nonnegative sequence $\{c_k\}$ with $\sum\limits_{k=1}^{\infty} c_k$ convergent and $|a_{n,k}| \leqslant c_k$ for all k, then $\sum\limits_{k=1}^{\infty} b_k$ is absolutely convergent and

$$\sum_{k=1}^{\infty} b_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}.$$

Exercise 23.

This is done in Subsection 3.5.1.

Exercise 24. Let $(X, \mathcal{M}, \mu)$ be measure space with $\mu(X) < \infty$ and let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. Suppose that $f\colon X \to \mathbb{R}$ is bounded. Then f is $\overline{\mathcal{M}}$-measurable (and hence in $L^1(\overline{\mu})$) iff there exist sequences $\{\phi_n\}$ and $\{\psi_n\}$ of $\mathcal{M}$-measurable simple functions such that $\phi_n \leqslant f \leqslant \psi_n$ and $\int(\psi_n - \phi_n)d\mu < n^{-1}$. In this case $\lim \int \phi_n d\mu = \lim \int \psi_n d\mu = \int f d\overline{\mu}$.

Solution. Assume that f if $\overline{\mathcal{M}}$-measurable. Then, by Proposition 3.1.24, there exists a $\mathcal{M}$-measurable function g such that $f = g$ $\overline{\mu}$-a.e., and since f is bounded, g is also bounded.

We will adapt the proof of Theorem 3.1.20. Choose n_0 such that $-2^{n_0} \leqslant g \leqslant 2^{n_0}$. For $n \geqslant n_0$ and we define

$$J_{n,0} = [-2^n, -2^n + 2^{-n}] \qquad \text{and} \qquad J_{n,k} = (-2^n + k2^{-n}, -2^n + (k+1)2^{-n}]$$

for $k = 1, \cdots, 2^{2n+1} - 1$. Hence $\bigcup_{k=1}^{2^{2n+1}-1} J_{n,k} = [-2^n, 2^n]$ for all n.

Define $E_{n,k} = g^{-1}(J_{n,k})$ for $k = 0, \cdots, 2^{2n+1} - 1$. Since $g(X) \subset [-2^{n_0}, 2^{n_0}] \subset [-2^n, 2^n]$ for $n \geqslant n_0$, we have $\bigcup_{k=0}^{2^{2n+1}-1} E_{n,k} = X$ and the finite sequence of sets $\{E_{n,k}\}_k$ is in $\mathcal{M}$ and is pairwise disjoint. Now we can define

$$\phi_n = \sum_{k=0}^{2^{2n+1}-1} (-2^n + k2^{-n})\chi_{E_{n,k}} \quad \text{and} \quad \psi_n = \sum_{k=0}^{2^{2n+1}-1} (-2^n + (k+1)2^{-n})\chi_{E_{n,k}}$$

for $k = 0, \cdots, 2^{2n+1} - 1$. Thus $\phi_{n_0} \leqslant \phi_{n_0+1} \leqslant \cdots \leqslant g \leqslant \cdots \leqslant \psi_{n_0+1} \leqslant \psi_{n_0}$ and moreover, for $n \geqslant n_0$ we have

$$\int (\psi_n - \phi_n)d\mu = \sum_{k=0}^{2^{2n+1}-1} 2^{-n}\mu(E_{n,k}) \leqslant 2^{-n}\mu(X).$$

If n_1 is such that for $n \geqslant n_1$ we have $n2^{-n}\mu(X) < 1$ then for $n \geqslant \max\{n_0, n_1\}$ we obtain

$$\int (\psi_n - \phi_n)d\mu < n^{-1}. \tag{$\star$}$$

Clearly, $\phi_n \leqslant f \leqslant \psi_n$ $\overline{\mu}$-a.e. Let A be the set on which $\phi_n \leqslant f$ fails, then $A \subset N$ where $N \in \mathcal{M}$ is a null set. We can redefine ϕ_n on N by setting $\phi_n(x) = \alpha = \inf_{x \in X} f(x)$ for $x \in N$, thus

$$\phi_n = \sum_{k=0}^{2^{2n+1}-1} (-2^n + k2^{-n})\chi_{E_{n,k}\cap N^c} + \alpha\chi_N,$$

is still a simple $\mathcal{M}$-measurable function and $\phi_n \leqslant f$. The same can be done with ψ_n using $\sup\limits_{x\in X} f(x)$, and hence $f \leqslant \psi_n$. Since integration does not see null sets, inequality $(\star)$ remains unchanged.

Now we prove the converse. To that end, if $\phi_n \leqslant f \leqslant \psi_n$ for all n and $\int(\psi_n - \phi_n)d\mu < n^{-1}$. Define $\phi = \limsup \phi_n$ and $\psi = \liminf \psi_n$, which are $\mathcal{M}$-measurable by Proposition 3.1.10, and well defined real-valued functions, since $\phi_n \leqslant f \leqslant \psi_n$ and f is bounded (hence $\phi_n \leqslant \inf\limits_{x\in X} f(x)$ and $\psi_n \geqslant \sup\limits_{x\in X} f(x)$ for all n). Thus $\phi \leqslant f \leqslant \psi$ and by Fatou's Lemma

$$\int(\psi - \phi)d\mu = \int(\liminf \psi_n - \limsup \phi_n)d\mu = \int \liminf(\psi_n - \psi_n)d\mu \leqslant \liminf \int(\psi_n - \phi_n)d\mu = 0,$$

hence $\int(\psi - \phi)d\mu = 0$. Using Proposition 3.5.7 we have $\psi = \phi$ μ-a.e. Since $\phi \leqslant f \leqslant \psi$, we have $\phi = \psi = f$ μ-a.e. Thus f is μ-a.e. equal to a $\mathcal{M}$-measurable function. Assume that A is a μ-null set such that $f \neq \phi$. Then if $B \subset \mathbb{R}$ is a Borel set, we have

$$f^{-1}(B) = (f^{-1}(B) \cap A^c) \cup (f^{-1}(B) \cap A) = \underbrace{(\phi^{-1}(B) \cap A^c)}_{\in\mathcal{M}} \cup \underbrace{(f^{-1}(B) \cap A)}_{\subset A},$$

hence $f^{-1}(B) \in \overline{\mathcal{M}}$, and f is $\overline{\mathcal{M}}$-measurable.

Exercise 25. Let $f(x) = x^{-1/2}$ if $0 < x < 1$, $f(x) = 0$ otherwise. Let $\{r_n\}$ be an enumeration of the rational, and set $g(x) = \sum\limits_{n=1}^{\infty} 2^{-n} f(x - r_n)$.

(a) $g \in L^1(m)$, and in particular $g < \infty$ a.e.

(b) g is discontinuous a.e. and unbounded on every interval, and it remains so after any modification on a Lebesgue null set.

(c) $g^2 < \infty$ a.e., but g^2 is not integrable on any interval.

Solution to (a). First note that $f|_{(-\infty,0]\cup[1,\infty)} \equiv 0$ and hence it is measurable. Also $f|_{(0,1)}$ is continuous, hence measurable. Thus by Exercise 5 of Section 3.2, f is measurable. For each n, the translation $h_n(x) = x - r_n$ is continuous, hence measurable. Thus $f_n = 2^{-n} f \circ h_n$ is measurable for each n, and thus g is measurable.

Clearly $g \geqslant 0$, and hence, by Proposition 3.3.9 we have

$$\int g dm = \sum_{n=1}^{\infty} 2^{-n} \int f(x - r_n) dm(x).$$

But note that $0 < x - r_n < 1$ iff $r_n < x < r_n + 1$, hence

$$\int f(x - r_n) dm(x) = \int_{r_n}^{r_n+1} (x - r_n)^{-1/2} dx = 2(x - r_n)\Big|_{r_n}^{r_n+1} = 2,$$

and hence

$$\int g dm = \sum_{n=1}^{\infty} 2^{-n+1} = \sum_{n=0}^{\infty} 2^{-n} = 2,$$

and $g \in L^1(m)$. By Proposition 3.3.14, $g < \infty$ a.e.

Solution to (b). First, we show that g is unbounded on every interval. Let $I \subset \mathbb{R}$ be a nondegenerated interval (that is, I is neither a single point nor empty). Thus there exists a rational r_n which is an interior point of I. Let $M > 0$ choose $\delta \in (0, 1)$ such that $(r_n, r_n + \delta) \subset I$ and $2^{-n}(x - r_n)^{-1/2} > M$ for $x \in (r_n, r_n + \delta)$. Hence, for $x \in (r_n, r_n + \delta)$ we have $0 < x - r_n < \delta < 1$ and

$$g(x) \geqslant 2^{-n} f(x - r_n) = 2^{-n}(x - r_n)^{-1/2} > M,$$

and this proves that g is unbounded in I. Furthermore $(r_n, r_n + \delta) \subset g^{-1}((M, \infty))$, and hence $m(g^{-1}((M, \infty))) \geqslant \delta$, so any redefinition of g in a Lebesgue null set will yield an unbounded function on every interval.

Since $g < \infty$ a.e. and g is unbounded on every interval, g is discontinuous a.e.

Solution to (c). Since $g < \infty$ a.e, then $g^2 < \infty$ a.e.

Now, fix a nondegenerated interval I, choose r_n a interior rational point of I and $\delta \in (0, 1)$ such that $(r_n, r_n + \delta) \subset I$. Thus we have

$$\begin{aligned}\int_I g^2 dm &\geqslant \int_I (2{-}n f(x - r_n))^2 dm(x) = 2^{-2n} \int_I f^2(x - r_n) dm(x) \\ &\geqslant 2^{-2n} \int_{r_n}^{r_n+1} (x - r_n)^{-}1 dm(x) = \infty,\end{aligned}$$

hence g is not integrable on I.

Exercise 26. If $f \in L^1(m)$ and $F(x) = \int_{-\infty}^{x} f(t) dt$, then F is continuous on $\mathbb{R}$.

Solution. Since $\chi_{(-\infty,x]}|f| \leqslant |f|$, $\chi_{(-\infty,x]} f$ is integrable and F is well defined for every

$x \in \mathbb{R}$. Now let $x \in \mathbb{R}$ and $x_n \to x$. Then

$$|F(x_n) - F(x)| = \left|\int_{-\infty}^{x_n} f(t)dt - \int_{-\infty}^{x} f(t)dt\right| = \left|\int \left(\chi_{(\infty,x_n]}f - \chi_{(-\infty,x]}f\right)dm\right|$$
$$\leqslant \int |\chi_{(\infty,x_n]}f - \chi_{(-\infty,x]}f|dm.$$

Let $g_n = |\chi_{(\infty,x_n]}f - \chi_{(-\infty,x]}f|$ for each n. We have

$$g_n(t) = \chi_{(\min\{x_n,x\},\max\{x_n,n\}]}|f(t)|.$$

Thus, since $x_n \to x$, we have $g_n(t) \to 0$ as $n \to \infty$ for all $t \neq x$. Hence $g_n \to 0$ a.e. Moreover $|g_n| \leqslant 2|f| \in L^1(m)$ and by the DCT, we have

$$|F(x_n) - F(x)| \leqslant \int |g_n|dm \to 0 \quad \text{as } n \to \infty,$$

and thus F is continuous at x.

Exercise 27. Let $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$.

(a) $\sum\limits_{n=1}^{\infty} \int\limits_{0}^{\infty} |f_n(x)|dx = \infty$.

(b) $\sum\limits_{n=1}^{\infty} \int\limits_{0}^{\infty} f_n(x)dx = 0$.

(c) $\sum\limits_{n=1}^{\infty} f_n \in L^1([0,\infty), m)$ and $\int\limits_{0}^{\infty} \sum\limits_{n=1}^{\infty} f_n(x)dx = \ln(b/a)$.

Solution to (a). We have

$$\int_0^{\infty} |f_n(x)|dx \geqslant \int_{1/na}^{\infty} |f_n(x)|dx \geqslant \left|\int_{1/na}^{\infty} f_n(x)dx\right|$$
$$= \left|\int_{1/na}^{\infty} (ae^{-nax} - be^{-nbx})dx\right| = \left|\left(\frac{e^{-nbx}}{n} - \frac{e^{-nax}}{n}\right)\Big|_{1/na}^{\infty}\right|$$
$$= \frac{1}{n} \cdot \left|e^{-b/a} - e^{-1}\right|,$$

and hence

$$\sum_{n=1}^{\infty} \int_0^{\infty} |f_n(x)|dx \geqslant |e^{-b/a} - e^{-1}| \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Solution to (b). Note that for each n we have

$$\int_0^\infty f_n(x)dx = \int_0^\infty (ae^{-nax} - be^{-nbx})dx = \Big(\frac{e^{-nbx}}{n} - \frac{e^{-nax}}{n}\Big)\Big|_0^\infty = 0,$$

and hence $\sum\limits_{n=1}^{\infty} \int\limits_0^{\infty} f_n(x)dx = 0$.

Solution to (c). Note that

$$\sum_{n=1}^\infty f_n(x) = \sum_{n=1}^\infty (ae^{-nax} - be^{-nbx}) = a\sum_{n=1}^\infty e^{-nax} - b\sum_{n=1}^\infty e^{-nbx},$$

since bot series in the right hand side converge absolutely for each $x > 0$, hence

$$\sum_{n=1}^\infty f_n(x) = a\Big(\frac{e^{-ax}}{1-e^{-ax}}\Big) - b\Big(\frac{e^{-bx}}{1-e^{-bx}}\Big) = \frac{a}{e^{ax}-1} - \frac{b}{e^{bx}-1}.$$

Set $f(x) = \dfrac{a}{e^{ax}-1} - \dfrac{b}{e^{bx}-1}$ for $x > 0$. But

$$\frac{e^{ax}-1}{a} = \sum_{n=1}^\infty \frac{a^{n-1}x^n}{n!} \leqslant \sum_{n=1}^\infty \frac{b^{n-1}x^n}{n!} = \frac{e^{bx}-1}{b},$$

hence $f(x) \geqslant 0$ for all $x > 0$. Define $g_n = \chi_{[1/n,\infty)} f$. Clearly $g_n \geqslant 0$, g_n is measurable for each n and g_n increases to f, and from the MCT we obtain

$$\int f = \lim_{n\to\infty} \int g_n = \lim_{n\to\infty} \int_{1/n}^\infty f(x)dx = \lim_{n\to\infty} \int_{1/n}^\infty \Big(\frac{a}{e^{ax}-1} - \frac{b}{e^{bx}-1}\Big)dx.$$

It just remains to compute the integral $\int \dfrac{c}{e^{cx}-1}dx$, for $c = a, b$. To that end, make the substituition $u = e^{cx} - 1$, hence $du = c(u+1)dx$ to obtain

$$\int \frac{c}{e^{cx}-1}dx = \int \frac{1}{u(u+1)}du = \int \Big(\frac{1}{u} - \frac{1}{u+1}\Big)du = \ln\Big|\frac{u}{u+1}\Big| + k = \ln\Big|1 - e^{-cx}\Big| + k,$$

and using the limits of integration, we obtain

$$\int_{1/n}^\infty \frac{c}{e^{cx}-1}dx = -\ln(1 - e^{-c/n}).$$

Therefore

$$\int f = \lim_{n\to\infty} \ln\Big(\frac{1-e^{-b/n}}{1-e^{-a/n}}\Big) = \ln(b/a),$$

where in the last equality the L'Hôpital Rule was used. Thus, the result is complete.

Exercise 28. Compute the following limits and justify the calculations.

(a) $\lim\limits_{n\to\infty}\int\limits_0^\infty \dfrac{\sin(x/n)}{(1+x/n)^n}dx.$

(b) $\lim\limits_{n\to\infty}\int\limits_0^1 \dfrac{1+nx^2}{(1+x^2)^n}dx.$

(c) $\lim\limits_{n\to\infty}\int\limits_0^\infty \dfrac{n\sin(x/n)}{x(1+x^2)}dx.$

(d) $\lim\limits_{n\to\infty}\int\limits_a^\infty \dfrac{n}{1+n^2x^2}dx.$ (The answer depends on whether $a>0$, $a=0$ or $a<0$. How does this accord with the various convergence theorems?)

Solution to (a). Define $f_n(x)=\dfrac{\sin(x/n)}{(1+x/n)^n}$ for each n and $x\geqslant 0$. Hence $f_n(0)=0$ for each n and for each $x>0$, $f_n(x)\to 0$ as $n\to\infty$. Also, since $(1+x/n)^n\geqslant 1+x+x^2/4$ for $n\geqslant 2$ and $x\geqslant 0$, we have

$$|f_n(x)|\leqslant\frac{1}{1+x+x^2/4}\qquad\text{and}\qquad\int_0^\infty\frac{1}{1+x+x^2/4}dx<\infty.$$

Hence, by the DCT we have $\int_0^\infty f_n(x)dx\to 0$.

Solution to (b). Define $f_n(x)=\dfrac{1+nx^2}{(1+x^2)^n}$ for each n and $x\geqslant 0$. We have $f_1(x)=1$ for all $x\geqslant 0$, $f_n(0)=1$ for all n and

$$|f_n(x)|\leqslant\frac{1+nx^2}{1+nx^2+\binom{n}{2}x^4}\leqslant 1,$$

and $1\in L^1([0,1],m)$. Since $f_n(x)\to 0$ for $0<x\leqslant 1$, from the DCT we have $\int_0^1 f_n(x)dx\to 0$.

Solution to (c). Define $f_n(x)=\dfrac{n\sin(x/n)}{x(1+x^2)}$ for each n and $x>0$. From the first fundamental limit, we have $f_n(x)\to\frac{1}{1+x^2}$ for all $x>0$. Since

$$|f_n(x)|\leqslant\frac{1}{1+x^2}\qquad\text{and}\qquad\int_0^\infty\frac{1}{1+x^2}dx=\frac{\pi}{2}<\infty,$$

by the DCT we have $\int_0^\infty f_n(x)dx\to\frac{\pi}{2}$.

Solution to (d). We can compute this limit directly:

$$\lim_{n\to\infty}\int_a^\infty \frac{n}{1+(nx)^2}dx = \lim_{n\to\infty}\left(\arctan(nx)\Big|_a^\infty\right) = \frac{\pi}{2} - \lim_{n\to\infty}\arctan(na),$$

and thus we have

$$\lim_{n\to\infty}\int_a^\infty \frac{n}{1+(nx)^2}dx = \begin{cases} 0 & \text{if } a>0 \\ \pi/2 & \text{if } a=0 \\ \pi & \text{if } a<0. \end{cases}$$

Now for the application of the convergence theorems in each case. Let

$$f(x) = \lim_{n\to\infty}\frac{n}{1+n^2x^2} = \begin{cases} 0 & \text{if } x\neq 0 \\ \infty & \text{if } x=0 \end{cases}$$

Since $\int_a^\infty f(x)dx = 0$ for each $a\in\mathbb{R}$, we could apply a convergence theorem for $a>0$, but there is no chance of applying a convergence theorem for $a\leqslant 0$.

Hence for $a\leqslant 0$, there can be no $L^1([a,\infty),m)$ function g such that $|f_n(x)|\leqslant g$ a.e. in $[a,\infty)$. Also, for $a\leqslant 0$ the sequence $\{f_n\}$ is not increasing. We can apply Fatou's Lemma to obtain

$$0\leqslant \int_a^\infty f(x)dx \leqslant \liminf_{n\to\infty}\int_a^\infty f_n(x)dx,$$

but this has no new information, since $\{f_n\}$ is nonnegative.

For $a>0$, since $|f_n(x)|\leqslant \dfrac{1}{x^2}$ for all $x>0$ and $\int_a^\infty \frac{1}{x^2}dx<\infty$, we can apply the DCT.

Exercise 29. Show that $\displaystyle\int_0^\infty x^n e^{-x}dx = n!$ by differentiating the equation $\displaystyle\int_0^\infty e^{-tx}dx = \frac{1}{t}$. Similarly, show that $\displaystyle\int_{-\infty}^\infty x^{2n}e^{-x^2}dx = (2n)!\sqrt{\pi}/4^n n!$ by differentiating the equation $\displaystyle\int_{-\infty}^\infty e^{-tx^2}dx = \sqrt{\pi/t}$.

Solution. Define $f(x,t) = e^{-tx}$ for $x\geqslant 0$ and $t\in[a,b]$ with $0<a<1<b$. Since f is continuous, f is measurable. Also $\int\limits_0^\infty e^{-tx}dx = 1/t<\infty$ for each $t\in[a,b]$. Moreover f is differentiable and $\partial f/\partial t = -xe^{-tx}$ for all $x,t>0$ and

$$\left|\frac{\partial f}{\partial t}(x,t)\right| \leqslant xe^{-ax} \qquad \text{and} \quad \int_0^\infty xe^{-ax}dx = 1/a^2<\infty,$$

hence from Theorem 3.5.16, applied for $F(t) = \int_0^\infty f(x,t)dx = 1/t$, we have

$$\frac{1}{t^2} = -F'(t) = \int_0^\infty xe^{-tx}dx.$$

Using induction on this last equation together with Theorem 3.5.16 we obtain

$$\frac{n!}{t^{n+1}} = \int_0^\infty x^n e^{-tx}dx,$$

and taking $t = 1$ we obtain the result. Analogously, we obtain the second part.

Exercise 30. Show that $\lim\limits_{k\to\infty} \int_0^k x^n \left(1 - \frac{x}{k}\right)^k dx = n!$.

Solution. Fix n and define $f_k(x) = \chi_{(0,k)}(x)x^n\left(1 - \frac{x}{k}\right)^k$. We have $f_k(x) \to x^n e^{-x}$ for $x > 0$. Since $f_k \geqslant 0$ for all k and $f_n \to f = x^n e^{-x} \geqslant 0$. Now for $0 < x < k$ we have

$$h(x) = k\ln\left(1 - \frac{x}{k}\right) + x \leqslant 0,$$

since $h(0) = 0$ and $h'(x) < 0$ for $0 < x < k$.

Applying the DCT and using Exercise 29 we have

$$\lim_{k\to\infty} \int_0^k x^n \left(1 - \frac{x}{k}\right)^k dx = \lim_{k\to\infty} \int_0^\infty f_k(x)dx = \int_0^\infty x^n e^{-x}dx = n!.$$

Exercise 31. Derive the following formulas by expanding part of the integrand into an infinite series and justifying the term-by-term integration. Exercise 29 may be useful (Note: in (d) and (e), term-by-term integration works, and the resulting series converges only for $a > 1$, but the formulas as stated are actually valid for all $a > 0$).

(a) For $a > 0$, $\displaystyle\int_{-\infty}^\infty e^{-x^2}\cos(ax) = \sqrt{\pi}e^{-a^2/4}$.

(b) For $a > -1$, $\displaystyle\int_0^1 x^a(1-x)^{-1}\ln(x)dx = -\sum_{k=1}^n \frac{1}{(a+k)^2}$.

(c) For $a > 1$, $\displaystyle\int_0^\infty x^{a-1}(e^x - 1)^{-1}dx = \Gamma(a)\zeta(a)$, where $\zeta(a) = \sum\limits_{n=1}^\infty \frac{1}{n^a}$.

(d) For $a > 1$, $\displaystyle\int_0^\infty e^{-ax}x^{-1}\sin(x)dx = \arctan(a^{-1})$.

(e) For $a > 1$, $\int_0^\infty e^{-ax} J_0(x)dx = \dfrac{1}{\sqrt{1+a^2}}$, where $J_0(x) = \sum_{n=0}^{\infty} \frac{(1-)^n x^{2n}}{4^n (n!)^2}$ is the Bessel function of order zero.

Solution to (a). For all $x, a \in \mathbb{R}$ we have $\cos(ax) = \sum_{n=0}^{\infty} \dfrac{(-a^2)^n x^{2n}}{(2n)!}$. Also using Exercise 29 we have

$$\int |f_n| \leqslant \int_{-\infty}^{\infty} e^{-x^2} \frac{a^{2n} x^{2n}}{(2n)!} dx = \frac{a^{2n}\sqrt{\pi}}{4^n n!},$$

and hence $\sum \int |f_n| < \infty$ where $f_n(x) = e^{-x^2} \dfrac{(-a^2)^n x^{2n}}{(2n)!}$. By Theorem 3.5.14 we have

$$\int_{-\infty}^{\infty} e^{-x^2} \cos(ax) = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(-a^2/4)^n}{n!} = \sqrt{\pi} e^{-a^2/4}.$$

Solution to (b). For $0 < x < 1$ we have

$$\frac{x^a \ln(x)}{1-x} = x^a \ln(x) \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{a+n} \ln(x) = -\sum_{n=0}^{\infty} x^{a+n} \ln(1/x).$$

Since the functions $(0,1) \ni x \mapsto x^{a+n} \ln(1/x)$ are nonnegative, by Theorem 3.3.9 we have

$$\int_0^1 x^a (1-x)^{-1} \ln(x) dx = -\sum_{n=0}^{\infty} \int_0^1 x^{a+n} \ln(1/x) dx.$$

Using integration by parts and the fact that $a > -1$ we obtain

$$\int_0^1 x^{a+n} \ln(1/x) dx = \frac{1}{(a+n+1)^2},$$

hence

$$\int_0^1 x^a (1-x)^{-1} \ln(x) dx = -\sum_{n=0}^{\infty} \frac{1}{(a+n+1)^2} = -\sum_{k=1}^{\infty} \frac{1}{(a+k)^2}.$$

Solution to (c). We write

$$\frac{x^{a-1}}{e^x - 1} = \frac{x^{a-1} e^{-x}}{1 - e^{-x}} = x^{a-1} e^{-x} \sum_{n=0}^{\infty} e^{-nx} = \sum_{n=0}^{\infty} x^{a-1} e^{-(n+1)x} = \sum_{k=1}^{\infty} x^{a-1} e^{-kx}.$$

Since $f_n(x) = x^{a-1} e^{-(n+1)x}$ are nonnegative functions for $x \geqslant 0$, Theorem 3.3.9 gives us

$$\int_0^\infty \frac{x^{a-1}}{e^x - 1} dx = \sum_{k=1}^{\infty} \int_0^\infty x^{a-1} e^{-kx} dx.$$

Making the change $u = kx$ we have $du = kdx$ and

$$\int_0^\infty \frac{x^{a-1}}{e^x - 1} dx = \sum_{k=1}^\infty \frac{1}{k^a} \int_0^\infty u^{a-1} e^{-u} du = \zeta(a)\Gamma(a).$$

Solution to (d). We write

$$e^{-ax} x^{-1} \sin(x) = \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n+1)!} e^{-ax},$$

and for $f_n(x) = \frac{(-1)^n x^{2n}}{(2n+1)!} e^{-ax}$ for $x > 0$, we have

$$\begin{aligned}\int_0^\infty |f_n(x)| dx &= \frac{1}{(2n+1)!} \int_0^\infty x^{2n} e^{-ax} dx = \frac{1}{(2n+1)! a^{2n+1}} \int_0^\infty u^{2n} e^{-u} du \\ &= \frac{\Gamma(2n+1)}{(2n+1)! a^{2n+1}} = \frac{1}{(2n+1) a^{2n+1}},\end{aligned}$$

and hence $\sum \int |f_n| < \infty$ if $a > 1$. Thus

$$\int_0^\infty e^{-ax} x^{-1} \sin(x) dx = \sum_{n=0}^\infty \int_0^\infty \frac{(-1)^n x^{2n}}{(2n+1)!} e^{-ax} dx = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1) a^{2n+1}} = \arctan(a^{-1}).$$

Solution to (e). For $f_n(x) = \dfrac{(-1)^n x^{2n}}{4^n (n!)^2} e^{-ax}$ for $x > 0$ we have

$$\int_0^\infty |f_n(x)| dx = \int_0^\infty \frac{x^{2n}}{4^n (n!)^2} e^{-ax} dx = \frac{\Gamma(2n+1)}{4^n (n!)^2 a^{2n+1}} = \frac{(2n)!}{4^n (n!)^2 a^{2n+1}}$$

and $\sum \int |f_n|$ is convergent (using the Ratio Test) if $a > 1$. Therefore

$$\int_0^\infty e^{-ax} J_0(x) dx = \sum_{n=0}^\infty \frac{(-1)^n (2n)!}{4^n (n!)^2 a^{2n+1}}.$$

To conclude, note that for $|x| < 1$ we have

$$\frac{1}{\sqrt{1+x}} = \sum_{n=0}^\infty \frac{(-1)^n 1.3.5 \ldots (2n-1) x^n}{2^n n!} = \sum_{n=0}^\infty \frac{(-1)^n (2n)! x^n}{4^n (n!)^2},$$

since $1.3.5 \ldots (2n-1) = \dfrac{(2n)!}{2.4.6 \ldots 2n} = \dfrac{(2n)!}{2^n n!}$. For $a > 1$, taking $x = 1/a^2$ we have $0 < x < 1$

and

$$\frac{1}{\sqrt{a^2+1}} = \frac{1}{a\sqrt{1+\frac{1}{a^2}}} = \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{4^n(n!)^2a^{2n+1}} = \int_0^{\infty} e^{-ax}J_0(x)dx.$$

3.7| Modes of convergence

If we consider a sequence $\{f_n\}$ of complex functions on a set X, the statement $f_n \to f$ can have several distinct meanings, for instance, uniformly or pointwise. In the case where X is a measure space, we can speak of a.e.-convergence or L^1-convergence. Clearly uniform convergence implies pointwise convergence, and the latter implies a.e convergence (the converses are not true in general). None of these convergences (without further hypotheses) imply L^1 convergence, neither vice versa.

Example 3.7.1. *Consider in $\mathbb{R}$ with the Lebesgue measure the following example.*

(i) $f_n = n^{-1}\chi_{(0,n)}$,

(ii) $f_n = \chi_{(n,n+1)}$,

(iii) $f_n = n\chi_{[0,1/n]}$,

(iv) $f_n = \chi_{[j/2^k,(j+1)/2^k]}$ *for* $n = 2^k + j$ *for* $0 \leqslant j < 2^k$.

In (i), $\sup_{x\in\mathbb{R}} |f_n(x)| = 1/n \to 0$ *as* $n \to 0$*, that is,* $f_n \to 0$ *uniformly. In (ii),* $f_n \to 0$ *pointwise, but* $\sup_{x\in\mathbb{R}} |f_n(x)| = 1$ *and* $\int f_n = 1$ *for all* n*. In (iii),* $f_n(x) \to 0$ *for all* $x \neq 0$*, but* $f_n(0) = n \to \infty$*, hence* $f_n \to 0$ *a.e. but* $\int f_n = 1$ *for all* n*.*

In (iv), since $\int |f_n| = 2^{-k} \to 0$ *with* $2^k \leqslant n < 2^{k+1}$ *and hence* $\int |f_n| \to 0$*, that is* $f_n \to 0$ *in* L^1*. But* $f_n(x)$ *does not converge for any* $x \in [0,1]$*, since* $f_n(x) = 0$ *for infinitely many* n *and* $f_n(x) = 1$ *for infinitely many* n*.*

We will see now another mode of convergence, that we be useful.

Definition 3.7.2. *Let* $\{f_n\}$ *be a sequence of measurable complex-valued functions on* $(X,\mathcal{M},\mu)$*. We say that* $\{f_n\}$ *is* **Cauchy in measure** *if for every* $\epsilon > 0$*,*

$$\mu(\{x \in X \colon |f_n(x) - f_m(x)| \geqslant \epsilon\}) \to 0 \qquad \textit{as } n,m \to \infty,$$

and that $\{f_n\}$ **converges in measure** *to* f *(which has to be measurable) if for every* $\epsilon > 0$*,*

$$\mu(\{x \in X \colon |f_n(x) - f(x)| \geqslant \epsilon\}) \to 0 \qquad \textit{as } n \to \infty.$$

Proposition 3.7.3. *If* $f_n \to f$ *in measure, then* $\{f_n\}$ *is Cauchy in measure.*

Proof. Let $\epsilon > 0$ be given. If $x \in X$ is such that $|f_n(x) - f_m(x)| \geqslant \epsilon$ then $\epsilon \leqslant |f_n(x) - f_m(x)| \leqslant |f_n(x) - f(x)| + |f(x) - f_m(x)|$. If $|f_n(x) - f(x)| < \epsilon/2$ and $|f(x) - f_m(x)| < \epsilon/2$ then $|f_n(x) - f_m(x)| < \epsilon$. Hence $x \notin \{x \in X \colon |f_n(x) - f(x)| < \epsilon/2\} \cap \{x \in X \colon |f_m(x) - f(x)| < \epsilon/2\}$ and therefore we have

$$\begin{aligned}&\{x \in X \colon |f_n(x) - f_m(x)| \geqslant \epsilon\}\\ &\qquad\qquad \subset \{x \in X \colon |f_n(x) - f(x)| \geqslant \epsilon/2\} \cup \{x \in X \colon |f(x) - f_m(x)| \geqslant \epsilon/2\},\end{aligned}$$

and since the measure of the two sets on the right hand side converge to zero as $n, m \to \infty$, $\{f_n\}$ is Cauchy in measure. ∎

In (i) of the previous example, given $\epsilon > 0$ we have

$$m(\{x \in \mathbb{R} \colon |f_n(x)| \geqslant \epsilon\}) = 0 \qquad \text{for all } n > \epsilon^{-1},$$

hence $f_n \to 0$ in measure.

In (iii) we have

$$m(\{x \in \mathbb{R} \colon |f_n(x)| \geqslant \epsilon\}) = 1/n \qquad \text{for } n \geqslant \epsilon,$$

hence $f_n \to 0$ in measure.

In (iv) we have

$$m(\{x \in \mathbb{R} \colon |f_n(x)| \geqslant \epsilon\}) = 2^{-k} \qquad \text{for } 2^k \leqslant n < 2^{k+1},$$

hence $f_n \to 0$ in measure.

But in (ii) we have for $n \neq m$ and $0 < \epsilon < 1$ then

$$m(\{x \in \mathbb{R} \colon |f_n(x) - f_m(x)| \geqslant \epsilon\}) = 2,$$

since $|f_n(x) - f_m(x)| = 1$ for $x \in (n, n+1) \cup (m, m+1)$. Therefore $\{f_n\}$ is not Cauchy in measure.

Proposition 3.7.4. *If* $f_n \to f$ *in* L^1 *then* $f_n \to f$ *in measure.*

Proof. Let $E_{n,\epsilon} = \{x \in X : |f_n(x) - f(x)| \geqslant \epsilon\}$. Then

$$\int |f_n - f| \geqslant \int_{E_{n,\epsilon}} |f_n - f| \geqslant \epsilon\mu(E_{n,\epsilon}),$$

thus $\mu(E_{n,\epsilon}) \leqslant \epsilon^{-1} \int |f_n - f| \to 0$ as $n \to \infty$, that is, $f_n \to f$ in measure. ■

The converse of this result is clearly false, by (i) and (iii) of the previous examples.

Theorem 3.7.5. *Suppose that $\{f_n\}$ is Cauchy in measure. Then there is a measurable function f such that $f_n \to f$ in measure, and there is a subsequence $\{f_{n_j}\}$ that converges to f a.e. Moreover, if also $f_n \to g$ in measure, then $g = f$ a.e.*

Proof. First we fix $j = 1$. Since $\{f_n\}$ is Cauchy in measure, we can find n_1 such that

$$\mu(\{x \in X : |f_n(x) - f_{n_1}(x)| \geqslant 2^{-1}\}) \leqslant 2^{-1} \quad \text{for } n, m \geqslant n_1. \tag{3.7.1}$$

Set $g_1 = f_{n_1}$. Likewise, we can choose $n_2 \geqslant n_1$ such that

$$\mu(\{x \in X : |f_n(x) - f_m(x)| \geqslant 2^{-2}\}) \leqslant 2^{-2} \quad \text{for } n, m \geqslant n_2.$$

Set $g_2 = f_{n_2}$ and $E_1 = \{x \in X : |g_1(x) - g_2(x)| \geqslant 2^{-1}\}$. Then by (3.7.1) we have $\mu(E_1) \leqslant 2^{-1}$. Inductively we can choose $n_{j+1} \geqslant n_j$, $g_j = f_{n_j}$ and $E_j = \{x \in X : |g_j(x) - g_{j+1}(x)| \geqslant 2^{-j}\}$ with $\mu(E_j) \leqslant 2^{-j}$.

Now for each k, set $F_k = \bigcup_{j=k}^{\infty} E_j$ then $\mu(F_k) \leqslant \sum\limits_{j=k}^{\infty} 2^{-j} = 2^{1-k}$, and for $x \notin F_k$ and $i \geqslant j \geqslant k$ we have

$$|g_j(x) - g_i(x)| \leqslant \sum_{p=j}^{i-1} |g_p(x) - g_{p+1}(x)| \leqslant \sum_{p=j}^{i-1} 2^{-p} \leqslant 2^{1-j}, \tag{3.7.2}$$

and thus $\{g_j\}$ is pointwise Cauchy on F_k^c. If $F = \bigcap_{k=1}^{\infty} F_k = \limsup E_j$, then $\mu(F) = \lim\limits_{j\to\infty} \mu(E_j) = 0$, and $\{g_j\}$ is pointwise Cauchy on F^c. Set $f(x) = \lim g_j(x)$ for $x \in F^c$ and $f(x) = 0$ for $x \in F$ (by Exercises 3 and 5, f is measurable). Hence $g_j \to f$ a.e.

Using (3.7.2) and making $i \to \infty$ for each $x \in F_k^c$, we have $|g_j(x) - f(x)| \leqslant 2^{1-j}$ and since $\mu(F_k) \to 0$ as $k \to \infty$, $g_j \to f$ in measure. Now

$$\{x \in X : |f_n(x) - f(x)| \geqslant \epsilon\} \subset \{x \in X : |f_n(x) - g_j(x)| \geqslant \epsilon/2\} \cup \{x \in X : |g_j(x) - f(x)| \geqslant \epsilon/2\},$$

and thus $f_n \to f$ in measure, since the measure of both sets on the right side converge to zero as $n, j \to \infty$.

Now assume that $f_n \to g$ in measure and fix $k \in \mathbb{N}$. We have

$$\{x \in X : |f(x) - g(x)| \geqslant k^{-1}\} \\ \subset \{x \in X : |f(x) - f_n(x)| \geqslant k^{-1}/2\} \cup \{x \in X : |f_n(x) - g(x)| \geqslant k^{-1}/2\},$$

for all n, and making $n \to \infty$ we obtain $\mu(\{x \in X : |f(x) - g(x)| \geqslant k^{-1}\}) = 0$. Thus, since $\{x \in X : f(x) \neq g(x)\} = \bigcap_{n=1}^{\infty}\{x \in X : |f(x) - g(x)| \geqslant k^{-1}\}$, we have $\mu(\{x \in X : f(x) \neq g(x)\}) = 0$ and hence $f = g$ a.e. ■

The fact that $f_n \to f$ a.e. alone is not enough to ensure that $f_n \to f$ in measure, as item (ii) of the previous example shows. However, this does hold, with even stronger conclusions, if X has finite measure, as we will show in the next result.

Theorem 3.7.6 (Egoroff's Theorem). *Suppose that $\mu(X) < \infty$ and $f_n \to f$ a.e. (f_n and f are all measurable complex-valued functions). Then for every $\epsilon > 0$ there exists $E \subset X$ with $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c.*

Proof. Assume first that $f_n \to f$ pointwise on X. For $k, n \in \mathbb{N}$ define

$$E_n(k) = \bigcup_{j=n}^{\infty}\{x \in X : |f_j(x) - f(x)| \geqslant k^{-1}\}.$$

If k is fixed, then $\{E_n(k)\}_n$ is a decreasing sequence and since $f_j(x) \to f(x)$ as $j \to \infty$ for each $x \in X$, we have $\bigcap_{n=1}^{\infty} E_n(k) = \varnothing$. Since $\mu(X) < \infty$, from the continuity from above, we have $\mu(E_n(k)) \to 0$ as $n \to \infty$. Given $\epsilon > 0$ and $k \in \mathbb{N}$, choose n_k such that $\mu(E_{n_k}(k)) < \epsilon 2^{-k}$ and let $E = \bigcup_{k=1}^{\infty} E_{n_k}(k)$. Then $\mu(E) < \epsilon$ and $|f_n(x) - f(x)| < k^{-1}$ for $n \geqslant n_k$ and $x \notin E$. Thus $f_n \to f$ uniformly on E^c.

Now if $f_n \to f$ a.e., let $F \subset X$ be the set with $\mu(F) = 0$ such that $f_n \to f$ everywhere on F^c. Thus, from the previous result (with F^c instead of X), given $\epsilon > 0$ there exists a set $E \subset F^c$ with $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c. Thus taking $A = E \cup F$ then $\mu(A) = \mu(E) + \mu(F) = \mu(E) < \epsilon$ and $A^c = E^c \cap F^c = E^c$, hence $f_n \to f$ uniformly on A^c. ■

The convergence in Egoroff's Theorem is often called **almost uniform convergence**, and it implies a.e. convergence and convergence in measure (see Exercise 39).

Proposition 3.7.7. *$f_n \to f$ in measure iff $\mathrm{Re} f_n \to \mathrm{Re} f$ and $\mathrm{Im} f_n \to \mathrm{Im} f$ in measure.*

Proof. Note that $|\mathrm{Re} f_n - \mathrm{Re} f| \leqslant |f_n - f|$, $|\mathrm{Im} f_n - \mathrm{Im} f| \leqslant |f_n - f|$ and $|f_n - f|^2 = |\mathrm{Re} f_n - \mathrm{Re} f|^2 + |\mathrm{Im} f_n - \mathrm{Im} f|^2$, hence

$$\{x \in X\colon |\mathrm{Re} f_n(x) - \mathrm{Re} f(x)| \geqslant \epsilon\} \subset \{x \in X\colon |f_n(x) - f(x)| \geqslant \epsilon\},$$

$$\{x \in X\colon |\mathrm{Im} f_n(x) - \mathrm{Im} f(x)| \geqslant \epsilon\} \subset \{x \in X\colon |f_n(x) - f(x)| \geqslant \epsilon\},$$

and

$$\begin{aligned}&\{x \in X\colon |f_n(x) - f(x)| \geqslant \epsilon\}\\ &\qquad\subset \{x \in X\colon |\mathrm{Re} f_n - \mathrm{Re} f| \geqslant \epsilon/\sqrt{2}\} \cup \{x \in X\colon |\mathrm{Im} f_n - \mathrm{Im} f| \geqslant \epsilon/\sqrt{2}\},\end{aligned}$$

which concludes the proof. ■

3.8 | Solved exercises from [1, Page 63]

Exercise 32. Suppose $\mu(X) < \infty$. If f and g are complex-valued measurable functions on X, define

$$\rho(f, g) = \int \frac{|f - g|}{1 + |f - g|} d\mu.$$

Then ρ is a metric on the space of complex-valued measurable functions defined on X, if we identify that function are equal a.e. and $f_n \to f$ with respect to this metric iff $f_n \to f$ in measure.

Solution. First of all, since $\mu(X) < \infty$ and $0 \leqslant \frac{|f-g|}{1+|f-g|} < 1$ for all f, g measurable we have

$$\int \frac{|f - g|}{1 + |f - g|} d\mu < \mu(X) < \infty,$$

and $\rho(f, g)$ is well defined.

We have $\rho(f, g) = 0$ iff $|f - g| = 0$ a.e., that is, iff $f = g$ a.e. Clearly $\rho(f, g) \geqslant 0$ and $\rho(f, g) = \rho(g, f)$ for all measurable f, g.

To show the triangle inequality, consider the real function $u(s) = s(1 + s)^{-1}$ for $s \geqslant 0$. Then $u'(s) = (1 + s)^{-2} > 0$, and hence u is increasing. Thus, since $|f - g| \leqslant |f - h| + |h - g|$

we have

$$\begin{aligned}\frac{|f-g|}{1+|f-g|} &= u(|f-g|) \leqslant u(|f-h|+|h-g|) = \frac{|f-h|+|h-g|}{1+|f-h|+|h-g|} \\ &= \frac{|f-h|}{1+|f-h|+|h-g|} + \frac{|h-g|}{1+|f-h|+|h-g|} \\ &\leqslant \frac{|f-h|}{1+|f-h|} + \frac{|h-g|}{1+|h-g|},\end{aligned}$$

and integration on both sides yields $\rho(f,g) \leqslant \rho(f,g) + \rho(h,g)$ for f,g,h measurable.

Now assume that $\rho(f_n, f) \to 0$. We will prove that $f_n \to f$ in measure. To this end let $\epsilon > 0$ and consider $E_n = \{x \in X : |f_n(x) - f(x)| \geqslant \epsilon\}$. Thus using the increasing property of the function u above, for $x \in E_n$ we have

$$\rho(f_n, f) \geqslant \int_{E_n} \frac{|f_n - f|}{1+|f_n - f|} = \int_{E_n} u(|f_n - f|) \geqslant \int_{E_n} u(\epsilon) = \int_{E_n} \frac{\epsilon}{1+\epsilon} = \frac{\epsilon}{1+\epsilon}\mu(E_n),$$

thus $\mu(E_n) \leqslant \epsilon^{-1}(1+\epsilon)\rho(f_n, f) \to 0$ as $n \to \infty$. Hence $f_n \to f$ in measure.

If $f_n \to f$ in measure, let $\epsilon > 0$, $n \in \mathbb{N}$ and $E_{n,\epsilon} = \{x \in X : |f_n(x) - f(x)| \geqslant \epsilon\}$. Thus

$$\rho(f_n, f) = \int \frac{|f_n - f|}{1+|f_n - f|} = \int_{E_{n,\epsilon}} \frac{|f_n - f|}{1+|f_n - f|} + \int_{E_{n,\epsilon}^c} \frac{|f_n - f|}{1+|f_n - f|} \leqslant \mu(E_{n,\epsilon}) + \epsilon\mu(X),$$

since $0 \leqslant \frac{|f_n-f|}{1+|f_n-f|} < 1$ in X, $\frac{|f_n-f|}{1+|f_n-f|} < \frac{\epsilon}{1+|f_n-f|} < \epsilon$ in $E_{n,\epsilon}^c$ and $\mu(E_{n,\epsilon}^c) \leqslant \mu(X)$. Therefore $\rho(f_n, f) \to 0$ as $n \to \infty$.

Exercise 33. If $f_n \geqslant 0$ is measurable for all n and $f_n \to f$ in measure then $\int f \leqslant \liminf \int f_n$.

Solution. Let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$ such that $\int f_{n_k} \to \liminf \int f_n =: \alpha$. Since $\int f_n \geqslant 0$, we have $\alpha \geqslant 0$. Since $f_n \to f$ in measure, $f_{n_k} \to f$ is measure as well. Hence by Theorem 3.7.5 there exists a subsequence $\{f_{n_{k_j}}\}$ of $\{f_{n_k}\}$ such that $f_{n_{k_j}} \to f$ a.e. as $j \to \infty$. Hence, by Fatou's Lemma

$$\int f \leqslant \liminf \int f_{n_{k_j}} = \lim \int f_{n_{k_j}} = \lim \int f_{k_n} = \alpha = \liminf \int f_n.$$

Exercise 34. Suppose $|f_n| \leqslant g \in L^1$ and $f_n \to f$ in measure.

(a) $\int f = \lim \int f_n$.

(b) $f_n \to f$ in L^1.

Using Proposition 3.7.7, we can assume that f_n and f are all real. Also we note that since $|\int f_n| \leqslant \int |f_n| \leqslant \int g < \infty$, the sequence $\{\int f_n\}$ is bounded and has a convergent subsequence (which we will call $\{\int f_{n_k}\}$). From this subsequence, since $f_{n_k} \to f$ in measure, there exists a subsequence (which we call $\{f_{n_{k_j}}\}$) and a measurable function h, with $h = f$ a.e., such that $f_{n_{k_j}} \to h$ a.e. But then by the DCT, since $|f_{n_{k_j}}| \to |h|$ a.e. and $|f_{n_{k_j}}| \leqslant g$ we have

$$\int |f| = \int |h| = \lim_{j\to\infty} \int |f_{n_{k_j}}|,$$

and since $\{\int |f_n|\}$ is bounded, $f \in L^1$. Now we can solve the exercise.

Solution to (a).

Now we have $g + f_n \geqslant 0$ and $g - f_n \geqslant 0$ for all n and since $f_n \to f$ in measure, we have $g + f_n \to g + f$ and $g - f_n \to g - f$ in measure. Using Exercise 33, we have

$$\int g + \int f = \int (g+f) \leqslant \liminf \int (g + f_n) = \int g + \liminf \int f_n$$

and

$$\int g - \int f = \int (g-f) \leqslant \liminf \int (g - f_n) = \int g - \limsup \int f_n,$$

and since $g \in L^1$, we have $\int f \leqslant \liminf \int f \leqslant \limsup f_n \leqslant \int f$, hence all inequalities are equalities and $\int f = \lim \int f_n$.

Solution to (b). Since for given $\epsilon > 0$ we have

$$\{x \in X \colon ||f_n(x) - f(x)| - 0| \geqslant \epsilon\} = \{x \in X \colon |f_n(x) - f(x)| \geqslant \epsilon\},$$

we see that $f_n \to f$ in measure iff $|f_n - f| \to 0$ in measure. Also $|f_n - f| \leqslant g + |f| \in L^1$, hence we can apply part (a) to $|f_n - f|$ to conclude that

$$\int |f_n - f| \to 0,$$

that is, $f_n \to f$ in L^1.

EXERCISE 35. $f_n \to f$ in measure iff for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\mu(\{x \in X \colon |f_n(x) - f(x)| \geqslant \epsilon\}) < \epsilon$ for every $n \geqslant N$.

Solution. If $f_n \to f$ in measure, the conclusion is straightforward from the definition of convergence in measure. Now for the converse, assume that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\mu(\{x \in X \colon |f_n(x) - f(x)| \geqslant \epsilon\}) < \epsilon$ for every $n \geqslant N$. Assume also that $\{f_n\}$ does not converge to f in measure. This implies that exist $\epsilon_0, \eta_0 > 0$ and a sequence $n_k \to \infty$ as

$k \to \infty$ such that

$$\mu(\{x \in X : |f_{n_k}(x) - f(x)| \geqslant \epsilon_0\}) \geqslant \eta_0 \text{ for all } k.$$

Set $\epsilon = \min\{\epsilon_0, \eta_0\} > 0$. We have

$$\{x \in X : |f_{n_k}(x) - f(x)| \geqslant \epsilon\} \supset \{x \in X : |f_{n_k}(x) - f(x)| \geqslant \epsilon_0\},$$

and thus

$$\mu(\{x \in X : |f_{n_k}(x) - f(x)| \geqslant \epsilon\}) \geqslant \mu(\{x \in X : |f_{n_k}(x) - f(x)| \geqslant \epsilon_0\}) \geqslant \eta_0 \geqslant \epsilon,$$

for all k, and contradicts the hypothesis.

EXERCISE 36. If $\mu(E_n) < \infty$ for $n \in \mathbb{N}$ and $\chi_{E_n} \to f$ in L^1, then f is (a.e. equal to) the characteristc function of a measurable set.

Solution. Since $\chi_{E_n} \to f$, then there exists a subsequence $\{E_{n_j}\}$ such that $\chi_{E_{n_j}} \to f$ a.e. Let $E \subset X$ be the measurable set on which $\chi_{E_{n_j}}(x) \to f(x)$ for $x \in E$, and $\mu(E^c) = 0$.

But since $\chi_{E_{n_j}}(x) = 0$ or 1 for every $x \in X$, we must have $f(x) = 0$ or $f(x) = 1$ on E. If $A = \{x \in E : f(x) = 1\} = E \cap f^{-1}(\{1\})$ then A is measurable and $f = \chi_A$ a.e., since $f(x) = \chi_A(x)$ for all $x \in E$.

Note. The hypothesis $\mu(E_n) < \infty$ for all n is only to ensure that $\chi_{E_n} \in L^1$ for all n.

EXERCISE 37. Suppose that f_n and f are measurable complex functions and $\phi : \mathbb{C} \to \mathbb{C}$.

(a) If ϕ is continuous and $f_n \to f$ a.e., then $\phi \circ f_n \to \phi \circ f$ a.e.

(b) If ϕ is uniformly continuous and $f_n \to f$ uniformly, almost uniformly, or in measure, then $\phi \circ f_n \to \phi \circ f$ uniformly, almost uniformly, or in measure, respectively.

(c) There are counterexamples when the continuity assumptions on ϕ are note satisfied.

Solution to (a). Let $E \subset X$ be the null set such that $f_n(x) \to f(x)$ for every $x \in E^c$. Then, since ϕ is continuous, $\phi(f_n(x)) \to \phi(f(x))$ for each $x \in E^c$, hence $\phi \circ f_n \to \phi \circ f$ a.e.

Solution to (b). Given $\epsilon > 0$ there exists $\delta > 0$ such that $|\phi(z) - \phi(w)| < \epsilon$ if $|z - w| < \delta$.

Assume that $f_n \to f$ uniformly. Then for $\delta > 0$ above, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \delta$ for all $x \in X$. Hence

$$|\phi(f_n(x)) - \phi(f(x))| < \epsilon \quad \text{ for all } x \in X,$$

hence $\phi \circ f_n \to \phi \circ f$ uniformly.

If $f_n \to f$ almost uniformly, there exists a measurable set E with $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c. From the above, $\phi \circ f_n \to \phi \circ f$ on E^c uniformly and hence $\phi \circ f_n \to \phi \circ f$ almost uniformly.

If $f_n \to f$ in measure, then

$$\{x \in X\colon |f_n(x) - f(x)| < \delta\} \subset \{x \in X\colon |\phi(f_n(x)) - \phi(f(x))| < \epsilon\},$$

thus

$$\{x \in X\colon |f_n(x) - f(x)| \geqslant \delta\} \supset \{x \in X\colon |\phi(f_n(x)) - \phi(f(x))| \geqslant \epsilon\},$$

and hence

$$\mu(\{x \in X\colon |f_n(x) - f(x)| \geqslant \delta\}) \geqslant \mu(\{x \in X\colon |\phi(f_n(x)) - \phi(f(x))| \geqslant \epsilon\}),$$

and since the measure on the left side converges to zero as $n \to \infty$ we have $\phi \circ f_n \to \phi \circ f$ in measure.

Solution to (c). We can assume, without loss of generality that we have real-valued functions and $\phi\colon \mathbb{R} \to \mathbb{R}$.

For a counterexample of (a), take $f_n(x) \equiv 1/n$ for all $n \in \mathbb{N}$ and $f \equiv 0$ (thus $f_n \to f$ uniformly in $\mathbb{R}$) and $\phi(x) = 0$ for $x \neq 0$ and $\phi(0) = 1$. Thus $\phi \circ f_n = 0$ and $\phi \circ f = 1$ and $\{(\phi \circ f_n)(x)\}$ does not converge to $(\phi \circ f)(x)$ for any $x \in \mathbb{R}$.

Counterexample of (b). Define $f_n(x) = x + 1/n$ and $f(x) = x$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Thus $f_n \to f$ uniformly (and hence almost uniformly and in measure). Take $\phi(x) = x^2$ (which is continuous but not uniformly continuous). But

$$\{x \in \mathbb{R}\colon |(\phi \circ f_n)(x) - (\phi \circ f)(x)| \geqslant \epsilon\} = \{x \in \mathbb{R}\colon |\tfrac{2x}{n} + \tfrac{1}{n^2}| \geqslant \epsilon\} \supset [\tfrac{\epsilon n^2 - 1}{2n}, \infty),$$

for all $\epsilon > 0$ and $n \in \mathbb{N}$, hence f_n does not converge to f in measure (and hence it does not converge neither almost uniformly nor uniformly, see Exercise 39).

Exercise 38. Suppose $f_n \to f$ and $g_n \to g$ in measure.

(a) $f_n + g_n \to f + g$ in measure.

(b) $f_n g_n \to fg$ in measure if $\mu(X) < \infty$, but not necessarily if $\mu(X) = \infty$.

Solution to (a). Given $\epsilon > 0$, we have

$$\{x \in X\colon |(f_n + g_n)(x) - (f+g)(x)| \geqslant \epsilon\} \\ \subset \{x \in X\colon |f_n(x) - f(x)| \geqslant \epsilon/2\} \cup \{x \in X\colon |g_n(x) - g(x)| \geqslant \epsilon/2\},$$

and from this it follows that $f_n + g_n \to f + g$ in measure.

Solution to (b). Recall that f_n, f, g_n, g are complex-valued functions. We will prove first that $f_n^2 \to f^2$ in measure, and to this end, we will brake the proof into a few claims.

We define for $M > 0$, the set $A_M(h) = \{x \in X\colon |h(x)| \geqslant M\}$, for $h = f_n$ or $h = g$.

Claim 1: given $\eta > 0$ we can choose $M > 0$ such that $\mu(A_M(f)) < \eta$.

We have $A_{m+1}(f) \subset A_m(f)$ for all m, and $\bigcap\limits_{m=1}^{\infty} A_m(f) = \varnothing$. Since $\mu(X) < \infty$, we have from the continuity from above $\lim\limits_{m\to\infty} \mu(A_m(f)) = \mu(\bigcap\limits_{m=1}^{\infty} A_m(f)) = \mu(\varnothing) = 0$, hence the claim holds.

Claim 2: given $\eta > 0$ and $M > 0$ as above, we can choose $N \in \mathbb{N}$ such that $\mu(A_{M+1}(f_n)) < 2\eta$ for all $n \geqslant N$.

Indeed, since $f_n \to f$ in measure, we have $\mu(\{x \in X\colon |f_n(x) - f(x)| \geqslant 1\}) \to 0$ as $n \to \infty$, hence there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < 1$ on a set B^c, with $\mu(B) < \eta$. Hence $|f_n(x)| < 1 + |f(x)|$ on B^c and

$$\begin{aligned} A_{M+1}(f_n) &= \{x \in B^c\colon |f_n(x)| \geqslant M+1\} \cup \{x \in B\colon |f_n(x)| \geqslant M+1\} \\ &\subset \{x \in B^c\colon |f(x)| \geqslant M\} \cup B \\ &\subset A_M(f) \cup B, \end{aligned}$$

for all $n \geqslant N$, hence $\mu(A_{M+1}(f_n)) < 2\eta$, which proves Claim 2.

Since $A_{M+1}(f) \subset A_M(f)$, we can join these two claims to obtain the following: given $\eta > 0$, we can choose $M, N > 0$ such that for all $n \geqslant N$ we have

$$\mu(A_M(f)) < \eta \quad \text{ and } \mu(A_M(f_n)) < 2\eta.$$

Given $\epsilon > 0$, since $f_n \to f$ in measure choose $N_1 \geqslant N$ such that for $n \geqslant N_1$ we have

$$\mu(\{x \in X\colon |f_n(x) - f(x)| \geqslant \epsilon/2M\}) < \eta.$$

Thus if $|f(x)| < M$, $|f_n(x)| < M$ and $|f_n(x) - f(x)| < \epsilon/2M$ we have

$$|f_n^2(x) - f^2(x)| = |f_n(x) - f(x)||f_n(x) + f(x)| < 2M|f_n(x) - f(x)| < \epsilon,$$

hence

$$\{x \in X\colon |f_n^2(x) - f^2(x)| \geqslant \epsilon\} \subset A_M(f) \cup A_M(f_n) \cup \{x \in X\colon |f_n(x) - f(x)| \geqslant \epsilon\},$$

and thus

$$\mu(\{x \in X\colon |f_n^2(x) - f^2(x)| \geqslant \epsilon\}) \leqslant 4\eta,$$

for $n \geqslant N_1$. Thus $f_n^2 \to f^2$ in measure.

Since $f_n g_n = \frac{1}{2}\Big[(f_n + g_n)^2 - f_n^2 - g_n^2\Big]$ and $fg = \frac{1}{2}\Big[(f+g)^2 - f^2 - g^2\Big]$, we have $f_n g_n \to fg$ in measure, using item (a).

Now for the counterexample, see the counterexample of Exercise 37 (c).

Exercise 39. If $f_n \to f$ almost uniformly, then $f_n \to f$ in measure.

Solution. Given $\epsilon > 0$, choose a measurable set E with $\mu(E) < \epsilon$ such that $f_n \to f$ uniformly on E^c. Also, choose $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for $n \geqslant N$ and all $x \in E^c$. Thus if $n \geqslant N$ we have

$$\{x \in X\colon |f_n(x) - f(x)| \geqslant \epsilon\} \subset E,$$

and hence $\mu(\{x \in X\colon |f_n(x) - f(x)| \geqslant \epsilon\}) < \epsilon$. Thus by Exercise 35, $f_n \to f$ in measure.

Exercise 40. In Egoroff's theorem, the hypothesis "$\mu(X) < \infty$" can be replaced by "$|f_n| \leqslant g$ for all n, where $g \in L^1(\mu)$".

Solution. From the DCT $\int |f| \leqslant \int g$ and $f \in L^1(\mu)$. As in the proof of Egoroff's Theorem, we can assume without loss of generality that $f_n \to f$ pointwise, and we set

$$E_n(k) = \bigcup_{j=n}^{\infty} \{x \in X\colon |f_j(x) - f(x)| \geqslant k^{-1}\}.$$

If we can prove that $\mu(E_1(k)) < \infty$ for all k, then as in the proof of Egoroff's Theorem, it will follow that $\mu(E_n(k)) \to 0$ as $n \to \infty$ and the rest of the proof remains unchanged.

Now, if $x \in E_1(k)$ then there exists $j \in \mathbb{N}$ such that $|f_j(x) - f(x)| \geqslant k^{-1}$. Hence

$$\begin{aligned}
\mu(E_1(k)) = \int \chi_{E_1(k)} d\mu = \int_{E_1(k)} d\mu \leqslant k \int_{E_1(k)} |f_j(x) - f(x)| d\mu \\
\leqslant k \int_{E_1(k)} (|f_j(x)| + |f(x)|) d\mu \leqslant 2k \int_{E_1(k)} g d\mu \leqslant 2k \int g d\mu < \infty,
\end{aligned}$$

since $g \in L^1(\mu)$. Therefore the result follows.

Exercise 41. If μ is σ-finite and $f_n \to f$ a.e., there exist measurable $E_1, E_2, \cdots \subset X$ such that $\mu\Big(\Big(\bigcup_{j=1}^{\infty} E_j\Big)^c\Big) = 0$ and $f_n \to f$ uniformly on each E_j.

Solution. Since μ is σ-finite, we can write $X = \bigcup_{j=1}^{\infty} A_j$ with $\mu(A_j) < \infty$ and the sequence $\{A_j\} \subset \mathcal{M}$ disjoint. Since $f_n \to f$ a.e. on X, $f_n \to f$ a.e. on each A_j.

Now we fix j. Given $k \in \mathbb{N}$, from Egoroff's Theorem applied to A_j, there exists a measurable set $E_{j,k} \subset A_j$ with $\mu(A_j \setminus E_{j,k}) < k^{-1}2^{-j}$ and $f_n \to f$ uniformly on $E_{j,k}$.

Hence

$$\mu\Big(\Big(\bigcup_{j=1}^{\infty} E_{j,k}\Big)^c\Big) \leqslant \mu\Big(\bigcup_{j=1}^{\infty}(A_j \setminus E_{j,k})\Big) = \sum_{j=1}^{\infty} \mu(A_j \setminus E_{j,k}) \leqslant \frac{1}{k} \quad \text{for all } j,$$

and thus $\mu\Big(\Big(\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{\infty} E_{j,k}\Big)^c\Big) = \mu\Big(\bigcap_{k=1}^{\infty}\Big(\bigcup_{j=1}^{\infty} E_{j,k}\Big)^c\Big) \leqslant \frac{1}{k}$ for each k.

Therefore $f_n \to f$ uniformly on each $E_{j,k}$ and $\mu\Big(\Big(\bigcup_{j,k=1}^{\infty} E_{j,k}\Big)^c\Big) = 0$. Relabelling the $E_{j,k}$'s we have the result.

Exercise 42. Let μ be the counting measure on $\mathbb{N}$. Then $f_n \to f$ in measure iff $f_n \to f$ uniformly.

Solution. Assume that $f_n \to f$ in measure. Thus, given $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\mu(\{m \in \mathbb{N} \colon |f_n(m) - f(m)| \geqslant \epsilon\}) < 1,$$

since μ is the counting measure, then $\mu(\{m \in \mathbb{N} \colon |f_n(m) - f(m)| \geqslant \epsilon\}) = 0$, which implies that $\{m \in \mathbb{N} \colon |f_n(m) - f(m)| \geqslant \epsilon\} = \varnothing$, and thus $|f_n(m) - f(m)| < \epsilon$ for all $m \in \mathbb{N}$ and $n \geqslant N$, that is, $f_n \to f$ uniformly.

The converse is straightforward.

Exercise 43. Suppose that $\mu(X) < \infty$ and $f \colon X \times [0,1] \to \mathbb{C}$ is a function such that $f(\cdot, y)$ is measurable for each $y \in [0,1]$ and $f(x, \cdot)$ is continuous for each $x \in X$.

(a) If $0 < \epsilon, \delta < 1$ then $E_{\epsilon,\delta} = \{x \in X \colon |f(x,y) - f(x,0)| \leqslant \epsilon \text{ for all } y < \delta\}$ is measurable.

(b) For any $\epsilon > 0$, there is a set $E \subset X$ such that $\mu(E) < \epsilon$ and $f(\cdot, y) \to f(\cdot, 0)$ uniformly on E^c as $y \to 0$.

Solution to (a). Define

$$F_{\epsilon,\delta} = \{x \in X \colon |f(x,y) - f(x,0)| \leqslant \epsilon \text{ for all } y < \delta \text{ with } y \in \mathbb{Q}\},$$

thus $F_{\epsilon,\delta} = \bigcap\{x \in X \colon |f(x,y) - f(x,0)| \leqslant \epsilon\}$ where the intersection is taken over rational $y < \delta$, and thus, since $f(\cdot, y)$ is measurable for each $y \in [0,1]$, it is a countable union of measurable sets, which is measurable.

To conclude (a), we will prove that $E_{\epsilon,\delta} = F_{\epsilon,\delta}$. Clearly $E_{\epsilon,\delta} \subset F_{\epsilon,\delta}$. Now if $x \in F_{\epsilon,\delta}$, then $|f(x,y) - f(x,0)| \leqslant \epsilon$ for all $y < \delta$ in $\mathbb{Q}$. But $f(x,\cdot)$ is continuous, and by density of $\mathbb{Q}$, it follows that $|f(x,y) - f(x,0)| \leqslant \epsilon$ for all $y < \delta$. Hence $F_{\epsilon,\delta} \subset E_{\epsilon,\delta}$.

Solution to (b). Fix $\epsilon, \eta > 0$. For each $n \in \mathbb{N}$, define $A_n = E_{\eta,1/n}$, which is measurable by item (a). We have $A_n \subset A_{n+1}$ for all n. Since $f(x,y) \to f(x,0)$ as $y \to 0$ for each $x \in X$ we have $\bigcap\limits_{n=1}^{\infty} A_n^c = \varnothing$ and from the continuity from above (since $\mu(X) < \infty$) we can choose $N \in \mathbb{N}$ such that $\mu(A_N^c) < \epsilon$. On A_N we have $|f(x,y) - f(x,0)| \leqslant \eta$ for $y < 1/N$.

Now we will use this procedure as follows. Fix $\epsilon > 0$ and $j \in \mathbb{N}$. From the previous construction, we can choose $N_j \in \mathbb{N}$ and a set $A_{N_j} = E_{1/j,1/N_j}$, with $\mu(A_{N_j}^c) \leqslant \epsilon 2^{-j}$ and $|f(x,y) - f(x,0)| \leqslant 1/j$ on A_{N_j} for $y < 1/N_j$.

Take $E = \bigcup\limits_{j=1}^{\infty} A_{N_j}^c$. Thus $\mu(E) \leqslant \epsilon$ and given $\eta > 0$, choose j such that $1/j < \eta$ and for $x \in E^c = \bigcap\limits_{j=1}^{\infty} A_{N_j}$ we have

$$|f(x,y) - f(x,0)| < 1/j < \eta \text{ for } y < 1/N_j,$$

which means that $f(\cdot, y) \to f(\cdot, 0)$ uniformly on E^c as $y \to 0$.

Exercise 44. **Lusin's Theorem.** If $f \colon [a,b] \to \mathbb{C}$ is Lebesgue measurable and $\epsilon > 0$, there is a compact set $E \subset [a,b]$ such that $m(E^c) < \epsilon$ and $f|_E$ is continuous (Use Egoroff's Theorem and Theorem 3.5.15).

Solution. Take $\epsilon > 0$. For each $j \in \mathbb{N}$, set $A_j = \{x \in [a,b] \colon |f(x)| \leqslant j\}$. Thus $[a,b] = \bigcup\limits_{j=1}^{\infty} A_j$, $A_j \subset A_{j+1}$ for all j and since $m([a,b]) = b - a < \infty$, using the lower semicontinuity of m we can choose j_0 such that $m([a,b] \setminus A_{j_0}) < \epsilon$.

Define $g = \chi_{A_{j_0}} f$. Using Theorem 3.5.15, for each n there exists a continuous function $g_n \colon [a,b] \to \mathbb{C}$ such that $\int_a^b |g_n(x) - g(x)| dx < 1/n$.

Thus $g_n \to g$ in $L^1([a,b], m)$ and hence it converges in measure. Thus there exists a subsequence $\{g_{n_k}\}$ that converges a.e. to g. Now we can use Egoroff's Theorem to ensure that there exists a set $F \subset [a,b]$ with $\mu(F) < \epsilon$ and $g_{n_k} \to g$ uniformly on $[a,b] \setminus F$.

By inner regularity of m, there exists a compact set $E \subset A_{j_0} \setminus F$ such that $m(A_{j_0} \setminus F) \leqslant m(E) + \epsilon$. Thus

$$m([a,b] \setminus E) = m([a,b]) - m(E) \leqslant \underbrace{m([a,b]) - m(A_{j_0})}_{<\epsilon} + \underbrace{m(F)}_{<\epsilon} + \epsilon < 3\epsilon.$$

Since $E \subset [a,b] \setminus F$, $g_{n_k} \to g$ uniformly on E, hence $g|_E$ is continuous. Also $E \subset A_{j_0}$, hence $f|_E = g|_E$, so $f|_E$ is continuous and concludes the result.

3.9| Product measures

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. We have already constructed the product σ-algebra $\mathcal{M} \otimes \mathcal{N}$, which is generated by the family $\mathcal{E} = \{E \times F \colon E \in \mathcal{M} \text{ and } F \in \mathcal{N}\}$. Now, using μ and ν, we want to define a measure on $\mathcal{M} \otimes \mathcal{N}$.

First, the elements on $\mathcal{E}$ are called **rectangles**.

Lemma 3.9.1. *The family $\mathcal{E}$ of rectangles is an elementary family.*

Proof. Clearly $\varnothing = \varnothing \times \varnothing$ is a rectangle, thus $\varnothing \in \mathcal{E}$. Now if $A \times B$, $C \times D$ are rectangles, we have

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D) \in \mathcal{E},$$

and

$$(A \times B)^c = (X \times B^c) \cup (A^c \times B),$$

which is a finite disjoint union of rectangles. Thus $\mathcal{E}$ is an elementary family. ■

Using Proposition 1.4.6, the family $\mathcal{A}$ given by finite disjoint union of rectangles is an algebra (and $\mathcal{A}$ also generates $\mathcal{M} \otimes \mathcal{N}$).

Assume that $A \times B$ is a rectangle given as a (finite or countable) disjoint union of rectangles $A_j \times B_j$. Then for $x \in X$ and $y \in Y$ we have

$$\chi_A(x)\chi_B(y) = \chi_{A\times B}(x,y) = \sum \chi_{A_j \times B_j}(x,y) = \sum \chi_{A_j}(x)\chi_{B_j}(y).$$

Integrating with respect to x and using Theorem 3.3.9, we obtain

$$\mu(A)\chi_B(y) = \int \chi_A(x)\chi_B(y)d\mu(x) = \sum \int \chi_{A_j}(x)\chi_{B_j}(y)d\mu(y) = \sum \mu(A_j)\chi_{B_j}(y).$$

Now integrating with respect to y and again using Theorem 3.3.9 we have

$$\mu(A)\nu(B) = \sum \mu(A_j)\nu(B_j).$$

Definition 3.9.2 (Product premeasure). *If $E \in \mathcal{A}$ is the disjoint union of rectangles $\{A_i \times B_i\}_{i=1}^n$, we define*

$$\pi(E) = \sum_{i=1}^{n} \mu(A_i)\nu(B_i),$$

with the convention $0 \cdot \infty = 0$.

Then π is well defined by our previous argument, since any two representations of E as a finite disjoint union of rectangles have a common refinement), and it is a premeasure on $\mathcal{A}$.

Exercise 3.9.3. *Prove that π is a premeasure on $\mathcal{A}$.*

Using Theorem 2.3.5, π generates an outer measure whose restriction to $\mathcal{M} \otimes \mathcal{N}$ is a measure that extends π. This measure is called the **product measure** of μ and ν, and it is denoted by $\mu \times \nu$.

If both μ and ν are σ-finite, say $X = \bigcup_{j=1}^{\infty} A_j$ and $Y = \bigcup_{k=1}^{\infty} B_k$ with $\mu(A_j) < \infty$ and $\nu(B_k) < \infty$ for all j, k, then $X \times Y = \bigcup_{j,k} A_j \times B_k$ and $\mu \times \nu(A_j \times B_k) = \mu(A_j)\nu(B_k) < \infty$ for all j, k, so $\mu \times \nu$ is also σ-finite, and Theorem 2.3.5 ensures us that $\mu \times \nu$ is the unique measure on $\mathcal{M} \otimes \mathcal{N}$ that $\mu \times \nu(A \times B) = \mu(A)\nu(B)$ for all rectangles $A \times B$.

Proposition 3.9.4. *If μ and ν are σ-finite, then there exists an increasing sequence of rectangles $\{A_j \times B_j\}$ with finite product measure, such that $X \times Y = \bigcup_j (A_j \times B_j)$.*

Proof. We can write $X = \bigcup_j A_j$ and $Y = \bigcup_j B_j$ with $\{A_j\}$ and $\{B_j\}$ increasing sequences of measurable sets of μ and ν finite measures on X and Y, respectively. Thus $\{A_j \times B_j\}$ is an incrasing sequence of rectangles in $X \times Y$ with finite $\mu \times \nu$ measure and $X \times Y = \bigcup_j (A_j \times B_j)$. ■

The same construction works for any finite number of factors. That is, if $(X_i, \mathcal{M}_i, \mu_i)$ are measure spaces for $i = 1, \cdots, n$, defining a rectangle as sets of the form $A_1 \times \cdots \times A_n$ with $A_i \in \mathcal{M}_i$, then the collection $\mathcal{A}$ of disjoint unions of rectangles is an algebra and the same procedure used above can be applied to produce a measure $\mu_1 \times \cdots \times \mu_n$ on $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ such that

$$\mu_1 \times \cdots \times \mu_n(A_1 \times \cdots \times A_n) = \prod_{i=1}^{n} \mu_i(A_i).$$

Moreover, if all μ_i's are σ-finite, $\mu_1 \times \cdots \times \mu_n$ is the unique measure on $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ that extends the defined premeasure on $\mathcal{A}$. In this case, the obvious associativity properties hols, for example, identifying $X_1 \times X_2 \times X_3$ with $(X_1 \times X_2) \times X_3$, we have $\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \mathcal{M}_3 = (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$ (the former is generated by sets of the form $A_1 \times A_2 \times A_3$ with $A_i \in \mathcal{M}_i$ and the latter by sets of the form $B \times A_3$ with $B \in \mathcal{M}_1 \times \mathcal{M}_2$ and $A_3 \in \mathcal{M}_3$), and $\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$ (since they agree on sets of the form $A_1 \times A_2 \times A_3$, and hence in general by uniqueness).

We will presente the results for $n = 2$, just for simplicity, but they hold for any finite number of factors. Hence from now on we will just consider the case of two measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$.

Definition 3.9.5. *If $E \subset X \times Y$ we define:*

(i) *for each $x \in X$ the* **x-section** *E_x of E as the subset of Y given by*

$$E_x = \{y \in Y : (x, y) \in E\},$$

(ii) *for each $y \in Y$ the* **y-section** *E^y of E as the subset of X given by*

$$E^y = \{x \in X : (x, y) \in E\}.$$

Also, if f is a function defined in $X \times Y$, we define the **x-section** *f_x and* **y-section** *f^y of f by*

$$f_x(y) = f^y(x) = f(x, y).$$

Example 3.9.6. *If $E \subset X \times Y$, then $(\chi_E)_x = \chi_{E_x}$ and $(\chi_E)^y = \chi_{E^y}$. In fact, if $x \in X$ then $\chi_E(x, y) = 1$ iff $y \in E_x$ and if $y \in Y$ then $\chi_E(x, y) = 1$ iff $x \in E^y$.*

Proposition 3.9.7.

(a) *If $E \in \mathcal{M} \otimes \mathcal{N}$ then $E_x \in \mathcal{N}$ for all $x \in X$ and $E^y \in \mathcal{M}$ for all $y \in Y$.*

(b) *If f is $\mathcal{M} \otimes \mathcal{N}$-measurable, then f_x is $\mathcal{N}$-measurable for all $x \in X$ and f^y is $\mathcal{M}$-measurable for all $y \in Y$.*

Proof. **(a).** Define

$$\mathcal{R} = \{E \subset X \times Y : E_x \in \mathcal{N} \text{ for all } y \in Y \text{ and } E^y \in \mathcal{M} \text{ for all } x \in X\}.$$

Since $\left(\bigcup_{j=1}^{\infty} E_i\right)_x = \bigcup_{j=1}^{\infty}(E_i)_x$ for all $x \in X$ and $\left(\bigcup_{j=1}^{\infty} E_i\right)^y = \bigcup_{j=1}^{\infty}(E_i)^y$, we see that $\mathcal{R}$ is closed under countable unions. Also, since for $E \in \mathcal{R}$ we have $(E^c)_x = (E_x)^c$ and $(E^c)^y = (E^y)^c$, $\mathcal{R}$ is closed under complements.

Also each rectangle $A \times B$ is in $\mathcal{R}$, since $(A \times B)_x = B$ if $x \in A$ and $(A \times B)_x = \varnothing$ if $x \in A^c$, and $(A \times B)^y = A$ if $y \in B$ and $(A \times B)^y - \varnothing$ if $y \in B^c$. Hence $\mathcal{R}$ is a σ-algebra that contains all rectangles, and thus $\mathcal{M} \otimes \mathcal{N} \subset \mathcal{R}$, and concludes the proof of (a).

(b). We have for any set J

$$(f_x)^{-1}(J) = \{y \in Y \colon f_x(y) \in J\} = \{y \in Y \colon f(x,y) \in J\} = (f^{-1}(J))_x$$

and $(f^y)^{-1}(J) = (f^{-1}(J))^y$, hence (b) follows from (a) and from the $\mathcal{M} \otimes \mathcal{N}$-measurability of f. ■

3.9.1 | Monotone classes

Before proceeding, we will need some technical lemmas, that will help us further.

Definition 3.9.8. *Let X be a nonempty set. A subset $\mathcal{C}$ of $\mathcal{P}(X)$ is called a* **monotone class** *if $\mathcal{C}$ is closed under countable increasing unions and countable decreasing intersections, that is, if $\{E_j\} \subset \mathcal{C}$ and $E_j \subset E_{j+1}$ for all j then $\bigcup_j E_j \in \mathcal{C}$ and if $\{E_j\} \subset \mathcal{C}$ and $E_{j+1} \subset E_j$ for all j then $\bigcap_j E_j \in \mathcal{C}$.*

Clearly, every σ-algebra is a monotone class. Also, it is simple to see that if $\{\mathcal{C}_\lambda\}_{\lambda\in\Lambda}$ is a family of monotone classes in X, then $\bigcap_{\lambda\in\Lambda} \mathcal{C}_\lambda$ is also a monotone class. Hence, given any subset $\mathcal{E}$ of $\mathcal{P}(X)$, there exists a unique smallest monotone class containing $\mathcal{E}$, called the monotone class **generated** by $\mathcal{E}$, denoted by $\mathcal{C}(\mathcal{E})$.

Lemma 3.9.9 (The monotone class lemma)**.** *If $\mathcal{A}$ is an algebra of subsets of X, then the monotone class $\mathcal{C}$ generated by $\mathcal{A}$ coincides with the σ-algebra $\mathcal{M}$ generated by $\mathcal{A}$.*

Proof. Since $\mathcal{M}$ is also a monotone class, we have $\mathcal{C} \subset \mathcal{M}$. If we can show that $\mathcal{C}$ is a σ-algebra, then $\mathcal{M} \subset \mathcal{C}$ and the result is proven.

To show that $\mathcal{C}$ is a σ-algebra, for each $E \in \mathcal{C}$ we define

$$\mathcal{C}(E) = \{F \in \mathcal{C} \colon E \setminus F, F \setminus E \text{ and } E \cap F \text{ are in } \mathcal{C}\}.$$

Clearly $\varnothing, E \in \mathcal{C}(E)$ and also $E \in \mathcal{C}(F)$ iff $F \in \mathcal{C}(E)$. It also follows easily that $\mathcal{C}(E)$ is a monotone class.

If $E \in \mathcal{A}$, then $F \in \mathcal{C}(E)$ for all $F \in \mathcal{A}$, since $\mathcal{A}$ is an algebra, that is, $\mathcal{A} \subset \mathcal{C}(E)$ for all $E \in \mathcal{A}$. Hence $\mathcal{C} \subset \mathcal{C}(E)$ for all $E \in \mathcal{A}$. Thus if $E \in \mathcal{A}$ and $F \in \mathcal{C}$, then $F \in \mathcal{C}(E)$ and hence $E \in \mathcal{C}(F)$. Therefore $\mathcal{A} \subset \mathcal{C}(F)$, for all $F \in \mathcal{C}$, which in turn implies that $\mathcal{A} \subset \mathcal{C}(F)$ for all $F \in \mathcal{C}$.

Conclusion: if $E, F \in \mathcal{C}$ then $E \setminus F$, $F \setminus E$ and $E \cap F$ are in $\mathcal{C}$. Since $X \in \mathcal{A} \subset \mathcal{C}$, $\mathcal{C}$ is an algebra.

Now if $\{E_j\} \subset \mathcal{C}$, we have $\bigcup_{j=1}^{n} E_j \in \mathcal{C}$ for each n (since $\mathcal{C}$ is an algebra), and since $\mathcal{C}$ is closed under countable increasing unions we have

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{n} E_j \in \mathcal{C},$$

that is, $\mathcal{C}$ is a σ-algebra. ■

3.9.2 | The Fubini-Tonelli Theorem

Now we will relate integrals in $X \times Y$ with integrals on X and Y.

Theorem 3.9.10. *Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$, then the functions $X \ni x \mapsto \nu(E_x)$ and $Y \ni y \mapsto \mu(E^y)$ are measurable on X and Y, respectively, and*

$$\mu \times \nu(E) = \int \nu(E_x) d\mu(x) = \int \mu(E^y) d\nu(y).$$

Proof. First we assume that both μ and ν are finite measures, and let $\mathcal{C}$ be the set of all $E \in \mathcal{M} \otimes \mathcal{N}$ for which the conclusions of the theorem are true.

If $E = A \times B$ is a rectangle, then since $E_x = B$ if $x \in A$ and $E_x = \varnothing$ if $x \in A^c$ we have

$$\nu(E_x) = \begin{cases} \nu(B) & \text{if } x \in A \\ 0 & \text{if } x \in A^c \end{cases} = \chi_A(x)\nu(B),$$

and analogously $\mu(E^y) = \mu(A)\chi_B(y)$. Hence the result holds for all the rectangles.

Now we show that $\mathcal{C}$ is a monotone class. To that end, let $\{E_n\} \subset \mathcal{C}$ be an increasing sequence and $E = \bigcup_n E_n$. Since $E_x = \bigcup_n (E_i)_x$, the functions $f_n(x) = \nu((E_n)_x)$ are measurable for all n and they increase pointwise to $f(x) = \nu(E_x)$. Hence f is measurable and by the MCT together with the continuity from below of $\mu \times \nu$ we have

$$\int \nu(E_x) d\mu(x) = \lim \int \nu((E_n)_x) d\mu(x) = \lim \mu \times \nu(E_n) = \mu \times \nu(E).$$

The same reasoning shows that $\mu \times \nu(E) = \int \mu(E^y) d\nu(y)$, and $E \in \mathcal{C}$.

Now let $\{E_n\} \subset \mathcal{C}$ be a decreasing sequence and $E = \bigcap_n E_n$. Since $E_x = \bigcap_n (E_n)_x$, the functions $f_n(x) = \nu((E_n)_x)$ are measurable for all n, decrease pointwise to $f(x) = \nu(E_x)$ and $f_n(x) \leqslant \nu((E_1)_x) \leqslant \nu(Y) < \infty$. Hence we can use the DCT together with the continuity from above of $\mu \times \nu$ to show that $\mu \times \nu(E) = \int \nu(E_x) d\mu(x)$. Analogously we show that $\mu \times \nu(E) = \int \mu(E^y) d\nu(y)$, and hence $E \in \mathcal{C}$.

Thus $\mathcal{C}$ is a monotone class that contain all rectangles, and by additivity of μ, ν and $\mu \times \nu$, it contains the algebra $\mathcal{M}$ of all the finite disjoint unions of rectangles. By the Monotone Class Lemma (Lemma 3.9.9) $\mathcal{C}$ contains $\mathcal{M} \otimes \mathcal{N}$. This concludes the case when μ and ν are finite.

If μ and ν are σ-finite, from Proposition 3.9.4 we can write $X \times Y = \bigcup_j (A_j \times B_j)$ with $\{A_j \times B_j\}$ an increasing sequences of rectangles with finite $\mu \times \nu$ measure. If $E \in \mathcal{M} \otimes \mathcal{N}$, then we can apply the previous argument to $E \cap (A_j \times B_j)$ for each j, which gives us

$$\mu \times \nu(E \cap (A_j \times B_j)) = \int \chi_{A_j}(x) \nu(E_x \cap B_j) d\mu(x),$$

since $(E \cap (A_j \times B_j))_x = E_x \cap B_j$ if $x \in A_j$ and $(E \cap (A_j \times B_j))_x = \varnothing$ if $x \in A_j^c$.

The same applies to show that

$$\mu \times \nu(E \cap (A_j \times B_j)) = \int \mu(E^y \cap A_j) \chi_{B_j}(y) d\nu(y),$$

and an application of the MCT on both equalities proves the result. ■

Theorem 3.9.11 (The Fubini-Tonelli Theorem). *Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces.*

(a) *(Tonelli) If $f \in L^+(X \times Y)$ then the functions $g(x) = \int f_x d\nu$ and $h(y) = \int f^y d\mu$ are in $L^+(X)$ and $L^+(Y)$, respectively, and*

$$\int f d(\mu \times \nu) = \int \left[\int f(x,y) d\nu(y) \right] d\mu(x) = \int \left[\int f(x,y) d\mu(x) \right] d\nu(y). \tag{3.9.1}$$

(b) *(Fubini) If $f \in L^1(\mu \times \nu)$ then $f_x \in L^1(\nu)$ a.e. $x \in X$, $f^y \in L^1(\mu)$ a.e. $y \in Y$,* *the a.e.-defined functions* $g(x) = \int f_x d\nu$ *and* $h(y) = \int f^y d\mu$ *are in* $L^1(\mu)$ *and* $L^1(\nu)$, *respectively, and* (3.9.1) *holds.*

Proof. **(a).** When $E \in \mathcal{M}\otimes\mathcal{N}$ and $f = \chi_E$, then $g(x) = \int \chi_{E_x} d\nu = \nu(E_x)$ and $h(y) = \mu(E^y)$, and item (a) reduces to Theorem 3.9.10. By additivity, item (a) holds for nonnegative simple functions.

Now if $f \in L^+(X \times Y)$ then consider a sequence $\{\phi_n\}$ of nonnegative simple functions that increases pointwise to f, hence $\{(\phi_n)_x\}$ increases to f_x for each $x \in X$ and $\{(\phi_n)^y\}$ increases pointwise to f^y for each $y \in Y$. Now define

$$g_n(x) = \int (\phi_n)_x d\nu \quad \text{ and } h_n(y) = \int (\phi_n)^y d\mu,$$

for $x \in X$ and $y \in Y$, respectively. The MCT implies that

$$\lim g_n(x) = \lim \int (\phi_n)_x d\nu = \int \lim (\phi_n)_x d\nu = \int f_x d\nu = g(x),$$

for each $x \in X$. Hence, g is measurable. Analogously, h is measurable.

Using that (a) holds for each (ϕ_n) and the MCT, we have

$$\int g d\mu = \lim \int (\phi_n)_x d\mu = \lim \int \phi_n d(\mu \times \nu) = \int f d(\mu \times \nu),$$

and analogously $\int h d\nu = \int f d(\mu \times \nu)$. This proves (a).

(b). From (a), we have if $f \in L^+(X \times Y)$ and $\int f d(\mu \times \nu) < \infty$ then $\int g d\mu < \infty$ and hence $g < \infty$ a.e., which in turn, since $g = \int f_x d\nu$, implies that $f_x \in L^1(\nu)$ a.e. $x \in X$. Analogously we show that $f^y \in L^1(\mu)$ a.e. $y \in Y$.

Thus if $f \in L^1(\mu \times \nu)$ is a real function, part (b) follows from part (a) applied to f^+ and f^-. If $f \in L^1(\mu \times \nu)$, then (b) follows from (b) for real functions applied to Ref and Imf. ■

We will omit the brackets from now on, that is,

$$\int \left[\int f(x,y) d\nu(y) \right] d\mu(x) = \iint f(x,y) d\nu(y) d\mu(x) = \iint f d\nu d\mu.$$

In general, the Fubini and Tonelli theorems are used in sequence: one wants to reverse the order of integration in a double integral $\iint f d\mu d\nu$. First we check that $\int |f| d(\mu \times \nu) < \infty$, using Tonelli's part to compute this integral as an iterated double integral, and only then we apply Fubini's part to conclude that $\iint f d\mu d\nu = \iint f d\nu d\mu$

Also, it is important to point out that the hypothesis of σ-finiteness is necessary (see Exercise 46). Also the hypothesis $f \in L^+(X \times Y)$ or $f \in L^1(\mu \times \nu)$ is necessary, in two respects.

First, we can have f_x and f^y measurable for all x, y and for the iterated integrals $\iint f d\mu d\nu$ and $\iint f d\nu d\mu$ to exist, even if f is not $\mathcal{M} \otimes \mathcal{N}$-measurable. However, the iterated integrals can be different (see Exercise 47).

Second, if f is not nonnegative, it is possible for f_x and f_y to be integrable for all x, y and for the iterated integrals $\iint f d\mu d\nu$ and $\iint f d\nu d\mu$ to exist, even if $\int |f| d(\mu \times \nu) = \infty$, but again, in this case, these integrals can be different (see Exercise 48).

Even when μ and ν are complete, $\mu \times \nu$ is almost never complete. For instance, suppose that $A \in \mathcal{M}$ is such that $\mu(A) = 0$, that $\mathcal{N} \neq \mathcal{P}(Y)$ and take $E \in \mathcal{P}(Y) \setminus \mathcal{N}$. Then using item (a) of Proposition 3.9.7, we have $A \times E \notin \mathcal{M} \otimes \mathcal{N}$, since $(A \times E)_x = E$ for all $x \in A$, which is not in $\mathcal{N}$. But $A \times E \subset A \times Y$ and $(\mu \times \nu)(A \times Y) = \mu(A)\nu(Y) = 0$, since $\mu(A) = 0$. Thus we have a nonmeasurable set inside a zero measure measurable set, which means that $\mu \times \nu$ is not complete. A concrete example is $X = Y = \mathbb{R}$ and $\mu = \nu = m$ the Lebesgue measure.

If one wants to work with complete measure, one can consider the completion of $\mu \times \nu$. In this scenario, the relationship between the measurability of a function on $X \times Y$ and the measurability of its x-sections and y-sections is not so simple. However, when correctly reformulated, the Fubini-Tonelli Theorem is still valid.

Theorem 3.9.12 (The Fubini-Tonelli Theorem for Complete Measures). *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be complete and σ-finite measure spaces, and let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$.*

(a) *If $f \in L^+(X \times Y, \lambda)$ then f_x is $\mathcal{N}$-measurable for a.e. x and f^y is $\mathcal{M}$-measurable for a.e. $y \in Y$, $X \ni x \mapsto \int f_x d\nu$ and $Y \ni y \mapsto \int f^y d\mu$ are measurable and*

$$\int f d\lambda = \iint f d\mu d\nu = \iint f d\nu d\mu. \tag{3.9.2}$$

(b) *If $f \in L^1(X \times Y, \lambda)$ then f_x is $\mathcal{N}$-integrable for a.e. x and f^y is $\mathcal{M}$-integrable for a.e. $y \in Y$, $X \ni x \mapsto \int f_x d\nu$ and $Y \ni y \mapsto \int f^y d\mu$ are integrable and* (3.9.2) *holds.*

The proof of this theorem is outlined in Exercise 49.

3.10 | Solved exercises from [1, Page 68]

Exercise 45. If (X_j, M_j) is a measurable space for $j = 1, 2, 3$ then $\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \mathcal{M}_3 = (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$. Moreover, if μ_j is a σ-finite measure on $(X_j, \mathcal{M}_j)$ then $\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$.

Solution. Since $\mathcal{F}_1 = \{E_1 \times E_2 \colon E_i \in \mathcal{M}_i,\ i = 1, 2\}$ generates $\mathcal{M}_1 \otimes \mathcal{M}_2$, using Proposition 1.2.2, $(\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$ is generated by $\mathcal{F}_2 = \{A \times E_3 \colon A \in \mathcal{F}_1,\ E_3 \in \mathcal{M}_3\}$.

But $\mathcal{F}_2$ is naturally identified with $\mathcal{E} = \{E_1 \times E_2 \times E_3 \colon E_i \in \mathcal{M}_i,\ i = 1, 2, 3\}$, which generated $\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \mathcal{M}_3$. Hence the equality follows.

We have

$$\begin{aligned}\mu_1 \times \mu_2 \times \mu_3(E_1 \times E_2 \times E_3) &= \mu_1(E_1)\mu_2(E_2)\mu_3(E_3) \\ &= \mu_1 \times \mu_2(E_1 \times E_2)\mu_3(E_3) = [(\mu_1 \times \mu_2) \times \mu_3][(E_1 \times E_2) \times E_3]\end{aligned}$$

for all rectangles $E_1 \times E_2 \times E_3$. Hence, using the countable additivity of both measures, $\mu_1 \times \mu_2 \times \mu_3(A) = [(\mu_1 \times \mu_2) \times \mu_3](A)$ if A is a disjoint finite union of rectangles. Hence, by uniqueness of the extension given in Theorem 2.3.5 (using the σ-additivity of μ_i, $i = 1, 2, 3$, we see that $\mu_1 \times \mu_2 \times \mu_3$ is σ-finite), $\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$.

Exercise 46. Let $X = Y = [0, 1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$, μ = Lebesgue measure and ν = counting measure. If $D = \{(x, x) \colon x \in [0, 1]\}$ is the diagonal in $X \times Y$, then $\iint \chi_D d\mu d\nu$, $\iint \chi_D d\nu d\mu$ and $\int \chi_D d(\mu \times \nu)$ are unequal. (To compute $\int \chi_D d(\mu \times \nu) = \mu \times \nu(D)$, go back to the definition of $\mu \times \nu$).

Solution. First we note that D is measurable. Indeed, given $n \in \mathbb{N}$, define $I_{n,k} = [\frac{k}{n}, \frac{k+1}{n}]$ for $k = 0, \cdots, n-1$ and $E_n = \bigcup\limits_{k=0}^{n-1} (I_{n,k} \times I_{n,k})$. Thus $D = \bigcap_{n=1}^{\infty} E_n \in \mathcal{M} \otimes \mathcal{N}$.

We have

$$\iint \chi_D d\mu d\nu = \int_{[0,1]} \int_0^1 \chi_D(x, y) dx d\nu(y),$$

but for each fixed $y \in [0, 1]$, we have $\chi_D(x, y) = 0$ if $x \neq y$ and $\chi_D(x, y) = 1$ if $x = y$, hence $\chi_D(\cdot, y) = 0$ μ-a.e., and thus

$$\iint \chi_D d\mu d\nu = \int_{[0,1]} 0 d\nu(y) = 0 \cdot \nu([0, 1]) = 0 \cdot \infty = 0.$$

Now since $\nu(\{x\}) = 1$ for each $x \in [0, 1]$ we have

$$\iint \chi_D d\nu d\mu = \int_0^1 \int_{[0,1]} \chi_D(x, y) d\nu(y) dx = \int_0^1 1 dx = 1.$$

Lastly, to compute $\mu\times\nu(D)$, we will use the outer measure. Assume that $D \subset \bigcup_{j=1}^{\infty}(A_j \times B_j)$ where $A_j, B_j \in \mathcal{B}_{[0,1]}$ for all j. Since $D = \bigcup_{j=1}^{\infty} D \cap (A_j \times B_j)$, then given $x \in [0,1]$ we have $(x,x) \in (A_j \times B_j)$ for some j, that is, $x \in A_j \cap B_j$, and hence $\bigcup_{j=1}^{\infty} A_j \cap B_j = [0,1]$. Therefore there exists $j \in \mathbb{N}$ such that $\mu(A_j \cap B_j) > 0$, thus $\mu(A_j) \geqslant \mu(A_j \cap B_j) > 0$ and $\mu(B_j) \geqslant \mu(A_j \cap B_j) > 0$, and in particular, $\nu(B_j) = \infty$ (since if $\nu(B_j) < \infty$ implies that $\mu(B_j) = 0$). Hence $\mu \times \nu(A_j \times B_j) = \infty$, and thus $\sum_{j=1}^{\infty} \mu \times \nu(A_j \times B_j) = \infty$. Since this is true for any cover of D by rectangles, we have $\mu \times \nu(D) = \infty$.

Exercise 47. Let X be an uncountable linearly ordered set such that for each $x \in X$, the set $\{y \in X : y < x\}$ is countable (Example: the set of countable ordinals). Let $\mathcal{M}$ be the σ-algebra of countable or co-countable sets, and let $\mu = \nu$ be defined on $\mathcal{M}$ by $\mu(A) = 0$ if A is countable or $\mu(A) = 1$ if A is co-countable. Let $E = \{(x,y) \in X \times X : y < x\}$. Then E_x and E^y are measurable for all $x, y \in X$ and $\iint \chi_E d\mu d\nu$ and $\iint \chi_E d\nu d\mu$ exist but are not equal.

Solution. We have for each $x \in X$

$$E_x = \{y \in X : (x,y) \in E\} = \{y \in X : y < x\},$$

which is countable (hence in $\mathcal{M}$), and $\nu(E_x) = 0$. Then

$$\iint \chi_E d\nu d\mu = \int \nu(E_x) d\mu(x) = \int 0 d\mu = 0.$$

But for each $y \in X$ we have $E^y = \{x \in X : y < x\} = (\{x \in X : x < y\} \cup \{y\})^c$, and hence E^y is co-countable (hence in $\mathcal{M}$), and $\mu(E^y) = 1$. Thus

$$\iint \chi_E d\mu d\nu = \int \mu(E^y) d\nu = \int 1 d\nu = 1\nu(X) = 1,$$

since $\nu(X) = 1$ (X is co-countable).

Exercise 48. Let $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(\mathbb{N})$ and $\mu = \nu$ = counting measure. Define $f(m,n) = 1$ if $m = n$, $f(m,n) = -1$ if $m = n+1$ and $f(m,n) = 0$ otherwise. Then $\int |f| d(\mu \times \mu) = \infty$, and $\iint f d\mu d\nu$ and $\iint f d\nu d\mu$ exist and are unequal.

Solution. First we note that $\mu \times \nu$ is also a counting measure, since $\mu \times \nu(A \times B) = \mu(A)\nu(B)$ on rectangles. Now if we let $E = \bigcup_{n=1}^{\infty}[(n,n) \cup (n+1,n)]$, we have $|f| = \chi_E$, and since $\mu \times \nu(E) = \infty$, we have

$$\int |f| d(\mu \times \nu) = \int \chi_E d(\mu \times \nu) = \mu \times \nu(E) = \infty.$$

But

$$\iint f d\mu d\nu = \iint f(m,n) d\mu(m) d\nu(n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty} 0 = 0,$$

and

$$\iint f d\nu d\mu \iint f(m,n) d\nu(n) d\mu(m) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m,n) = f(1,1) = 1.$$

Exercise 49. Prove Theorem 3.9.12 by using Theorem 3.9.11 and Proposition 3.1.24 together with the following lemmas.

(a) If $E \in \mathcal{M} \otimes \mathcal{N}$ and $\mu \times \nu(E) = 0$ then $\nu(E_x) = \mu(E^y) = 0$ for a.e. x and y.

(b) If f is $\mathcal{L}$-measurable and $f = 0$ λ-a.e. then f_x and f^y are integrable for a.e. x and y, and $\int f_x d\nu = \int f^y d\mu = 0$ for a.e. x and y (Here the completeness of μ and ν are needed).

Solution to (a). This follows directly from Theorem 3.9.10.

Solution to (b). Let $F = \{(x,y) \in X \times Y \colon f(x,y) \neq 0\}$. Since $f = 0$ λ-a.e. and λ is complete we have $\lambda(F) = 0$. But λ is the completion of $\mu \times \nu$, and by definition, there exists $E \in \mathcal{M} \otimes \mathcal{N}$ such that $F \subset E$ and $\mu \times \nu(E) = 0$. By item (a), $\nu(E_x) = \nu(E^y) = 0$ a.e. x and y. Since $F_x \subset E_x$ and ν is complete, we have $F_x \in \mathcal{N}$ and $\nu(F_x) = 0$ μ-a.e. x. Analogously $F^y \in \mathcal{M}$ and $\mu(F^y) = 0$ ν-a.e. y.

Now $\{y \in Y \colon f_x(y) \neq 0\} = F_x$ and $\nu(F_x) = 0$ μ-a.e. x, it follows that for μ-a.e. x, $f_x = 0$ ν-a.e. By Proposition 3.1.23, since ν is complete, f_x is measurable, and $|f_x| = 0$ ν-a.e., hence $\int |f_x| d\nu = 0$. Therefore f_x is integrable and $\int f_x d\nu = 0$. Analogously f^y is integrable and $\int f^y d\mu = 0$.

Proof of Theorem 3.9.12. We prove first item (a). To that end, let $f \in L^+(X \times Y, \lambda)$. Hence by Proposition 3.1.24, there exists a $\mathcal{M} \otimes \mathcal{N}$-measurable function h such that $f = h$ λ-a.e., and we can assume $h \geqslant 0$, after possible redefinition of h in a $\mu \times \nu$-null set.

We define $g = f - h$. Then $g = 0$ λ-a.e. and by Proposition 3.1.23, g is measurable. Using part (b), g_x and g^y are integrable for a.e. x and y and $\int g_x d\nu = \int g^y d\mu = 0$. Thus $f_x = g_x + h_x$ is $\mathcal{N}$-measurable for a.e. $x \in X$ and $f^y = g^y + h^y$ is $\mathcal{M}$-measurable for a.e. $y \in Y$.

Also $x \mapsto \int f_x d\nu = \int (g_x + h_x) d\nu = \int h_x d\nu$ is $\mathcal{M}$-measurable for a.e. x and $y \mapsto \int f^y d\mu = \int (g^y + h^y) d\mu = \int h^y d\mu$ is $\mathcal{N}$-measurable for a.e. y, using the fact that g_x and g^y are integrable for a.e. x, y, $\int g_x d\nu = \int g^y d\mu = 0$ a.e. x, y and Theorem 3.9.11. Moreover

$$\iint f d\mu d\nu = \int \int f^y d\mu d\nu = \int \int h^y d\mu d\nu = \int hd(\mu \times \nu)$$

and analogously $\iint f d\nu d\mu = \int hd(\mu \times \nu)$.

Now it remains to prove that $\int f d\lambda = \int hd(\mu \times \nu)$. If $E \in \mathcal{M} \otimes \mathcal{N}$ then $\mu \times \nu(E) = \lambda(E)$ and hence $\int \chi_E d(\mu \times \nu) = \int \chi_E d\lambda$. By linearity, the result follows for $\mathcal{M} \otimes \mathcal{N}$-measurable nonnegative simple functions. Using the MCT, the result follows for all $\mathcal{M} \otimes \mathcal{N}$-measurable nonnegative functions. Since $f = h$ λ-a.e. we have

$$\int f d\lambda = \int hd\lambda = \int hd(\mu \times \nu).$$

Thus part (a) of Theorem 3.9.12 follows. Part (b) follows as in Theorem 3.9.11.

Exercise 50. Suppose $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and $f \in L^+(X)$ such that $f < \infty$ everywhere. Let

$$G_f = \{(x, y) \in X \times [0, \infty) \colon y \leqslant f(x)\}.$$

Then G_f is $\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$-measurable and $\mu \times m(G_f) = \int f d\mu$. The same is true if the inequality $y \leqslant f(x)$ is replaced by $y < f(x)$ in the definition of G_f. (To show the measurability of G_f, note that the map $(x, y) \mapsto f(x) - y$ is the composition of $(x, y) \mapsto (f(x), y)$ and $(z, y) \mapsto z - y$). This is the definitive statement of the familiar theorem from calculus: "the integral of a function is the area under its graph".

Solution. First we show the measurability of G_f. Consider the functions $\psi \colon X \times [0, \infty) \times \mathbb{R}^2$ and $\phi \colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$\psi(x, y) = (f(x), y) \quad \text{ and } \phi(z, w) = z - w,$$

for all $x \in X$, $y \in [0, \infty)$ and $z, w \in \mathbb{R}$. By Proposition 3.1.7, ψ is $\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$-measurable. Now if $F(x, y) = \phi \circ \psi(x, y)$ for all $x \in X$ and $y \in [0, \infty)$, we have

$$G_f = F^{-1}([0, \infty)) = \psi^{-1}(\phi^{-1}([0, \infty))),$$

but since ϕ is continuous, $\phi^{-1}([0, \infty))$ is closed in $\mathbb{R}^2$, and hence a borelian set. Since ψ is measurable, $G_f \in \mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$ (for the inequality $y < f(x)$, use $(0, \infty)$ which is open, instead of $[0, \infty)$).

Now using the Fubini Tonelli's Theorem, we have

$$\begin{aligned}\mu \times m(G_f) &= \int \chi_{G_f} d(\mu \times m) = \iint \chi_{G_f}(x, y) dm(y) d\mu(x) \\ &= \int m([0, f(x)]) d\mu(x) = \int f(x) d\mu(x) = \int f d\mu.\end{aligned}$$

With the inequality $y < f(x)$, just note that $m([0, f(x))) = f(x)$, and the result follow the same.

Exercise 51. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be arbitrary measure spaces (not necessarily σ-finite).

(a) If $f \colon X \to \mathbb{C}$ is $\mathcal{M}$-measurable, $g \colon Y \to \mathcal{N}$ is $\mathcal{N}$-measurable and $h(x, y) = f(x)g(y)$, then h is $\mathcal{M} \otimes \mathcal{N}$-measurable.

(b) If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and $\int h d(\mu \times \nu) = [\int f d\mu][\int g d\nu]$.

Solution to (a). Define $\psi \colon X \times Y \to \mathbb{C}^2$ by $\psi(x, y) = (f(x), g(y))$ for all $x \in X$ and $y \in Y$, and also define $\phi \colon \mathbb{C}^2 \to \mathbb{C}$ by $\phi(z, w) = zw$, for all $z, w \in \mathbb{C}$. Hence ψ is $\mathcal{M} \otimes \mathcal{N}$-measurable by Proposition 3.1.7 and ϕ is continuous. Hence $h = \phi \circ \psi$ is $\mathcal{M} \otimes \mathcal{N}$-measurable.
Solution to (b). Assume that $f = \chi_A$ and $g = \chi_B$ for $A \in \mathcal{M}$ and $B \in \mathcal{N}$. Hence $h = \chi_A \times \chi_B = \chi_{A \times B}$ and

$$\int h d(\mu \times \nu) = \mu \times \nu(A \times B) = \mu(A) \times \nu(B) = \Big[\int \chi_A d\mu\Big]\Big[\int \chi_B d\nu\Big] = \Big[\int f d\mu\Big]\Big[\int g d\nu\Big],$$

since $\mu \times \nu(A \times B) = \mu(A) \times \nu(B)$ by definition of the product measure.

Now assume that f is a simple nonnegative function $f = \sum_{i=1}^{n} c_i \chi_{A_i}$ and $g = \chi_B$. We have $h = \sum_{i=1}^{n} c_i \chi_{A_i \times B}$ and thus

$$\int h d(\mu \times \nu) = \sum_{i=1}^{n} c_i \mu \times \nu(A_i \times B) = \sum_{i=1}^{n} c_i \mu(A_i)\nu(B) = \Big[\int f d\mu\Big]\Big[\int g d\nu\Big].$$

Now assume that both f and g are $\mathcal{M}$ and $\mathcal{N}$ measurable (respectively) simple nonnegative functions. Hence $g = \sum_{k=1}^{m} d_k \chi_{B_k}$ and we set $h_k = d_k f \chi_{B_k}$ for each $k = 1, \cdots, m$. Using the previous case, for each $k = 1, \cdots, m$ we have

$$\int h_k d(\mu \times \nu) = d_k \int f\chi_{B_k} d(\mu \times \nu) = d_k \Big[\int f d\mu\Big]\Big[\int \chi_{B_k} d\nu\Big] = \Big[\int f\Big]\Big[\int d_k \chi_{B_k} d\nu\Big],$$

and summing up for $k = 1, \cdots, m$, we obtain the result for f and g in this case.

Assume that $f \in L^+(X)$ and $g \in L^+(Y)$ are real. Then we have sequences $\{s_n\}$ and $\{r_n\}$ of nonnegative simple $\mathcal{M}$ and $\mathcal{N}$-measurable functions that increase pointwise to f and g, respectively. Consider $h_n = f_n g_n$, then each h_n is in $L^+(X \times Y)$ by Proposition 3.1.9 and $\{h_n\}$ increases to fg. Therefore the MCT implies that

$$\int h d(\mu \times \nu) = \lim_{n\to\infty} \int h_n d(\mu \times \nu) = \lim_{n\to\infty} \Big[\int f_n d\mu\Big]\Big[\int g_n d\mu\Big] = \Big[\int f d\mu\Big]\Big[\int g d\nu\Big].$$

For $f \in L^1(\mu)$ and $g \in L^1(\nu)$ real functions, the result follows by applying the previous case to f^+g^+, f^+g^-, f^-g^+ and f^-g^-. For complex functions, just apply the real L^1 case to $\mathrm{Re}f\mathrm{Re}g$, $\mathrm{Re}f\mathrm{Im}g$, $\mathrm{Im}f\mathrm{Re}g$ and $\mathrm{Im}f\mathrm{Im}g$.

Exercise 52. The Fubini-Tonelli Theorem if valid we $(X, \mathcal{M}, \mu)$ is an arbitrary measure space and Y is a countable set, $\mathcal{N} = \mathcal{P}(Y)$ and ν is the counting measure on Y. (Cf. Theorems 3.3.9 and 3.5.14).

Solution. By using an enumeration $\{r_n\}$ of Y, we can assume that $Y = \mathbb{N}$. We prove first two lemmas:

Lemma 1. If f is a function defined in $\mathbb{N}$ then $\int f d\nu = \sum_{n=1}^{\infty} f(n)$.

Proof. Clearly every function defined in $\mathbb{N}$ is $\mathcal{N} = \mathcal{P}(\mathbb{N})$-measurable. Now if $J \subset \mathbb{N}$ and $f = \chi_J$, then

$$\int f d\nu = \int \chi_J d\nu = \nu(J) = \sum_{n=1}^{\infty} \chi_J(n).$$

Now, by additivity it holds for nonnegative simple functions. Using the MCT it holds for nonnegative functions. Applying to f^+ and f^- we have the result for real $L^1(\nu)$ functions, and applying it to the real and imaginary parts of f we have the result for $L^1(\nu)$ functions. □

Lemma 2. For $f \in L^+(X \times Y)$ or $f \in L^1(\mu \times \nu)$, we have

$$\int f d(\mu \times \nu) = \iint f d\mu d\nu.$$

Proof. Let $A \in \mathcal{N}$ and $J \subset \mathbb{N}$. For $f = \chi_{A\times J}$ we have

$$\int fd(\mu\times\nu) = \mu\times\nu(A\times J) = \mu(A)\nu(J) = \sum_{n=1}^{\infty}\mu(A)\chi_J(n) = \sum_{n=1}^{\infty}\int \chi_{A\times J}(x,n)d\mu(x),$$

and using Lemma 1, we have $\int fd(\mu\times\nu) = \iint fd\mu d\nu$. Using the same steps of Lemma 2 (additivity, MCT, f^+ and f^-, real and imaginary parts of f), we prove the result. □

Now for any function f defined on $X\times\mathbb{N}$, we have

$$f_x(n) = f^n(x) = f(x,n) \quad \text{for all } n\in\mathbb{N}.$$

Hence if $f \in L^+(X\times Y)$, Theorem 3.3.9 implies that $x \mapsto \int f_x(n)d\nu(n) = \sum_{n=1}^{\infty} f_x(n) = \sum_{n=1}^{\infty} f^n(x)$ is $\mathcal{M}$-measurable and that

$$\iint f(x,n)d\nu(n)d\mu(x) = \int\sum_{n=1}^{\infty} f^n(x)d\mu(x) = \sum_{n=1}^{\infty}\int f^n(x)d\mu(x) = \iint f(x,n)d\mu(x)d\nu(n).$$

The same conclusion holds if $f \in L^1(\mu\times\nu)$, using Theorem 3.5.14.

3.11 | The n-dimensional Lebesgue integral

The **Lebesgue measure** m^n on $\mathbb{R}^n$ is the completion of the product $m\times\cdots\times m$, n-times, of the Lebesgue measure m in $\mathbb{R}$, that is, the completions of $m\times\cdots\times m$ on $\mathcal{B}_{\mathbb{R}}\otimes\cdots\otimes\mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^n}$, or equivalently, the completion of $m\times\cdots\times m$ on $\mathcal{L}\otimes\cdots\otimes\mathcal{L}$. The domain $\mathcal{L}^n$ of m^n is called the class of **Lebesgue measurable sets** in $\mathbb{R}^n$; and we will sometimes consider m^n on the smaller domain $\mathcal{B}_{\mathbb{R}^n}$. When there is no danger of confusion, we will omit the superscript n of m^n, and write m for m^n as in the case $n = 1$, and write $\int f(x)dx$ for $\int fdm$.

We begin establishing extensions of the results for Borel measures on the real line for the Lebesgue measure m in $\mathbb{R}^n$. If $R = \prod_{j=1}^n R_j$ is a rectangle in $\mathbb{R}^n$, that is, each $R_j \in \mathcal{L}$ for $j = 1,\cdots,n$, we will call the sets $R_j \subset \mathbb{R}$ the **sides** of R.

Theorem 3.11.1. *Suppose $E \in \mathcal{L}^n$. Then*

(a) $m(E) = \inf\{m(U)\colon E\subset U,\ U \text{ open}\} = \sup\{m(K)\colon K\subset E,\ K \text{ compact}\}$.

(b) $E = A_1\cup N_1 = A_2\setminus N_2$, *where A_1 is an F_σ set, A_2 is a G_δ set and* $m(N_1) = m(N_2) = 0$.

(c) *If $m(E) < \infty$, for any $\epsilon > 0$ there exists a finite collection $\{R_j\}_{j=1}^p$ of disjoint rectangles whose sides are open intervals such that $m(E \Delta \bigcup_{j=1}^p R_j) < \epsilon$.*

Proof. **(a).** As in Proposition 2.5.9, set $\mu_{op}(E) = \inf\{m(U) \colon E \subset U,\ U \text{ open}\}$. By monotonicity of m, if $E \subset U$ then $m(E) \leqslant m(U)$, hence $m(E) \leqslant \mu_{op}(E)$. If $m(E) = \infty$, then by monotonicity $\mu_{op}(E) = \infty$. Now assume that $m(E) < \infty$. By the definition of product measures, for $E \in \mathcal{L}^n$ and $\epsilon > 0$, there exists a sequence $\{T_j\}$ of rectangles in $\mathbb{R}^n$, with $T_j = \prod_{k=1}^n R_{j,k}$, $R_{j,k} \in \mathcal{L}$ for each $k = 1, \cdots, p$ and $j \in \mathbb{N}$, and $E \subset \bigcup_{j=1}^\infty T_j$ such that

$$\sum_{j=1}^\infty \prod_{k=1}^n m(R_{j,k}) = \sum_{j=1}^\infty m(T_j) \leqslant m(E) + \frac{\epsilon}{2}.$$

Now fix $j \in \mathbb{N}$. Given $\eta > 0$, for each $k = 1, \cdots, n$, using Proposition 2.5.9 applied to $R_{j,k} \in \mathcal{L}$, there exists an open set $U_{j,k}$ such that $R_{j,k} \subset U_{j,k}$ and $m(U_{j,k}) \leqslant m(R_{j,k}) + \eta$. Set $U_j = \prod_{k=1}^n U_{j,k}$, which is open in $\mathbb{R}^n$ and $T_j \subset U_j$. We have

$$m(U_j) = \prod_{k=1}^n m(U_{j,k}) \leqslant \prod_{k=1}^n (m(R_{j,k}) + \eta) = \prod_{k=1}^n m(R_{j,k}) + \eta \cdot Z(R_{j,1}, \cdots, R_{j,n}, \eta),$$

where $Z(R_{j,1}, \cdots, R_{j,n}, \eta)$ is a function involving products of $m(R_{j,k})$, $k = 1, \cdots, n$ and powers of η up to the power $n-1$. Since $m(E) < \infty$ and $\{R_{j,k}\}_{k=1}^n$ is a finite collection (remember that j is fixed for the moment), have $Z(R_{j,1}, \cdots, R_{j,n}, \eta)$ bounded, and hence we can choose $\eta > 0$ small so that $\eta \cdot Z(R_{j,1}, \cdots, R_{j,n}, \eta) < \epsilon 2^{-j-1}$.

Hence, for each $j \in \mathbb{N}$, we have an open set U_j with $T_j \subset U_j$ and $m(U_j) \leqslant m(T_j) + \epsilon 2^{-j}$. Defining $U = \bigcup_{j=1}^\infty U_j$, we have U open, $E \subset U$ and hence

$$\mu_{op}(E) \leqslant m(U) \leqslant \sum_{j=1}^\infty m(U_j) \leqslant \sum_{j=1}^\infty m(T_j) + \frac{\epsilon}{2} \leqslant m(E) + \epsilon,$$

and since this is true for any $\epsilon > 0$, we obtain $\mu_{op}(E) \leqslant m(E)$.

The proof that $m(E) = \sup\{m(K) \colon K \subset E,\ K \text{ compact}\}$ is completely analogous to the proof of Proposition 2.5.10. The only change is that, when $m(E) = \infty$, we consider $\mathbb{R}^n = \bigcup_{j=1}^\infty B_j$ where $B_j = \{x \in \mathbb{R}^n \colon j - 1 \leqslant \|x\| < j\}$ for each $j \in \mathbb{N}$, and set $E_j = E \cap B_j$ for each $j \in \mathbb{N}$.

(b). The proof of (b) is analogous to the proof of Theorem 2.5.11.

(c). Suppose $m(E) < \infty$. Using the construction made in (a), given $\epsilon > 0$, there exists an open set $U = \bigcup_{j=1}^{\infty} U_j$, such that $U_j = \prod_{k=1}^{n} U_{j,k}$ for each $j \in \mathbb{N}$, with $E \subset U$ and

$$m(U) \leqslant \sum_{j=1}^{\infty} m(U_j) = \sum_{j=1}^{\infty} \prod_{k=1}^{n} m(U_{j,k}) \leqslant m(E) + \epsilon.$$

Now fix $j \in \mathbb{N}$. Since $U_{j,k}$ is open, we have $U_{j,k} = \bigcup_{r=1}^{\infty} I_{j,k}^r$ where $\{I_{j,k}^r\}_{r=1}^{\infty}$ is a disjoint countable collection of open intervals. Since $m(E) < \infty$, we have $m(U_{j,k}) < \infty$ for all $j \in \mathbb{N}$ and $k = 1, \cdots, n$, and thus

$$\sum_{r=1}^{\infty} m(I_{j,k}^r) = m(U_{j,k}) < \infty.$$

Therefore there exists $r_0 = r_0(j) \in \mathbb{N}$ such that $\sum_{r=r_0+1}^{\infty} m(I_{j,k}^r) < (\epsilon 2^{-j})^{1/n}$, for each $k = 1, \cdots, p$. Set $V_{j,k} = \bigcup_{r=1}^{r_0} I_{j,k}^r$ and $V_j = \prod_{k=1}^{n} V_{j,k}$. We have $V_j \subset U_j$ and

$$m(U_j \setminus V_j) = \prod_{k=1}^{n} m(U_{j,k} \setminus V_{j,k}) = \prod_{k=1}^{n} \sum_{r=r_0+1}^{\infty} m(I_{j,k}^r) \leqslant \epsilon 2^{-j}.$$

Since $\sum_{j=1}^{\infty} m(U_j) < \infty$, choose $N \in \mathbb{N}$ such that $\sum_{j=N+1}^{\infty} m(U_j) < \epsilon$. For $V = \bigcup_{j=1}^{N} V_j$, we have $V \subset U$, $m(V \setminus E) \leqslant m(U \setminus E) \leqslant \epsilon$ and

$$m(E \setminus V) \leqslant m(U \setminus V) \leqslant m\Big(\bigcup_{j=1}^{N} (U_j \setminus V_j)\Big) + m\Big(\bigcup_{j=N+1}^{\infty} U_j\Big) \leqslant 2\epsilon,$$

hence $m(E \Delta V) \leqslant 3\epsilon$. Now note that since each V_j, $j = 1, \cdots, N$, is a rectangle whose sides are finite unions of disjoint intervals, we can write $\bigcup_{j=1}^{N} V_j$ as a finite disjoint union of rectangles whose sides are open intervals, and (c) is proven. ■

Theorem 3.11.2. *If $f \in L^1(m)$ and $\epsilon > 0$, there is a simple function $\phi = \sum_{j=1}^{N} a_j \chi_{R_j}$, where each R_j is a product of intervals, such that $\int |f - \phi| < \epsilon$, and there is a continuous function g that vanishes outside a bounded set such that $\int |f - g| < \epsilon$.*

Proof. Using Proposition 3.1.22, we can find a sequence $\{\phi_j\}$ of simple measurable functions such that $0 \leqslant |\phi_1| \leqslant |\phi_2| \leqslant \cdots \leqslant |f|$, $\phi_j \to f_i$ pointwise and $\phi_j \to f_i$ uniformly on any set on which f is bounded.

Since $f \in L^1(m)$, using the DCT, since $|\phi_j - f| \to 0$ pointwise and $|\phi_j - f| \leqslant 2|f| \in L^1(m)$ for all j, we obtain $\int |\phi_j - f| \to 0$ as $j \to \infty$. Thus given $\eta > 0$, we can choose j large enough so that $\int |\phi_j - f| < \eta$.

Assume that $\phi_j = \sum_{k=1}^{p} c_k \chi_{E_k}$, where each $E_k \in \mathcal{L}$. As in Theorem 3.5.15, we can assume that all a_k are nonzero and the E_k are disjoint. Aside from the E_k where ϕ_j is zero, we have

$$m(E_k) = \int \chi_{E_k} = |a_k|^{-1} \int_{E_k} |\phi_j| \leqslant |a_k|^{-1} \int |f| < \infty.$$

Now we can use part (c) of the previous theorem to find, for each $k = 1, \cdots, p$, a finite collection $\{R_{i,k}\}_{i=1}^{n_k}$ of disjoint rectangles whose sides are open intervals, such that $m(E_k \Delta \bigcup_{i=1}^{n_k} R_{i,k}) < \frac{\eta}{p}$. Setting $\phi = \sum_{k=1}^{p} \sum_{i=1}^{n_k} a_k \chi_{R_{i,k}}$ we have

$$\int |\phi - f| \leqslant \int |\phi - \phi_j| + \int |\phi_j - f| \leqslant \sum_{k=1}^{p} m\Big(E_k \Delta \bigcup_{i=1}^{n_k} R_{i,k}\Big) + \eta \leqslant 2\eta.$$

Now, for each rectangle $R_{i,k} = \prod_{j=1}^{n}(a_{i,k,j}, b_{i,k,j})$ we choose $0 < \delta_{i,k,j} < \min\{\frac{1}{2}(b_{i,k,j} - a_{i,k,j}), \eta\}$ and define $T_{i,k} = \prod_{j=1}^{n}[a_{i,k,j} + \delta_{i,k,j}, b_{i,k,j} - \delta_{i,k,j}]$. Hence $T_{i,k} \subset R_{i,k}$ is a rectangle with closed intervals as sides. Thus, using Urysohn's Lemma, we can construct a continuous function g that vanishes outside $\bigcup_{k=1}^{p} \bigcup_{i=1}^{n_k} R_{i,k}$ and coincides with ϕ on $\bigcup_{k=1}^{p} \bigcup_{i=1}^{n_k} T_{i,k}$, hence

$$\int |g - f| \leqslant \int |g - \phi| + \int |\phi - f| \leqslant (C+1)\eta,$$

where $C = \sum_{k=1}^{p} \sum_{i=1}^{n_k} \sum_{j=1}^{n} \delta_{i,k,j}$ is a bounded constant. Taking $\eta < \min\{\frac{\epsilon}{C+1}, \frac{\epsilon}{2}\}$, the result follows. ■

With theses results, we can prove that the Lebesgue measure m in $\mathbb{R}^n$ is also translation invariant.

Theorem 3.11.3. *The Lebesgue measure m is translation invariant in $\mathbb{R}^n$. More precisely, for $a \in \mathbb{R}^n$, define $\tau_a \colon \mathbb{R}^n \to \mathbb{R}^n$ by $\tau_a(x) = x + a$ for all $x \in \mathbb{R}^n$, and we have:*

(a) *if $E \in \mathcal{L}^n$ then $\tau_a(E) \in \mathcal{L}^n$ and $m(\tau_a(E)) = m(E)$,*

(b) *if $f \colon \mathbb{R}^n \to \mathbb{C}$ is Lebesgue measurable, then so if $f \circ \tau_a$. Moreover, if either $f \geqslant 0$ or $f \in L^1(m)$ then $\int (f \circ \tau_a) dm = \int f dm$.*

Proof. **(a).** First note that τ_a is invertible, with inverse τ_{-a} for each $a \in \mathbb{R}$. Also τ_a is continuous for each $a \in \mathbb{R}$, it preserves the class of open sets of $\mathbb{R}^n$. Thus, the class of Borel sets is preserved by τ_a. If $E = \prod_{i=1}^{n} E_i$ and $a = (a_1, \cdots, a_n)$ then $\tau_a(E) = \prod_{i=1}^{n}(E_i + a_i)$ and hence $m(\tau_a(E)) = \prod_{i=1}^{n} m(E_i + a_i) = \prod_{i=1}^{n} m(E_i) = m(E)$, since the Lebesgue measure in $\mathbb{R}$ is translation invariant. By the uniqueness of the Lebesgue product measure on Borelians, this formula remains true for all Borel sets. As for the one dimensional case, the set of Lebesgue null sets is preserved by τ_a, and the result follows from Proposition 3.11.1, item (b).

(b). If f is Lebesgue measurable and B is a Borel set in C, then $f^{-1}(B) = E \cup N$ where E is e Borel set in $\mathbb{R}^n$ and $N \in \mathcal{L}^n$ with $m(N) = 0$. But $\tau_a^{-1}(E) = \tau_{-a}(E)$ is also a Borel set and $m(\tau_a^{-1}(N)) = m(\tau_{-a}(N)) = 0$, so $(f \circ \tau_a)^{-1}(B) = \tau_a^{-1}(E) \cup \tau_a^{-1}(N) \in \mathcal{L}^n$, and $f \circ \tau_a$ is Lebesgue measurable.

The equality $\int (f \circ \tau_a) dm = \int f dm$ reduces to $m(\tau_a(E)) = m(E)$ when $f = \chi_E$. By linearity, it follows for all simple functions, and by the MCT it follows for all nonnegative measurable functions. Taking the positive and negative parts of real and imaginary parts when $f \in L^1(m)$, we conclude the result. ■

3.11.1 | The Jordan content meets Lebesgue measure

In this subsection we compare the Lebesgue measure on $\mathbb{R}^n$ with the notion of Jordan content from calculus of several variables.

Definition 3.11.4. *A* **cube** Q *in* $\mathbb{R}^n$ *is a set of the form* $Q = \prod_{i=1}^n [a_i, b_i]$ *with* $-\infty < a_i \leqslant b_i < \infty$ *for each* $i = 1, \cdots, n$, *with* $b_i - a_i = b_1 - a_1$ *for each* $i = 2, \cdots, n$. *That is, a cube is a product of equal length closed intervals.*

For each $k \in \mathbb{Z}$ we set $\mathcal{Q}_k$ as the collection of all cubes of the form $\prod_{i=1}^n [m_i 2^{-k}, (m_i+1)2^{-k}]$, where $m_i \in \mathbb{Z}$ for $i = 1, \cdots, n$. So the length of the intervals of a cube in $\mathcal{Q}_k$ is 2^{-k}, hence its *volume* is 2^{-nk}. Clearly, the Lebesgue measure of such a cube is also 2^{-nk}.

We note the following two properties of this collection:

(P1) any two distinct cubes in $\mathcal{Q}_k$ have disjoint interiors;

(P2) $\bigcup \{Q : Q \in \mathcal{Q}_k\} = \mathbb{R}^n$;

(P3) cubes in $\mathcal{Q}_{k+1}$ are obtained from cubes in $\mathcal{Q}_k$ by bisecting each side of the cube.

Definition 3.11.5. *If* $E \subset \mathbb{R}^n$ *we define the:*

(a) **inner approximation** *of* E *by* $\mathcal{Q}_k$ *as*

$$\underline{A}(E, k) = \bigcup \{Q \in \mathcal{Q}_k : Q \subset E\}.$$

(b) **outer approximation** *of* E *by* $\mathcal{Q}_k$ *as*

$$\overline{A}(E, k) = \bigcup \{Q \in \mathcal{Q}_k : Q \cap E \neq \varnothing\}.$$

We always have $\underline{A}(E,k) \subset E \subset \overline{A}(E,k)$, and $\underline{A}(E,k)$ may be empty, even if E is not empty. However, if E is not empty, $\overline{A}(E,k)$ will always be nonempty. Here, we have again that the *volume* of $\underline{A}(E,k)$ is simply $p2^{-nk}$, where p is the number of cubes of $\mathcal{Q}_k$ that lie inside E. Also, this coincides with its Lebesgue measure $m(\underline{A}(E,k))$, that is, $m(\underline{A}(E,k)) = p2^{-nk}$.

The same conclusion holds for $\overline{A}(E,k)$, but with p being the number of cubes that intersect E. When E is unbounded, $p = \infty$, and hence both the volume and the Lebesgue measure of $\overline{A}(E,k)$ are ∞. We also have the following for each $k \in \mathbb{Z}$:

$$\underline{A}(E,k) \subset \underline{A}(E,k+1) \qquad \text{and} \qquad \overline{A}(E,k) \supset \underline{A}(E,k+1),$$

since each cube in $\mathcal{Q}_{k+1}$ is the union of 2^n cubes in $\mathcal{Q}_k$ with disjoint interiors. Hence, we can define the limits

$$\underline{\kappa}(E) = \lim_{k\to\infty} m(\underline{A}(E,k)) \qquad \text{and} \qquad \overline{\kappa}(E) = \lim_{k\to\infty} m(\overline{A}(E,k)),$$

called, respectively, the **inner** and **outer content** of E, and if they are equal, their common value $\kappa(E)$ is called the **Jordan content** of E. We note that if $E \subset \mathbb{R}^n$ is not bounded, then $\overline{\kappa}(E) = \infty$, so the theory of Jordan content makes sense, and it meaningful, only when E is bounded.

If $E \subset \mathbb{R}$ is bounded, define

$$\underline{A}(E) = \bigcup_{k=1}^{\infty} \underline{A}(E,k) \qquad \text{and} \qquad \overline{A}(E) = \bigcap_{k=1}^{\infty} \overline{A}(E,k).$$

Thus $\underline{A}(E) \subset E \subset \overline{A}(E)$, both $\underline{A}(E)$ and $\overline{A}(E)$ are Borel sets, and using continuity from below and above of m we have

$$\underline{\kappa}(E) = m(\underline{A}(E)) \qquad \text{and} \qquad \overline{\kappa}(E) = m(\overline{A}(E)).$$

Therefore, since $m(\overline{A}(E)) < \infty$ (recall that E is bounded), the Jordan content of E exists iff $m(\overline{A}(E) \setminus \underline{A}(E)) = 0$. We have then the following result.

Proposition 3.11.6. *Let $E \subset \mathbb{R}^n$ be a bounded set. If the Jordan content of E exists then $E \in \mathcal{L}^n$ and $m(E) = \kappa(E)$.*

Proof. We write $E = \underline{A}(E) \cup (E \setminus \underline{A}(E))$. If the Jordan content of E exists, then $m(\overline{A}(E) \setminus \underline{A}(E)) = 0$, and since $E \setminus \underline{A}(E) \subset \overline{A}(E) \setminus \underline{A}(E)$, the completeness of m shows that $E \setminus \underline{A}(E) \in$

$\mathcal{L}^n$ and $m(E \setminus \underline{A}(E)) = 0$. Hence, using item (b) of Theorem 3.11.1, $E \in \mathcal{L}^n$. Lastly

$$m(E) = m(\underline{A}(E)) + m(E \setminus \underline{A}(E)) = \kappa(E) + 0 = \kappa(E),$$

and the result is complete. ■

Now we will establish further relations between the Lebesgue measure and the Jordan content.

Lemma 3.11.7. *If $U \subset \mathbb{R}^n$ is open, then $U = \underline{A}(U)$. Moreover, U is a countable union of cubes with disjoint interiors.*

Proof. If $x \in U$, since U is open, there exists $\delta > 0$ such that $\{y \in \mathbb{R}^n : \|y - x\| < \delta\} \subset U$. Let $k \in \mathbb{Z}$ be large enough so that $2^{-k}\sqrt{n} < \delta$. Then if Q is a cube in $\mathcal{Q}_k$ that contains x and $y \in Q$ we have $\|y - x\| \leqslant 2^{-k}\sqrt{n} < \delta$ (the worst case scenario is when x and y are complete opposite vertices of the cube Q, and their distance is exactly $2^{-k}\sqrt{n}$). Hence $Q \subset U$, which implies that $x \in \underline{A}(U,k) \subset \underline{A}(U)$ ad therefore $U \subset \underline{A}(U)$. Since $\underline{A}(U) \subset U$, we have the equality.

For the second assertion, first write

$$U = \underline{A}(U) = \bigcup_{k=0}^{\infty} \underline{A}(U,k) = \underline{A}(U,0) \cup \bigcup_{k=1}^{\infty} [\underline{A}(U,k) \setminus \underline{A}(U,k-1)].$$

Now $\underline{A}(U,0)$ is a countable (or possibly finite) union of cubes in $\mathcal{Q}_0$ with disjoint interiors. Also the closure of $\underline{A}(U,k) \setminus \underline{A}(U,k-1)$ is a countable (or possibly finite) union of cubes in $\mathbb{Q}_k$ with disjoint interiors, and hence the result follows. ■

Corollary 3.11.8. *If $U \subset \mathbb{R}^n$ is open we have $m(U) = \underline{\kappa}(U)$.*

Corollary 3.11.9. *If $K \subset \mathbb{R}^n$ is compact, then $m(K) = \overline{\kappa}(K)$.*

Proof. Since K is compact, K is bounded, and we can find $k_0 \in \mathbb{Z}$ and a cube $Q_0 \in \mathcal{Q}_{k_0}$ such that $K \subset \operatorname{int}(Q_0)$. If $Q \in \mathcal{Q}_k$, for $k \geqslant k_0$, then either $Q \cap K = \varnothing$ or $Q \subset Q_0 \setminus K$, hence

$$m(\overline{A}(K,k)) + m(\underline{A}(Q_0 \setminus K, k)) = m(Q_0),$$

and letting $k \to \infty$ we obtain

$$\overline{\kappa}(K) + \underline{\kappa}(Q_0 \setminus K) = m(Q_0).$$

But $Q_0 \setminus K = (\text{int}(Q_0) \setminus K) \cup \partial Q_0$, with disjoint union, where ∂Q_0 is the boundary of Q_0 which has zero Jordan content, so $\underline{\kappa}(Q_0 \setminus K) = \underline{\kappa}(\text{int}(Q_0) \setminus K) = m(\text{int}(Q_0) \setminus K) = m(Q_0)$. Here we used the fact that $\text{int}(Q_0) \setminus K$ is open, used the previous corollary and also the fact the the Lebesgue measure of ∂Q_0 is zero. From this it follows that

$$\overline{\kappa}(K) = m(Q_0) - m(Q_0 \setminus K) = m(K),$$

and concludes the result. ■

Now we can see the true relationship between the Lebesgue measure and Jordan content. The Jordan content is obtained approximating a bounded set E from the inside and outside by a finite union of cubes with disjoint interiors. The Lebesgue measure, however, is obtained with a two-step approximation process: first we approximate E from the outside with an open set, and from the inside with a compact set, then we approximate this open set from the inside and the compact set from the outside by finite unions of cubes with disjoint interiors. The Lebesgue measurable sets are precisely those for which these outer-inner and inner-outer approximations give the same answer in the limit.

3.11.2 | The Change of Variables Theorem

In this subsection we will see what happens to a measurable function f and its integral when we compose it with a diffeomorphism G. But first, we need to investigate the simpler case where G is a invertible linear transformation T.

Definition 3.11.10. *We consider $\{e_j\}_{j=1}^n$ the standard basis of $\mathbb{R}^n$. If $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, we consider the square matrix $(T_{ij}) = (e_i \cdot Te_j)$. The* **determinant** *of this matrix will be denoted by $\det T$. We will denote by $GL(n, \mathbb{R})$ the group of invertible linear transformations of $\mathbb{R}^n$.*

We recall that $\det(T \circ S) = \det T \det S$ for linear transformations $T, S\colon \mathbb{R}^n \to \mathbb{R}^n$.

Definition 3.11.11. *The following three kinds of linear transformations are called* **elementary type transformations***:*

Type 1. *Multiply one coordinate by a nonzero number:*

$$T(x_1, \cdots, x_j, \cdots, x_n) = (x_1, \cdots, cx_j, \cdots, x_n) \qquad c \neq 0;$$

Type 2. *Add a multiple of one coordinate in another coordinate:*

$$T(x_1, \cdots, x_j, \cdots, x_n) = (x_1, \cdots, x_j + cx_k, \cdots, x_n) \qquad k \neq j;$$

Type 3. *Interchange two coordinates:*

$$T(x_1, \cdots, x_j, \cdots, x_k, \cdots, x_n) = (x_1, \cdots, x_k, \cdots, x_j, \cdots, x_n).$$

We sumarize the main properties of elementary types transformations.

Proposition 3.11.12. *We have the following:*

(a) *Any elementary type transformation is invertible, and its inverse has the same type.*

(b) *If T is of type 1 then $\det T = c$, if T is of type 2 then $\det T = 1$, and if T is of type 3 then $\det T = -1$.*

(c) *If $T \in GL(n, \mathbb{R})$, there exists $T_1, T_2, \cdots, T_m$ elementary type transformations such that*

$$T = T_m \circ \cdots T_2 \circ T_1,$$

that is, any invertible transformation is a finite composition of elementary type transformations.

Proof. **(a).** This is clear from the fact that the inverse of a type 1 elementary transformation is $T(x_1, \cdots, x_j, \cdots, x_n) = (x_1, \cdots, c^{-1}x_j, \cdots, x_n)$, the inverse of a type 2 transformation is $T(x_1, \cdots, x_j, \cdots, x_n) = (x_1, \cdots, x_j - cx_k, \cdots, x_n)$, and the inverse of a type 3 transformation is itself.

(b). This follows easily by computing the determinant of the matrix (T_{ij}) in each of the three cases.

(c). This follows from the fact that any invertible matrix can be row reduced to the identity matrix. ■

Now we will see how the Lebesgue integral behaves under linear transformations. We begin with a lemma in $\mathbb{R}$.

Lemma 3.11.13. *Let f be a Lebesgue measurable function in $\mathbb{R}$.*

(a) *If $T_1(x) = cx$ for all $x \in \mathbb{R}$, with $c \neq 0$, then $f \circ T_1$ is Lebesgue measurable. If $f \geqslant 0$ or $f \in L^1(m)$ then*

$$\int f(x)dx = |c| \int f(cx)dx.$$

(b) *If $T_2(x) = x + a$ for all $x \in \mathbb{R}$, with $a \in \mathbb{R}$, then $f \circ T_2$ is Lebesgue measurable. If either $f \geqslant 0$ and $f \in L^1(m)$ then*

$$\int f(x)dx = \int f(x+a)dx.$$

Proof. If $E \in \mathcal{L}$ and $f = \chi_E$, then for $c \neq 0$ we have $f(cx) = \chi_E(cx) = \chi_{c^{-1}E}(x)$. Using Theorem 2.5.13, $c^{-1}E \in \mathcal{L}$ and

$$\int f(x)dx = \int \chi_E(x)dx = m(E) = |c|m(c^{-1}E) = |c| \int \chi_{c^{-1}E}(x)dx = |c| \int f(cx)dx.$$

By additivity, the result follows for simple functions. Using the MCT, the result follows for nonnegative measurable function. Using the positive and negative parts of the real and imaginary parts, the result follows for $L^1(m)$-functions, and (a) is proved. The proof of (b) is completely analogous. ■

THEOREM 3.11.14. *Suppose $T \in GL(n, \mathbb{R})$.*

(a) *If f is a Lebesgue measurable function on $\mathbb{R}^n$, the so is $f \circ T$. If $f \geqslant 0$ or $f \in L^1(m)$, then*

$$\int f(x)dx = |\det T| \int (f \circ T)(x)dx \tag{3.11.1}$$

(b) *If $E \in \mathcal{L}^n$, then $T(E) \in \mathcal{L}^n$ and $m(T(E)) = |\det T|m(E)$.*

Proof. First suppose that f is Borel measurable. Then $f \circ T$ is also Borel measurable, since T is continuous. If (3.11.1) holds for linear transformations T and S, we have

$$\begin{aligned}\int f(x)dx &= |\det T| \int (f \circ T)(x)dx = |\det T||\det S| \int ((f \circ T) \circ S)(x)dx \\ &= |\det(T \circ S)| \int (f \circ (T \circ S))(x)dx,\end{aligned}$$

and hence (3.11.1) holds also for $T \circ S$. Thus, it suffices to prove (3.11.1) for elementary type transformations.

If T is a type 3 elementary transformation, then using the Fubini-Tonelli Theorem we obtain

$$\begin{aligned}\int f(x)dx &= \int \cdots \int f(x_1, \cdots, x_j, \cdots, x_k, \cdots, x_n)dx_1 \cdots dx_j \cdots dx_k \cdots dx_n \\ &= \int \cdots \int f(x_1, \cdots, x_k, \cdots, x_j, \cdots, x_n)dx_1 \cdots dx_k \cdots dx_j \cdots dx_n \\ &= \int (f \circ T)(x)dx = |\det T| \int (f \circ T)(x)dx,\end{aligned}$$

since $\det T = -1$ for an elementary type 3 transformation.

If T is a type 1 transformation, again by using the Fubini-Tonelli Theorem we have

$$\begin{aligned}\int f(x)dx &= \int \cdots \int f(x_1, \cdots, x_j, \cdots, x_n)dx_1 \cdots dx_j \cdots dx_n \\ &= \int \cdots \int \Big[\int f(x_1, \cdots, x_j, \cdots, x_n)dx_j \Big] dx_1 \cdots dx_{j-1}dx_{j+1} \cdots dx_n,\end{aligned}$$

and using Lemma 3.11.13 we obtain

$$\begin{aligned}\int f(x)dx &= \int \cdots \int \Big[|c| \int f(x_1, \cdots, cx_j, \cdots, x_n)dx_j \Big] dx_1 \cdots dx_{j-1}dx_{j+1} \cdots dx_n \\ &= |c| \int \cdots \int f(x_1, \cdots, cx_j, \cdots, x_n)dx_1 \cdots dx_j \cdots dx_n \\ &= |c| \int (f \circ T)(x)dx = |\det T| \int (f \circ T)(x)dx,\end{aligned}$$

since $\det T = c$ for en elementary type 1 transformation.

Analogously for a type 2 elementary transformation, we use the Fubini-Tonelli Theorem to integrate first in x_j and Lemma 3.11.13 to conclude the result.

Thus, so far, we have proven that if f is Borel measurable and $T \in GL(n, \mathbb{R})$, then $f \circ T$ is Borel measurable and (3.11.1) holds.

To continue, we note that both T and T^{-1} are continuous, so if A is a Borel set, then so is $T(A)$. Applying what we have just proved to $f = \chi_{T(A)}$ and T we obtain

$$\begin{aligned}m(T(A)) &= \int \chi_{T(A)}(x)dx = \int f(x)dx = |\det T| \int (f \circ T)(x)dx \\ &= |\det T| \int \chi_A(x)dx = |\det T| m(A),\end{aligned}$$

since $(f \circ T)(x) = \chi_{T(A)}(Tx) = \chi_A(x)$ for all $x \in \mathbb{R}^n$. Hence if $m(A) = 0$ then $m(T(A)) = 0$, and the class of Borel null sets is invariant under T (and hence also under T^{-1}). If $E \in \mathcal{L}^n$ and $m(E) = 0$, then there exists a Borel set A such that $E \subset A$ and $m(A) = 0$. Hence $T(E) \subset T(A)$ and $T(A)$ is a Borel null set, and since m is complete, $T(E) \in \mathcal{L}^n$ is a Lebesgue null set. Writing a Lebesgue measurable set $E = A \cup N$ where A is a Borel set and N a Lebesgue null set (using Theorem 3.11.1, item (b)) and we can assume $A \cap N = \varnothing$, we have $T(E) = T(A) \cup T(N)$, where $T(A)$ is a Borel set and $T(N)$ is a Lebesgue null set, which implies that $T(E) \in \mathcal{L}^n$ and

$$m(T(E)) = m(T(A)) + m(T(N)) = m(T(A)) = |\det T| m(A) = |\det T| m(E),$$

and we conclude (b).

Now if f is Lebesgue measurable, $T \in GL(n,\mathbb{R})$ and B is a Borel set in $\mathbb{C}$ we have $f^{-1}(B) = A \cup N$, where A is a Borel set and $m(N) = 0$. But $T^{-1}(A)$ is a Borel set and $T^{-1}(N)$ is a Lebesgue null set, hence

$$(f \circ T)^{-1}(B) = T^{-1}(f^{-1}(B)) = T^{-1}(A) \cup T^{-1}(N) \in \mathcal{L}^n,$$

and f is Lebesgue measurable.

Now from what we have proved above, (3.11.1) holds for characteristic functions of Lebesgue measurable sets. By linearity it holds for simple functions. Using the MCT it holds for nonnegative measurable functions and taking positive and negative parts of real and imaginary parts it holds for $L^1(m)$-functions. ■

Corollary 3.11.15. *Lebesgue measure is invariant under rotations.*

Proof. A rotation is a linear maps satysfying $TT^* = I$, where T^* is the transpose of T. Since $\det T = \det T^*$, we must have $|\det T| = 1$, and hence if $E \in \mathcal{L}^n$ we have $m(T(E)) = m(E)$. ■

Now we will treat the general case of C^1 diffeomorphisms.

Definition 3.11.16. *Let $G = (g_1, \cdots, g_n)$ be a map from an open set $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$, whose components g_j are C^1, that is, have continuous first order partial derivatives. We denote by D_xG the linear map defined by the matrix $((\partial g_i/\partial x_j)(x))$ of partial derivatives at x, for each $x \in \Omega$. Note that if G is linear, then $D_xG = G$ for all $x \in \Omega$. The map G is called a* **C^1 diffeomorphism** *if G is injective and D_xG is invertible for all $x \in \Omega$. In this case, the inverse function theorem guarantees that $G^{-1} \colon G(\Omega) \to \Omega$ is also a C^1 diffeomorphism and that $D_x(G^{-1}) = [D_{G^{-1}(x)}G]^{-1}$ for all $x \in G(\Omega)$.*

Before stating and proving the Change of Variables Theorem, we set some notation and prove a lemma. For $x \in \mathbb{R}^n$ and $T = (T_{ij}) \in GL(n,\mathbb{R}))$ we take

$$\|x\| = \max_{i=1,\cdots,n} |x_i| \qquad \text{and} \qquad \|T\| = \max_{i=1,\cdots,n} \sum_{j=1}^{n} |T_{ij}|.$$

Then we have $\|Tx\| \leqslant \|T\|\|x\|$, and if $a \in \mathbb{R}^n$ and $h > 0$ the set $\{x \in \mathbb{R}^n \colon \|x - a\| \leqslant h\}$ is the cube centered in a with side length $2h$.

Lemma 3.11.17. *Suppose that Ω is an open set in $\mathbb{R}^n$ and $G \colon \Omega \to \mathbb{R}^n$ is a C^1 diffeomorphism. If $A \subset \Omega$ is a Borel set we have*

$$m(G(A)) \leqslant \int_A |\det D_xG| dx.$$

Proof. First consider a cube $Q = \{x \in \mathbb{R}^n \colon \|x - a\| \leqslant h\} \subset \Omega$. Given $x \in Q$, fix $j = 1, \cdots, n$ and consider the function $[0,1] \ni t \mapsto g_j(tx + (1-t)a) \in \mathbb{R}$, which is a C^1 function in $\mathbb{R}$. Hence, by the Mean Value Theorem, there exists $t_0 \in (0,1)$ such that

$$g_j(x) - g_j(a) = \frac{d}{dt} g_j(t_0 x + (1-t_0)a) = \sum_{i=1}^{n} \frac{\partial g_j}{\partial x_i}(t_0 x + (1-t_0)a)(x_i - a_i),$$

and therefore

$$\|G(x) - G(a)\| = \max_{j=1,\cdots,n} |g_j(x) - g_j(a)| \leqslant (\sup_{y \in Q} \|D_y G\|) \max_{i=1,\cdots,n} |x_i - a_i| \leqslant h \sup_{y \in Q} \|D_y G\|,$$

since $t_0 x + (1-t_0)a \in Q$ for any $t_0 \in (0,1)$. This implies that $G(Q)$ is contained in the cube $\tilde{Q} = \{y \in \mathbb{R}^n \colon \|y - G(a)\| \leqslant h \sup_{y \in Q} \|D_y G\|\}$, and thus

$$m(G(Q)) \leqslant m(\tilde{Q}) = (2h \sup_{y \in Q} \|D_y G\|)^n = (\sup_{y \in Q} \|D_y G\|)^n m(Q).$$

If $T \in GL(n, \mathbb{R})$, then we can apply this formula for $T^{-1} \circ G$ instead of G and use Theorem 3.11.14 to obtain

$$m(G(Q)) = |\det T| m(T^{-1}(G(Q))) \leqslant |\det T| (\sup_{y \in Q} \|T^{-1} D_y G\|)^n m(Q), \tag{3.11.2}$$

since $T^{-1} \circ G$ is a diffeomorphism and $D_y(T^{-1}G) = T^{-1} D_y(G)$ for all $y \in \Omega$.

Now fix $\epsilon > 0$. Since $\Omega \ni y \mapsto D_y G$ is continuous (and hence uniformly continuous in Q), we can choose $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ we have

$$\|D_z G - D_y G\| \leqslant \frac{\epsilon}{1 + \sup_{w \in Q} \|(D_w G)^{-1}\|} \text{ for all } y, z \in Q \text{ with } \|z - y\| < \delta,$$

and since $D_z G - D_y G = D_z G(I - (D_z G)^{-1} D_y G)$ we obtain

$$\|(D_z G)^{-1} D_y G\| \leqslant \|I - (D_z G)^{-1} D_y G\| + 1 = \|(D_z G)^{-1}\| \|D_z G - D_y G\| + 1 < \epsilon + 1,$$

for $z, y \in Q$ with $\|z - y\| < \delta$. Now we subdivide Q into N cubes, $Q_1, \cdots, Q_N$, with disjoint interiors, whose side lengths are smaller than δ, and we name $x_1, \cdots, x_N$ the centers of such cubes. Using (3.11.2) for Q_j instead of Q and $D_{x_j}G$ instead of T, we obtain

$$\begin{aligned} m(G(Q)) &= m\Big(\bigcup_{j=1}^{N} G(Q_j)\Big) \leqslant \sum_{j=1}^{N} m(G(Q_j)) \\ &\leqslant \sum_{j=1}^{N} |\det D_{x_j}G|(\sup_{y\in Q_j} \|(D_{x_j}G)^{-1}D_yG\|^n)m(Q_j) \\ &\leqslant (1+\epsilon)^n \sum_{j=1}^{N} |\det D_{x_j}G| m(Q_j). \end{aligned}$$

Now, the last sum of these inequalities is the integral (on Q) of the simple function $\phi_\delta = \sum_{j=1}^{N} |\det D_{x_j}G|\chi_{Q_j\setminus\partial Q_j}$ (the dependence of δ is the dependence on the subdivision $\{Q_j\}$ of Q), since $m(\partial Q_j) = 0$ for all $j = 1, \cdots, N$. But from the uniform continuity of the map $Q \ni x \mapsto D_xG$, we have $\phi_\delta(x) \to |\det D_xG|$ m-a.e. in Q as $\delta \to 0^+$, and the Dominated Convergence Theorem shows us that making $\delta \to 0^+$ (along a countable subsequence) we obtain

$$\sum_{j=1}^{N} |\det D_{x_j}G| m(Q_j) = \int_Q \phi_\delta(x)dx \to \int_Q |\det D_xG|dx \text{ as } \delta \to 0^+.$$

Hence

$$m(G(Q)) \leqslant (1+\epsilon)^n \int_Q |\det D_xG|dx,$$

and making $\epsilon \to 0$ we obtain $m(G(Q)) \leqslant \int_Q |\det D_xG|dx$.

Now if $U \subset \Omega$ is open, then by Lemma 3.11.7, we can write $U = \bigcup_{j=1}^{\infty} Q_j$ where the Q_j's are cubes with disjoint interiors Thus

$$m(G(U)) \leqslant \sum_{j=1}^{\infty} m(G(Q_j)) \leqslant \sum_{j=1}^{\infty} \int_{Q_j} |\det D_xG|dx,$$

and since the boundaries ∂Q_j have zero Lebesgue measure (and thus their union also have zero Lebesgue measure), we obtain

$$\begin{aligned} \sum_{j=1}^{\infty} \int_{Q_j} |\det D_xG|dx &= \sum_{j=1}^{\infty} \int_{Q_j\setminus\partial Q_j} |\det D_xG|dx = \int_{\bigcup_{j=1}^{\infty}(Q_j\setminus\partial Q_j)} |\det D_xG|dx \\ &= \int_{U\setminus\bigcup_{j=1}^{\infty}\partial Q_j} |\det D_xG|dx = \int_U |\det D_xG|dx, \end{aligned}$$

and therefore $m(G(U)) \leqslant \int_U |\det D_xG|dx$.

Now let $E \subset \Omega$ be a Borel set with $m(E) < \infty$. Using Theorem 3.11.1 item (a), we can construct a sequence $\{U_j\}$ of open subsets of Ω with $E \subset U_j$ and $m(U_j) < \infty$ for all j, and $m(\bigcap_{j=1}^{\infty} U_j \setminus E) = 0$ and thus $\chi_{U_j} \to \chi_E$ m-a.e. Hence, by the DCT we have

$$\begin{aligned} m(G(E)) &\leqslant m\Big(G\Big(\bigcap_{j=1}^{\infty} U_j\Big)\Big) = \lim_{j\to\infty} m(G(U_j)) \\ &\leqslant \lim_{j\to\infty} \int_{U_j} |\det D_x G| dx = \int_E |\det D_x G| dx. \end{aligned}$$

Lastly, if $E \subset \Omega$ is any Borel set, since m is σ-finite, we can write $E = \bigcup_{j=1}^{\infty} E_j$ with disjont Borel sets with $m(E_j) < \infty$, then

$$m(G(E)) \leqslant \sum_{j=1}^{\infty} m(G(E_j)) \leqslant \sum_{j=1}^{\infty} \int_{E_j} |\det D_x G| dx = \int_E |\det D_x G| dx.$$

■

Theorem 3.11.18. *Suppose that Ω is an open set in $\mathbb{R}^n$ and $G\colon \Omega \to \mathbb{R}^n$ is a C^1 diffeomorphism.*

(a) *If f is a Lebesgue measurable function on $G(\Omega)$ then $f \circ G$ is a Lebesgue measurable function on Ω. If $f \geqslant 0$ or $f \in L^1(G(\Omega), m)$, then*

$$\int_{G(\Omega)} f(x) dx = \int_\Omega (f \circ G)(x) |\det D_x G| dx.$$

(b) *If $E \subset \Omega$ and $E \in \mathcal{L}^n$, then $G(E) \in \mathcal{L}^n$ and $m(G(E)) = \int_E |\det D_x G| dx$.*

Proof. **(a).** Consider f a simple nonnegative Borel measurable function on $G(\Omega)$, that is, $f = \sum_{j=1}^{m} a_j \chi_{A_j}$, which A_j is a Borel set for each j. Using the previous lemma, we have

$$\int_{G(\Omega)} f(x) dx = \sum_{j=1}^{m} a_j m(A_j) \leqslant \sum_{j=1}^{m} a_j \int_{G^{-1}(A_j)} |\det D_x G| dx = \int_\Omega (f \circ G)(x) |\det D_x G| dx.$$

Thus, if $f \in L^+(G(\Omega))$ then we can choose a sequence $\{s_n\}$ of nonnegative simple Borel measurable functions that increases to f, and

$$\int_{G(\Omega)} s_n(x) dx \leqslant \int_\Omega (s_n \circ G)(x) |\det D_x G| dx,$$

and since $(s_n \circ G)(x)|\det D_x G|$ increases to $(f \circ G)(x)|\det D_x G|$ pointwise, the MCT implies

that
$$\int_{G(\Omega)} f(x)dx \leqslant \int_{\Omega} (f \circ G)(x)|\det D_x G| dx.$$

Applying this reasoning to Ω replaced by $G(\Omega)$, G replaced by G^{-1} and $f(x)$ replaced by $(f \circ G)(x)|\det D_x G|$, so that

$$\begin{aligned}\int_{\Omega} (f \circ G)(x)|\det D_x G| dx &\leqslant \int_{G(\Omega)} (f \circ G \circ G^{-1})(x)|\det D_{G^{-1}(x)} G||\det D_x G^{-1}| dx \\ &= \int_{G(\Omega)} f(x) dx,\end{aligned}$$

which concludes the case for $f \geqslant 0$ Borel measurable. For f real valued integrable Borel measurable function, we see that $(f \circ G)^+ = f^+ \circ G$ and $(f \circ G)^- = f^- \circ G$, and the result follows easily. Using real and imaginary parts, the result is true for complex integrable Borel measurable functions. The proof for Lebesgue measurable functions is analogous to the proof of Theorem 3.11.14.

(b). This statement is just item (a) applied to $f = \chi_{G(E)}$. ■

3.12| Solved exercises from [1, Page 76]

Exercise 53. Fill in the details of the proof of Theorem 3.11.2.

Solution. It is already done.

Exercise 54. How much of Theorem 3.11.14 remains valid if T is not invertible?

Solution. First we see that applying a finite sequence of elementary type transformations, we can assume that $T(x_1, \cdots, x_n) = (x_1, \cdots, x_k, 0, \cdots, 0)$ for some $0 \leqslant k < n$ (for $k = 0$, $T \equiv 0$). Setting $A = \{x \in \mathbb{R}^n \colon x_n = 0\}$ we have $A \in \mathcal{L}^n$ (it is closed) and $m(A) = 0$.

If $E \in \mathcal{L}^n$, then $T(E) \in \mathcal{L}^n$ since $T(E) \subset A$ and m is complete. Moreover

$$m(T(E)) = 0 = |\det T| m(E),$$

since $\det T = 0$. Now we note that $f \circ T$ may not be Lebesgue measurable, even if f is. Let $B \subset \mathcal{R}^k$ not Lebesgue measurable in $\mathbb{R}^k$. We have $B \times \{0\}$ measurable in $\mathcal{L}^n$ (here $0 \in \mathbb{R}^{n-k}$), since m is complete. Consider $f = \chi_{B \times \{0\}}$. Thus

$$T^{-1}(f^{-1}(\{1\}) = T^{-1}(B \times \{0\}) = B \times \mathbb{R}^{n-k}$$

which is not measurable in $\mathbb{R}^n$ (since at least one of its sections is B, which is not measurable in $\mathbb{R}^k$).

In fact, even when $f \circ T$ is Lebesgue measurable, (3.11.1) may not be true. Consider $f = \chi_{[0,1]^n}$, which is in $L^1(m) \cap L^+(\mathbb{R}^n)$, and take $T \equiv 0$. Then $f \circ T = f(0) = 1$ which is not in $L^1(m)$. In this case we have $f \circ T \in L^+(\mathbb{R}^n)$ but

$$1 = \int f(x)dx \neq 0 = |\det T| \int f \circ T(x)dx.$$

Exercise 55. Let $E = [0,1] \times [0,1]$. Investigate the existence and equality of $\int_E f dm^2$, $\int_0^1 \int_0^1 f(x,y)dxdy$ and $\int_0^1 \int_0^1 f(x,y)dydx$ for the following f:

(a) $f(x,y) = (x^2 - y^2)(x^2 + y^2)^{-2}$.

(b) $f(x,y) = (1 - xy)^{-a}$, with $a > 0$.

(c) $f(x,y) = (x - \frac{1}{2})^{-3}$ if $0 < |y| < |x - \frac{1}{2}|$ and $f(x,y) = 0$ otherwise.

Solution to (a). Firstly, note that f is continuous from $E \setminus \{(0,0)\}$ to R, and hence f is measurable. Now we fix $y \in (0,1]$ and we define $F\colon [0,1] \to \mathbb{R}$ by $F(x) = -x(x^2 + y^2)^{-1}$. Then using the quotient rule we have

$$F'(x) = \frac{-x^2 - y^2 + 2x^2}{(x^2 + y^2)^2} = f(x,y),$$

hence

$$\int_0^1 f^+(x,y)dx = \int_y^1 f(x,y) = \int_y^1 F'(x)dx = F(1) - F(y) = \frac{1}{2y} - \frac{1}{1+y^2},$$

for all $y \in (0,1]$, since $f^+(x,y) = 0$ for $0 \leqslant x < y$. Analogously, for $y \in (0,1]$, we have

$$\int_0^1 f^-(x,y)dx = \int_0^y (-f(x,y))dx = -\int_0^y F'(x)dx = F(0) - F(y) = \frac{1}{2y}.$$

Applying Tonelli's Theorem to f^+ we obtain

$$\int_E f^+ dm^2 = \int_0^1 \int_0^1 f^+(x,y)dxdy = \int_0^1 \Big(\frac{1}{2y} - \frac{1}{1+y^2}\Big)dy.$$

From the MCT, we have

$$\int_0^1 \frac{1}{2y}dy = \lim_{n\to\infty}\int_0^1 \frac{1}{2y}\chi_{[1/n,1)}(y)dy = \lim_{n\to\infty}\int_{1/n}^1 \frac{1}{2y}dy = \lim_{n\to\infty}\frac{1}{2}\ln n = \infty,$$

and since $\frac{1}{1+y^2} \leqslant 1$ for all $y \in [0,1]$ we have $\int_0^1 \frac{1}{1+y^2}dy \leqslant 1 < \infty$, which implies that $\int_E f^+ dm^2 = \infty$. Again from Tonelli's Theorem we have

$$\int_E f^- dm^2 = \int_0^1 \frac{1}{2y}dy = \infty,$$

hence $\int_E f dm^2$ is not defined. However

$$\begin{aligned}\int_0^1\int_0^1 f(x,y)dxdy &= \int_0^1\left[\int_0^1 f^+(x,y)dx + \int_0^1 f^-(x,y)dx\right]dy\\ &= -\int_0^1 \frac{1}{1+y^2}dy = \arctan(0) - \arctan(1) = -\frac{\pi}{4},\end{aligned}$$

and since $f(x,y) = -f(y,x)$ for all $x, y \in [0,1]$ we have

$$\int_0^1\int_0^1 f(x,y)dydx = \frac{\pi}{4}.$$

Solution to (b). Note that $f(x,y) \geqslant 0$ for all $(x,y) \in E$ and since f is continuous, $f \in L^+(E, m^2)$ and hence by Tonelli's Theorem, all three integrals exist and are equal.

Solution to (c). Note that f is measurable and

$$\begin{aligned}f^+(x,y) &= \begin{cases}(x-\frac{1}{2})^{-3} & \text{if } x \in (\frac{1}{2},1] \text{ and } 0 < y < x - \frac{1}{2}\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases}(x-\frac{1}{2})^{-3} & \text{if } y \in [0,\frac{1}{2}] \text{ and } y + \frac{1}{2} < x \leqslant 1\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

and

$$\begin{aligned}f^-(x,y) &= \begin{cases}(\frac{1}{2}-x)^{-3} & \text{if } x \in [0,\frac{1}{2}) \text{ and } 0 < y < \frac{1}{2} - x\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases}(\frac{1}{2}-x)^{-3} & \text{if } y \in [0,\frac{1}{2}] \text{ and } 0 \leqslant x < \frac{1}{2} - y\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

Hence we have

$$\begin{aligned}\int_E f^+ dm^2 &\overset{\text{Tonelli's Th.}}{=} \int_0^1\int_0^1 f^+(x,y)dxdy = \int_0^{1/2}\int_{y+1/2}^1 (x-\tfrac{1}{2})^{-3}dxdy \\ &\overset{\text{Change of Var.}}{=} \int_0^{1/2}\int_y^{1/2} x^{-3}dxdy \overset{\text{Fund. Th. Calc.}}{=} \frac{1}{2}\int_0^{1/2}\Big(\frac{1}{y^2}-4\Big)dy \\ &\overset{\text{MCT}}{=} \lim_{n\to\infty}\frac{1}{2}\int_{1/n}^{1/2}\Big(\frac{1}{y^2}-4\Big)dy = \lim_{n\to\infty}\frac{n^2-4n+4}{2n} = \infty.\end{aligned}$$

Similarly we have

$$\begin{aligned}\int_E f^- dm^2 &= \int_0^1\int_0^1 f^-(x,y)dxdy = \int_0^{1/2}\int_0^{1/2-y}\Big(\frac{1}{2}-x\Big)^{-3}dxdy \\ &= \int_0^{1/2}\int_y^{1/2} x^{-3}dxdy = \infty,\end{aligned}$$

and hence $\int_E f dm^2$ is not defined. We also prove above that $\int_0^1 f^+(x,y)dx = \frac{1}{2}(\frac{1}{y^2}-4) = \int_0^1 f^-(x,y)dx$ for $y\in[0,\frac{1}{2}]$ and hence

$$\int_0^1\int_0^1 f(x,y)dxdy = \int_0^{1/2}\left[\int_0^1 f^+(x,y)dx - \int_0^1 f^-(x,y)dx\right]dy = 0.$$

Note that for $x\in[0,\frac{1}{2})$ we have $f^+(x,y)=f(x,y)$ and hence

$$\int_0^1 f(x,y)dy = \int_0^1 f^+(x,y)dy = \int_0^{x-1/2}\Big(x-\frac{1}{2}\Big)^{-3}dy = \Big(x-\frac{1}{2}\Big)^{-2},$$

and for $x\in(\frac{1}{2},1]$ we have $f^-(x,y)=-f(x,y)$ and thus

$$\int_0^1 f(x,y)dy = -\int_0^1 f^-(x,y)dy = -\int_0^{1/2-x}\Big(\frac{1}{2}-x\Big)^{-3}dy = -\Big(\frac{1}{2}-x\Big)^{-2}.$$

Thus

$$\int_0^1\Big(\int_0^1 f(x,y)dy\Big)^+dx = \int_{1/2}^1\Big(x-\frac{1}{2}\Big)^{-2}dx = \int_0^{1/2}x^{-2}dx \overset{\text{MCT}}{=} \infty,$$

and

$$\int_0^1\Big(\int_0^1 f(x,y)dy\Big)^-dx = \int_0^{1/2}\Big(\frac{1}{2}-x\Big)^{-2}dx = \int_0^{1/2}x^{-2}dx \overset{\text{MCT}}{=} \infty,$$

and therefore $\int_0^1\int_0^1 f(x,y)dydx$ is not defined.

Exercise 56. If f is Lebesgue integrable on $(0,a)$ and $g(x) = \int_x^a t^{-1}f(t)dt$, then g is integrable on $(0,a)$ and $\int_0^a g(x)dx = \int_0^a f(x)dx$.

Solution. Define $F\colon (0,a)\times(0,a) \to \mathbb{C}$ by

$$F(t,x) = \frac{f(t)}{t}\chi_A(t,x), \text{ where } A = \{(t,x) \in (0,a)\times(0,a) \colon x < t\}.$$

Since f is measurable, $\frac{1}{t}$ is continuous (hence, measurable) and A is open, F is measurable. Using Tonelli's Theorem we have

$$\begin{aligned}\int |F| dm^2 &= \int_0^a \int_0^a |F(t,x)| dxdt = \int_0^a \int_0^a \frac{|f(t)|}{t}\chi_A(t,x)dxdt \\ &= \int_0^a \int_0^t \frac{|f(t)|}{t} dxdt = \int_0^a |f(t)|dt < \infty,\end{aligned}$$

hence $F \in L^1$ and hence

$$g(x) = \int_x^a \frac{f(t)}{t}dt = \int_0^a \frac{f(t)}{t}\chi_{(x,a)}(t)dt = \int_0^a \frac{f(t)}{t}\chi_A(t,x)dt = \int_0^a F_x(t)dt$$

is integrable (since for each $x \in (0,a)$ we have $\chi_A(t,x) = \chi_{(x,a)}(t)$) by Fubini's Theorem, and also

$$\int_0^a g(x)dx = \int_0^a \int_0^a F(t,x)dxdt = \int_0^a \int_0^a F(t,x)dtdx = \int_0^a \int_0^t \frac{f(t)}{t}dxdt = \int_0^a f(t)dt.$$

Exercise 57. Show that $\int_0^\infty e^{-sx}x^{-1}\sin x dx = \arctan(s^{-1})$ for $s > 0$ by integrating $e^{-sxy}\sin x$ with respect to x and y. (It may be useful to recall that $\tan(\frac{\pi}{2} - \theta) = (\tan\theta)^{-1}$ Cf. Exercise 31d.)

Solution. We have $|e^{-sxy}\sin x| \leqslant xe^{-sxy}$ for all $x > 0$ and $y \geqslant 1$. Hence $f(x,y) = e^{-sxy}\sin x$ is in $L^1(m^2)$ on $E = (0,\infty)\times[1,\infty)$ since

$$\int_0^\infty xe^{-sxy}dx = \frac{1}{(sy)^2} \text{ for all } y \geqslant 1,$$

and hence by Tonelli' Theorem we have

$$\int_E xe^{-sxy}dm^2 = \int_1^\infty \int_0^\infty xe^{-sxy}dxdy = \int_1^\infty \frac{1}{(sy)^2}dy = \frac{1}{s^2} < \infty.$$

We have

$$\int_1^\infty e^{-sxy}\sin x dy = s^{-1}e^{-sx}x^{-1}\sin x \text{ for all } x > 0,$$

and using integration by parts we have

$$\int_0^\infty e^{-sxy}\sin x dx = \frac{1}{1+(sy)^2} \text{ for all } y \geqslant 1.$$

Using Fubini's Theorem we obtain

$$\begin{aligned} s^{-1}\int_0^\infty e^{-sx}x^{-1}\sin x dx &= \int_0^\infty \int_1^\infty e^{-sxy}\sin x dy dx = \int_1^\infty \int_0^\infty e^{-sxy}\sin x dx dy \\ &= \int_1^\infty \frac{1}{1+(sy)^2} dy = s^{-1}\left[\frac{\pi}{2} - \arctan(s)\right] \\ &= s^{-1}\arctan(s^{-1}). \end{aligned}$$

Exercise 58. Show that $\int_0^\infty e^{-sx}x^{-1}\sin^2 x dx = \frac{1}{4}\log(1+4s^{-2})$ for $s > 0$ by integrating $e^{-sx}\sin 2xy$ with respect to x and y.

Solution. Note that $|e^{-sx}\sin 2xy| \leqslant e^{-sx}$ for all $x > 0$ and $y \in [0,1]$, hence $e^{-sx}\sin 2xy$ is in $L^1(m^2)$ in $E = (0,\infty)\times[0,1]$. Also

$$\int_0^1 e^{-sx}\sin 2xy dy = e^{-sx}x^{-1}\sin^2 x,$$

and using integration by parts we have

$$\int_0^\infty e^{-sx}\sin 2xy dx = \frac{2y}{s^2+4y^2} \text{ for all } y \in [0,1],$$

and

$$\int_0^1 \frac{2y}{s^2+4y^2} dy \overset{t=s^2+4y^2}{=} \frac{1}{4}\int_{s^2}^{s^2+4} \frac{1}{t} dt = \frac{1}{4}\log(1+4^{-s^2}),$$

and the result follows using Fubini's Theorem.

Exercise 59. Let $f(x) = x^{-1}\sin x$.

(a) Show that $\int_0^\infty |f(x)| dx = \infty$

(b) Show that $\lim_{b\to\infty}\int_0^b f(x)dx = \frac{1}{2}\pi$ by integrating $e^{-xy}\sin x$ with respect to x and y. (In view of part (a), some care is needed in passing to the limit as $b \to \infty$).

Solution to (a). Note that for $n \in \mathbb{N}_0$ and $x \in [(n+\frac{1}{6})\pi, (n+\frac{5}{6})\pi]$ we have $|\sin x| \geqslant \frac{1}{2}$

and $\frac{1}{(n+1)\pi} \leqslant \frac{1}{(n+\frac{5}{6})\pi} \leqslant \frac{1}{x}$. Hence we have

$$\begin{aligned}
\int_0^\infty |f(x)|dx &\geqslant \int_0^\infty \sum_{n=0}^\infty \left[\frac{1}{2(n+1)\pi}\chi_{[(n+\frac{1}{6})\pi,(n+\frac{5}{6})\pi]}(x)\right] dx \\
&\overset{(\star)}{=} \sum_{n=0}^\infty \int_0^\infty \frac{1}{2(n+1)\pi}\chi_{[(n+\frac{1}{6})\pi,(n+\frac{5}{6})\pi]}(x)dx \\
&= \frac{1}{3}\sum_{n=0}^\infty \frac{1}{n+1} = \infty,
\end{aligned}$$

where in $(\star)$ we used Theorem 3.3.9.

Solution to (b). Now fix $b > 0$. Since $|e^{-xy}\sin x| \leqslant 1$ for all $x, y \in (0,b)$, $e^{-xy}\sin x$ is in $L^1(m^2)$ in $E = (0,b)\times(0,b)$. We have

$$\int_0^b e^{-xy}\sin x dy = \frac{\sin x}{x} - \frac{e^{-bx}\sin x}{x} \text{ for all } x \in (0,b),$$

and integration by parts result in

$$\int_0^b e^{-xy}\sin x dx = \frac{1-e^{-by}\cos b - ye^{-by}\sin b}{1+y^2} \text{ for all } y \in (0,b).$$

Using Fubini's Theorem we have

$$\begin{aligned}
&\int_0^b \frac{\sin x}{x}dx - \int_0^b \frac{e^{-bx}\sin x}{x}dx = \int_0^b\int_0^b e^{-xy}\sin x dy dx \\
&= \int_0^b\int_0^b e^{-xy}\sin x dx dy = \int_0^b \frac{1}{1+y^2}dy - \int_0^b e^{-by}\frac{\cos b - y\sin b}{1+y^2}dy \\
&= \arctan(b) - \int_0^b e^{-by}\frac{\cos b - y\sin b}{1+y^2}dy \quad \text{ for all } b>0.
\end{aligned} \tag{3.12.1}$$

To conclude we will note that

$$\left|\int_0^b \frac{e^{-bx}\sin x}{x}dx\right| \leqslant \int_0^b e^{-bx}dx = \frac{1-e^{-b^2}}{b} \to 0 \text{ as } b\to\infty,$$

and

$$\left|\int_0^b e^{-by}\frac{\cos b - y\sin b}{1+y^2}dy\right| \leqslant \int_0^b e^{-by}\frac{1+y}{1+y^2}dy \overset{(\clubsuit)}{\leqslant} C\int_0^b e^{-by}dy = C\frac{1-e^{-b^2}}{b} \to 0 \text{ as } b\to\infty,$$

where in $(\clubsuit)$ we used that fact that $(0,\infty) \ni y \mapsto \frac{1+y}{1+y^2}$ is a bounded function (hence the bound C does not depend on b). Hence taking the limit in (3.12.1) we obtain

$$\lim_{b\to\infty}\int_0^b \frac{\sin x}{x}dx = \lim_{b\to\infty}\arctan(b) = \frac{\pi}{2}.$$

Exercise 60. Show that $\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ for all $x, y > 0$. (Write $\Gamma(x)\Gamma(y)$ as a double integral and use the argument of the exponential as a new variable of integration. See Definition 3.5.24 and Proposition 3.5.27).

Solution. Using Exercise 51, for $f(u,v) = u^{x-1}e^{-u}v^{y-1}e^{-v}$ $(u,v) \in E = (0,\infty)\times(0,\infty)$, we have $f \in L^1(E, m^2)$ and

$$\Gamma(x)\Gamma(y) = \int_0^\infty u^{x-1}e^{-u}du \cdot \int_0^\infty v^{y-1}e^{-v}dv = \int_E f dm^2.$$

Set $\Omega = (0,\infty)\times(0,1)$ and consider the map $G : \Omega \times \mathbb{R}^2$ given by $G(s,t) = (st, s(1-t))$. Hence $G(\Omega) = E$ and G is a diffeomorphism. Hence

$$\begin{aligned}
\int_E f dm^2 &= \int_\Omega (f\circ G)(s,t)|\det D_{(s,t)}\cdot G|dm^2 \\
&= \int_0^1\int_0^\infty (st)^{x-1}e^{-st}s^{y-1}(1-t)^{y-1}e^{-s(1-t)}sdsdt = \\
&= \int_0^1\int_0^\infty s^{x+y-1}t^{x-1}(1-t)^{y-1}e^{-s}dsdt \\
&\overset{(\spadesuit)}{=} \int_0^1 t^{x-1}(1-t)^{y-1}dt \cdot \int_0^\infty s^{x+y-1}e^{-s}ds \\
&= \Gamma(x+y)\int_0^1 t^{x-1}(1-t)^{y-1}dt,
\end{aligned}$$

where in $(\spadesuit)$ we used again Exercise 51, and we conclude the result.

Exercise 61. If f is continuous on $[0,\infty)$, for $\alpha > 0$ and $x \geqslant 0$ let

$$I_\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt.$$

$I_\alpha f$ is called the α^{th} **fractional integral** of f.

(a) $I_{\alpha+\beta}f = I_\alpha(I_\beta f)$ for all $\alpha, \beta > 0$. (Use Exercise 60)

(b) If $n \in \mathbb{N}$, $I_n f$ is an n^{th}-order antiderivative of f.

First we prove that for $x > 0$, the function $[0,x) \ni t \mapsto (x-t)^{\alpha-1}$, denoted by f_0, is in $L^1([0,x),m)$. Clearly $f_0 \in L^+([0,x))$. Consider $f_n\colon [0,x) \to \mathbb{R}$ given by $f_n(t) = \chi_{[0,x-\frac{1}{n}]}(t)(x-t)^{\alpha-1}$ (for sufficiently large n). Then $\{f_n\} \subset L^+([0,x))$ increases to f_0 pointwise, and using the MCT we have

$$\int_{[0,x)} (x-t)^{\alpha-1}dt = \lim_{n\to\infty} \int_{[0,x)} \chi_{[0,x-\frac{1}{n}]}(t)(x-t)^{\alpha-1}dt =$$
$$= \lim_{n\to\infty} \int_0^{x-\frac{1}{n}} (x-t)^{\alpha-1}dt = \lim_{n\to\infty} \int_{\frac{1}{n}}^{x} u^{\alpha-1}du = \lim_{n\to\infty} \left(\frac{x^\alpha}{\alpha} - \frac{1}{n^\alpha \alpha}\right) = \frac{x^\alpha}{\alpha} < \infty,$$

and thus $f_0 \in L^1([0,x),m)$.

Moreover, using Theorem 3.5.16 item (a), $I_\alpha f$ is continuous in $[0,\infty)$.

Solution to (a). Using Fubini and Change of Variables we have

$$\begin{aligned}
I_\beta(I_\alpha f)(x) &= \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)dsdt \\
&= \frac{1}{\Gamma(\beta)\Gamma(\alpha)} \int_0^x \int_0^t (x-t)^{\beta-1}(t-s)^{\alpha-1} f(s)dsdt \\
&= \frac{1}{\Gamma(\beta)\Gamma(\alpha)} \int_0^x \int_s^x (x-t)^{\beta-1}(t-s)^{\alpha-1} dt f(s)ds \\
&\overset{(\star)}{=} \frac{1}{\Gamma(\beta)\Gamma(\alpha)} \int_0^x (x-s)^{\beta+\alpha-1} \int_0^1 (1-u)^{\beta-1}u^{\alpha-1}du f(s)ds = I_{\alpha+\beta}f(x),
\end{aligned}$$

using Exercise 60, and in $(\star)$ the change of variable $u = (t-s)(x-s)^{-1}$.

Solution to (b). For $n = 1$ this result is just the Fundamental Theorem of Calculus. If the result holds for n, i.e. $\frac{d^n}{dx^n} I_n f(x) = f(x)$ then

$$\frac{d^{n+1}}{dx^{n+1}} I_{n+1}f(x) = \frac{d^n}{dx^n}\frac{d}{dx} I_1(I_n f)(x) = \frac{d^n}{dx^n} I_n f(x) = f(x),$$

and the result is complete.

CHAPTER 4

SIGNED MEASURES AND DIFFERENTIATION

In this chapter, given two measures μ, ν on X, we want to give meaning to the expression $\dfrac{d\nu}{d\mu}$, that is, we want to differentiate ν with respect to μ.

4.1 | Signed measures

Definition 4.1.1. *Let $(X, \mathcal{M})$ a measurable space. A* **signed measure** *on $(X, \mathcal{M})$ is a function $\nu\colon \mathcal{M} \to [-\infty, \infty]$ such that*

(i) $\nu(\varnothing) = 0$;

(ii) *ν assumes at most one of the values $\pm\infty$;*

(iii) *if $\{E_j\}$ is a disjoint sequence in $\mathcal{M}$, then*

$$\nu\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \sum_{j=1}^{\infty} \nu(E_j),$$

where the series above converges absolutely when $\nu(\bigcup_{j=1}^{\infty} E_j)$ is finite.

It is clear from this definition that every measure is a signed measure. To make the distinction clear, sometimes we will refer to measures as **positive measures**.

Example 4.1.2. 1. *Let μ_1, μ_2 measures in $\mathcal{M}$ such that at least one of them is finite. Then $\nu = \mu_1 - \mu_2$ is a signed measure on $\mathcal{M}$.*

2. *We say that a function $f\colon X \to [-\infty, \infty]$ is* **extended μ-integrable** *if at least one of $\int f^+ d\mu$ and $\int f^- d\mu$ is finite. Then the function*

$$\nu(E) = \int_E f d\mu,$$

is a signed measure.

We will see that, indeed, the examples in Example 4.1.2 exhaust *all possible signed measures*. That is, every signed measure can be represented by one of the two previous cases.

Proposition 4.1.3. *Let ν be a signed measure on $(X, \mathcal{M})$. Then ν is continuous from below and from above.*

Proof. Continuity from below: to show this, let $\{E_j\}$ be an increasing sequence in $\mathcal{M}$ and let $E = \bigcup_{j=1}^{\infty} E_j$. If $\nu(E_{j_0}) = \pm\infty$ for some E_{j_0}, since

$$\nu(E_{j+1}) = \nu(E_j) + \nu(E_{j+1} \setminus E_j), \tag{4.1.1}$$

(and recalling that ν can only assume one of $\pm\infty$) we have $\nu(E_{j+1}) = \pm\infty$ for all $j \geqslant j_0$, and also since $\nu(E) = \nu(E_{j_0}) + \nu(E \setminus E_{j_0})$ we have $\nu(E) = \pm\infty$. Hence $\nu(E) = \pm\infty = \lim_{j\to\infty} \nu(E_j)$.

Assume that $\nu(E_j) \in (-\infty, \infty)$ for all $j \in \mathbb{N}$. From (4.1.1) we have $\nu(E_{j+1} \setminus E_j) = \nu(E_{j+1}) - \nu(E_j)$, and the proof is analogous to the case of positive measures.

Continuity from above: this is analogous to the case of positive measures (see Theorem 2.1.8). ∎

Definition 4.1.4. *If ν is a signed measure on $(X, \mathcal{M})$, a set $E \in \mathcal{M}$ is called*

(a) **positive** *for ν is $\nu(F) \geqslant 0$ for all $F \in \mathcal{M}$ with $F \subset E$.*

(b) **negative** *for ν is $\nu(F) \leqslant 0$ for all $F \in \mathcal{M}$ with $F \subset E$.*

(c) **null** *for ν is $\nu(F) = 0$ for all $F \in \mathcal{M}$ with $F \subset E$.*

For example, when $\nu(E) = \int_E f d\mu$ for some extended μ-integrable function f, then E is positive/negative/null for ν precisely when $f \geqslant 0/f \leqslant 0/\ f = 0$ μ-a.e. on E.

Lemma 4.1.5. *Any measurable subset of a positive set is positive and the union of any countable family of positive sets is positive.*

Proof. The first assertion is clear from the definitions of positivity. Now let $\{P_j\}$ be a countable family of positive sets. Considering $Q_j = P_j \setminus \bigcup_{k=1}^{j-1} P_k$, we have $Q_j \subset P_j$ and hence Q_j is positive for ν, and also $\{Q_j\}$ is a countable disjoint family in $\mathcal{M}$. Thus if $E \in \mathcal{M}$ is such that $E \subset \bigcup_{j=1}^{\infty} P_j = \bigcup_{j=1}^{\infty} Q_j$ then $E = \bigcup_{j=1}^{\infty} E \cap Q_j$ and therefore

$$\nu(E) = \sum_{j=1}^{\infty} \nu(E \cap Q_j) \geqslant 0,$$

since $\nu(E \cap Q_j) \geqslant 0$ for all j (recall that Q_j is positive for ν and $E \cap Q_j \in \mathcal{M}$ and $E \cap Q_j \subset Q_j$). ■

THEOREM 4.1.6 (The Hahn Decomposition Theorem). *If ν is a signed measure on $(X, \mathcal{M})$, there exists a positive set P and a negative set N for ν such that $P \cup N = X$ and $P \cap N = \varnothing$ If $\tilde{P}, \tilde{N}$ is another such pair, then $P \Delta \tilde{P} = N \Delta \tilde{N}$ is null for ν.*

Proof. We can assume, without loss of generality, that ν does not assume that value ∞, for otherwise we can consider $-\nu$. Since $\nu(\varnothing) = 0$, we can define

$$\alpha = \sup\{\nu(E) \colon E \text{ is a positive set for } \nu\} \geqslant 0.$$

There exists a sequence $\{P_j\}$ of positive sets for ν such that $\lim_{j\to\infty} \nu(P_j) = \alpha$. Set $P = \bigcup_{j=1}^{\infty} P_j$, which is positive by the previous lemma, and hence $\nu(P) \leqslant \alpha$. Also setting $Q_j = \bigcup_{k=1}^{j} P_k$, $\{Q_j\}$ is an increasing sequence of positive sets for ν, $P_j \subset Q_j$ for all j. Now $\nu(Q_j) = \nu(P_j) + \nu(Q_j \setminus P_j)$ and since $Q_j \setminus P_j \subset Q_j$ and Q_j is positive for ν we have $\nu(Q_j \setminus P_j) \geqslant 0$ and hence $\nu(Q_j) \geqslant \nu(P_j)$.

Now, using the continuity from below we obtain

$$\nu(P) = \nu\Big(\bigcup_{j=1}^{\infty} P_j\Big) = \Big(\bigcup_{j=1}^{\infty} Q_j\Big) = \lim_{j\to\infty} \nu(Q_j) \geqslant \lim_{j\to\infty} \nu(P_j) = \alpha,$$

and therefore $\nu(P) = \alpha$. Since ν does not assume the value ∞, we conclude that $\alpha < \infty$. *Claim:* the set $N = X \setminus P$ is negative for ν.

In fact, assume that N is not negative for ν. First, note that if $E \subset N$ is positive and $\nu(E) > 0$, then $E \cup P$ is positive and $\nu(E \cup P) = \nu(E) + \nu(P) > \alpha$, which is a contradiction. Hence N cannot contain any nonnull positive set.

Now if $A \subset N$ and $\nu(A) > 0$, since A cannot be positive, there exists $C \subset A$ with $\nu(C) < 0$ and thus setting $B = A \setminus C$ we have $\nu(B) = \nu(A) - \nu(C) > \nu(A)$. In conclusion, if $A \subset N$ and $\nu(A) > 0$, there exists $B \subset A$ with $\nu(B) > \nu(A)$.

Now, since N is not negative for ν, let n_1 be the smallest positive integer for which there exists a set $B \subset N$ with $\nu(B) > n_1^{-1}$ (if no such integer exists, this means that $\nu(B) \leqslant 0$ for all $B \subset N$, and N would be negative for ν). Define A_1 a set such that $\nu(A_1) > n_1^{-1}$. Inductively, set n_j as the smallest positive integer for which there exists $B \subset A_{j-1}$ with $\nu(B) > \nu(A_{j-1}) + n_j^{-1}$ and name A_j such a set.

Define $A = \bigcap_{j=1}^{\infty} A_j$. Then $\infty > \nu(A) = \lim_{j\to\infty} \nu(A_j) \geqslant \sum_{j=1}^{\infty} n_j^{-1} > 0$, and hence $n_j \to \infty$ as $j \to \infty$. Also, since $\nu(A) > 0$ there exists $B \subset A$ such that $\nu(B) > \nu(A) + 2n^{-1}$ for some integer n. Since $B \subset A_{j-1}$ for all j, we can find j sufficiently large such that $n < n_j$ and $\nu(A_{j-1}) < \nu(A) + n^{-1}$, which leads us to

$$\nu(B) > \nu(A) + n^{-1} + n^{-1} > \nu(A_{j-1}) + n^{-1},$$

and since $n < n_j$, this contradicts the definition of n_j. Therefore N must be a negative set for ν.

Finally, assume that $\tilde{P}, \tilde{N}$ is such another pair. Thus $P \setminus \tilde{P} \subset P$, hence $P \setminus \tilde{P}$ is positive for ν. Also $P \setminus \tilde{P} \subset \tilde{N}$, and hence $P \setminus \tilde{P}$ is negative for ν, which implies that it must be null for ν. Likewise $\tilde{P} \setminus P$ is null for ν. The fact that $P \Delta \tilde{P} = N \Delta \tilde{N}$ is straightforward and the proof is complete. ■

A decomposition $X = P \cup N$ of X as the disjoint union of a positive and a negative set for ν is called **Hahn decomposition of ν**. It is usually not unique, since ν-null sets can be transferred from P to N or vice-versa, but it leads to a canonical representation of ν as the different of two positive measures, with at least one of them finite.

Definition 4.1.7. *We say that two measures μ and ν on $(X, \mathcal{M})$ are* **mutually singular**, *or that ν is* **singular with respect to μ** *(or vice-versa), if there exists $E, F \in \mathcal{M}$, $E \cap F = \varnothing$, $E \cup F = X$ such that E is null for μ and F is null for ν. In this case, we denote this relationship by*

$$\mu \perp \nu.$$

Informally speaking, mutually singularity means that μ and ν live in disjoint subsets of X.

Theorem 4.1.8 (The Jordan Decomposition Theorem)**.** *If ν is a signed measure, there exist unique positive measures ν^+ and ν^- on $(X, \mathcal{M})$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.*

Proof. Let $X = P \cup N$ a Hanh decomposition of ν. For $E \in \mathcal{M}$, define

$$\nu^+(E) = \nu(E \cap P) \quad \text{and} \quad \nu^-(E) = -\nu(E \cap N).$$

Then clearly we have $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$. Now if $\nu = \mu^+ - \mu^-$ with $\mu^+ \perp \mu^-$, then let $E, F \in \mathcal{M}$ be such that $X = E \cup F$, $E \cap F = \varnothing$, F is null for μ^+ and E is null for μ^-. Then $X = E \cup F$ is a Hahn decomposition of ν, and since E is positive for ν and F is negative for ν we have $P \Delta E$ and $N \Delta F$ null sets for ν. Therefore, if $A \in \mathcal{M}$ we have

$$\mu^+(A) = \mu^(A \cap E) + \underbrace{\mu^+(E \cap F)}_{=0} = \mu^+(A \cap E) = \nu(A \cap E) = \nu(A \cap P) + \underbrace{\nu(A \cap (E \setminus P))}_{=0} = \nu^+(A),$$

hence $\mu^+ = \nu^+$, and analogously we show that $\mu^- = \nu^-$ and conclude the proof. ■

The measures ν^+ and ν^- are called the **positive** and **negative variations** of ν, and $\nu = \nu^+ - \nu^-$ is called the **Jordan decomposition of ν**. Furthermore, we define the **total variation of ν** as the positive measure $|\nu|$ defined by

$$|\nu| = \nu^+ + \nu^-.$$

Proposition 4.1.9. *We have the following:*

(a) *$E \in \mathcal{M}$ is ν-null iff $|\nu|(E) = 0$;*

(b) *$\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.*

Proof. **(a).** Assume that $E \in \mathcal{M}$ is null for ν, that is, $\nu(F) = 0$ for all $F \in \mathcal{M}$ with $F \subset E$. Let $X = P \cup N$ a Hahn decomposition of ν, then

$$\nu^+(E) = \nu(E \cap P) = 0 \quad \text{ and } \nu^-(E) = -\nu(E \cap N) = 0,$$

and thus $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$. Reciprocally, if $|\nu|(E) = 0$, then $\nu^+(E) = \nu^-(E) = 0$. If $F \in \mathcal{M}$ and $F \subset E$ we have $\nu^+(F) = \nu^-(F) = 0$ and $\nu(F) = \nu^+(F) - \nu^-(F) = 0$, thus E is ν-null.

(b). Assume that $\nu \perp \mu$. Then $X = E \cup F$ with $E, F \in \mathcal{M}$, $E \cap F = \varnothing$, E μ-null and F ν-null. From item (a), $|\nu|(F) = 0$ and hence $|\nu| \perp \mu$.

Now if $|\nu| \perp \mu$, let $X = E \cup F$ with $E, F \in \mathcal{M}$, $E \cap F = \varnothing$, E is μ-null and $|\nu|(F) = 0$. Then $\nu^+(F) = 0$ and $\nu^-(F) = 0$, which implies that $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Lastly, assume that $X = E_1 \cup F_1 = E_2 \cup F_2$ with $E_1, E_2, F_1, F_2 \in \mathcal{M}$, $E_1 \cap F_1 = E_2 \cap F_2 = \varnothing$, E_1, E_2 μ-null, $\nu^+(F_1) = \nu^-(F_2) = 0$. Set $E = E_1 \cup E_2$ and $F = X \setminus E = F_1 \cap F_2$. Hence E is a μ-null set (union of two μ-null sets) and if $A \subset F$ with $A \in \mathcal{M}$ we have $A \subset F_1$ and $A \subset F_2$, which implies that $\nu^+(A) = \nu^-(A) = 0$, thus $\nu(A) = 0$ and F is a ν-null set. Therefore $\nu \perp \mu$, and the proof is complete. ■

We observe that if ν omits the value ∞, then $\nu^+(X) = \nu(P) < \infty$, so that ν^+ is a finite measure, and ν is bounded above by $\nu^+(X)$. Similarly if ν omits the value $-\infty$. In particular, if the range of ν is contained in $\mathbb{R}$, ν is bounded.

For a Hahn decomposition $X = P \cup N$ of ν, we can write

$$\nu(E) = \int_E f d\mu \quad \text{for } E \in \mathcal{M},$$

where $\mu = |\nu|$ and $f = \chi_P - \chi_N$, since

$$\begin{aligned}\nu(E) &= \nu(E \cap P) + \nu(E \cap N) = \nu^+(E \cap P) - \nu^-(E \cap N) \\ &= \nu^+(E \cap P) + \underbrace{\nu^-(E \cap P)}_{=0} - [\nu^-(E \cap N) + \underbrace{\nu^+(E \cap N)}_{=0}] \\ &= |\nu|(E \cap P) + |\nu|(E \cap N) = \int_E \chi_P d|\nu| - \int_E \chi_N d|\nu|.\end{aligned}$$

Definition 4.1.10. *If ν is a signed measure on $(X, \mathcal{M})$, we define*

$$L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-),$$

and for $f \in L^1(\nu)$ we define the **integral** *of f by*

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-.$$

Also, we say that a signed measure ν is **finite** (or **σ-finite**) is $|\nu|$ is finite (or σ-finite).

4.2 | Solved exercises from [1, Page 88]

Exercise 1. Prove Proposition 4.1.3.

Solution. Already proven in the text.

Exercise 2. Prove Proposition 4.1.9.

Solution. Already proven in the text.

Exercise 3. Let ν be a signed measure on $(X, \mathcal{M})$.

(a) $L^1(\nu) = L^1(|\nu|)$.

(b) If $f \in L^1(\nu)$, $\left|\int f d\nu\right| \leqslant \int |f| d|\nu|$.

(c) If $E \in \mathcal{M}$, $|\nu|(E) = \sup\{\left|\int_E f d\nu\right| : |f| \leqslant 1\}$

Solution to (a). Let $E \in \mathcal{M}$. Then

$$\int \chi_E d|\nu| = |\nu|(E) = \nu^+(E) + \nu^-(E) = \int \chi_E d\nu^+ + \int \chi_E d\nu^-.$$

Thus this result holds for simples functions by additivity of the integral, by the MCT for nonnegative measurable functions, and hence for functions in $L^1(|\nu|)$ (using the positive and negative parts for real functions, and then the real and imaginary parts for complex functions).

Hence if f is a measurable function, we have

$$\int |f| d\nu^+ + \int |f| d\nu^- = \int |f| d|\nu|,$$

which shows that $f \in L^1(|\nu|)$ iff $f \in L^1(\nu^+) \cap L^1(\nu^-) = L^1(\nu)$.

Solution to (b). If $f \in L^1(\nu)$ then

$$\begin{aligned}\left|\int f d\nu\right| &= \left|\int f d\nu^+ - \int f d\nu^-\right| \leqslant \left|\int f d\nu^+\right| + \left|\int f d\nu^-\right| \\ &\leqslant \int |f| d\nu^+ + \int |f| d\nu^- = \int |f| d|\nu|.\end{aligned}$$

Solution to (c). Define $\alpha(E) = \sup\{\left|\int_E f d\nu\right| : |f| \leqslant 1\}$. If $\nu(E) \in (-\infty, \infty)$, from item (b) applied to $f\chi_E$, we have $\alpha(E) \leqslant |\nu|(E)$. Note now that $\nu(E) \in \{\pm\infty\}$ iff $|\nu|(E) = \infty$, and in this case taking $f \equiv 1$ we obtain $\alpha(E) = \infty = |\nu|(E)$. Now if $X = P \cup N$ is a Hanh decomposition for ν, define $f = \chi_P - \chi_N$. Then $|f| \leqslant 1$ and

$$\left|\int_E f d\nu\right| = ||\nu(E \cap P) + \nu(E \cap N)| = |\nu^+(E) + \nu^-(E)| = \nu^+(E) + \nu^-(E) = |\nu|(E),$$

and hence $|\nu|(E) \leqslant \alpha(E)$.

Exercise 4. If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$ then $\lambda \geqslant \nu^+$ and $\mu \geqslant \nu^-$.

Solution. Let $X = P \cup N$ be a Hahn decomposition of ν. We have

$$\nu^+(E) = \nu(E \cap P) = \lambda(E \cap P) - \mu(E \cap P) \leqslant \lambda(E \cap P) \leqslant \lambda(E), \text{ and}$$
$$\nu^-(E) = -\nu(E \cap N) = \mu(E \cap N) - \lambda(E \cap N) \leqslant \mu(E \cap N) \leqslant \mu(E).$$

EXERCISE 5. If ν_1, ν_2 are signed measures that both omit the value ∞ or $-\infty$, then $|\nu_1 + \nu_2| \leqslant |\nu_1| + |\nu_2|$ (Use Exercise 4).

Solution. Writing

$$\nu_1 + \nu_2 = \underbrace{\nu_1^+ + \nu_2^+}_{=\lambda} - \underbrace{(\nu_1^- + \nu_2^-)}_{=\mu},$$

we have $\nu_1 + \nu_2 = \lambda - \mu$ where λ, μ are positive measures. From Exercise 4 we have

$$\lambda \geqslant (\nu_1 + \nu_2)^+ \quad \text{and} \quad \mu \geqslant (\nu_1 + \nu_2)^-,$$

hence

$$|\nu_1 + \nu_2| = (\nu_1 + \nu_2)^+(\nu_1 + \nu_2)^- \leqslant \lambda + \mu = |\nu_1| + |\nu_2|.$$

EXERCISE 6. Suppose $\nu(E) = \int_E f d\mu$ where μ is a positive measure and f is an extended μ-integrable function. Describe the Hahn decompositions of ν and the positive, negative, and total variations of ν in terms of f and μ.

Solution. Let $P^+ = \{x \in X : f(x) > 0\}$, $Z = \{x \in X : f(x) = 0\}$ and $N^- = \{x \in X : f(x) < 0\}$. If $P_0, N_0 \in \mathcal{M}$ are subsets of X such that $Z = P_0 \cup N_0$ and $P_0 \cap N_0 = \varnothing$, then $X = P \cup N$ is a Hahn decomposition of ν, where $P = P^+ \cup P_0$ and $N = N^- \cup N_0$. Clearly, these are all the possible Hahn decompositions of ν.

We have also

$$\nu^+(E) = \int_E f^+ d\mu, \quad \nu^-(E) = \int_E f^- d\mu \text{ and } |\nu|(E) = \int_E |f| d\mu,$$

for every $E \in \mathcal{M}$.

EXERCISE 7. Suppose that ν is a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$.

(a) $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{M},\ F \subset E\}$ and $\nu^-(E) = -\inf\{\nu(F) : F \in \mathcal{M},\ F \subset E\}$.

(b) $|\nu|(E) = \sup\{\sum_{j=1}^n |\nu(E_j)| : n \in \mathbb{N},\ E_1, \cdots, E_n$ are disjoint, and $\bigcup_{j=1}^n E_j = E\}$.

Solution to (a). Let $X = P \cup N$ be a Hahn decomposition of ν. If $F \in \mathcal{M}$ and $F \subset E$ we have

$$\nu(F) = \nu(F \cap P) + \nu(F \cap N) \leqslant \nu(F \cap P) = \nu^+(F) \leqslant \nu^+(E),$$

hence $\sup\{\nu(F) \colon F \in \mathcal{M},\ F \subset E\} \leqslant \nu^+(E)$. On the other hand $E \cap P \in \mathcal{M}$, $E \cap P \subset E$ and $\nu(E \cap P) = \nu^+(E)$, and thus the equality follows. Analogously we have $\nu^-(E) = -\inf\{\nu(F) \colon F \in \mathcal{M},\ F \subset E\}$.

Solution to (b). We consider the same Hahn decomposition of ν. If $E_1, \cdots, E_n$ are disjoint and $\bigcup_{j=1}^n E_j$ we have

$$|\nu(E_j)| = |\nu(E_j \cap P) + \nu(E_j \cap N)| = |\nu^+(E_j) - \nu^-(E_j)| \leqslant |\nu|(E_j)$$

and hence

$$\sum_{j=1}^n |\nu(E_j)| \leqslant \sum_{j=1}^n |\nu|(E_j) = |\nu|(E),$$

and thus $\sup\{\sum_{j=1}^n |\nu(E_j)| \colon n \in \mathbb{N},\ E_1, \cdots, E_n \text{ are disjoint, and } \bigcup_{j=1}^n E_j = E\} \leqslant |\nu|(E)$. For the converse, let $E = E_1 \cup E_2$ with $E_1 = E \cap P$ and $E_2 = E \cap N$. Then E_1, E_2 are disjoint, $E = E_1 \cup E_2$ and

$$|\nu(E_1)| + |\nu(E_2)| = |\nu(E \cap P)| + |\nu(E \cap N)| = |\nu^+(E)| + |-\nu^-(E \cap N)| = \nu^+(E) + \nu^-(E) = |\nu|(E),$$

and concludes the proof.

4.3 | The Lebesgue-Radon-Nikodym Theorem

Definition 4.3.1. *Assume that ν is a signed measure and μ is a positive measure in $(X, \mathcal{M})$. We say that ν is* **absolutely continuous** *with respect to μ, and we write $\nu \ll \mu$, if $\nu(E) = 0$ for all $E \in \mathcal{M}$ for which $\mu(E) = 0$.*

Proposition 4.3.2. *$\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.*

Proof. Consider $X = P \cup N$ a Hanh decomposition for ν. Assume that $\nu \ll \mu$ and $E \in \mathcal{M}$ is such $\mu(E) = 0$. Then $\mu(E \cap P) = \mu(E \cap N) = 0$ and hence $\nu^+(E) = \nu(E \cap P) = 0$ and $\nu^-(E) = -\nu(E \cap N) = 0$. Therefore $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

If $\nu^+ \ll \mu$ and $\nu^- \ll \mu$, then given $E \in \mathcal{M}$ with $\mu(E) = 0$ we have $\nu^+(E) = \nu^-(E) = 0$ and thus $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$, that is, $|\nu| \ll \mu$.

Finally, if $|\nu| \ll \mu$ and $E \in \mathcal{M}$ is such that $\mu(E) = 0$, we have $|\nu|(E) = 0$ and thus $\nu^+(E) = \nu^-(E) = 0$, which in turn implies that $\nu(E) = \nu^+(E) - \nu^-(E) = 0$, and $\nu \ll \mu$. ■

Absolute continuity is, in some sense, the antithesis of mutual singularity, that is, we have the following result.

Proposition 4.3.3. *If $\nu \perp \mu$ and $\nu \ll \mu$ then $\nu = 0$.*

Proof. Since $\nu \perp \mu$, there exists $E, F \in \mathcal{M}$ such that $X = E \cup F$, $E \cap F = \varnothing$, E is ν-null and $\mu(F) = 0$. But since $\nu \ll \mu$, we have $|\nu| \ll \mu$ and this implies that $|\nu|(F) = 0$, hence $X = E \cup F$ is ν-null, that is, $\nu = 0$. ■

The name *absolute continuity* comes from the next result.

Theorem 4.3.4. *Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{M})$. Then $\nu \ll \mu$ iff for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$.*

Proof. Since $\nu \ll \mu$ iff $|\nu| \ll \mu$ and $|\nu(E)| \leqslant |\nu|(E)$, it is sufficient to assume that ν is a positive measure. It is clear that the ϵ-δ condition implies that $\nu \ll \mu$. For the converse, assume that $\nu \ll \mu$ and assume that the ϵ-δ condition does not hold. Then there exists $\epsilon_0 > 0$ such that for all n, we can find $E_n \in \mathcal{M}$ with $\mu(E_n) \leqslant 2^{-n}$ and $\nu(E_n) \geqslant \epsilon_0$.

Define $F_k = \bigcup_{n=k}^{\infty} E_n$. Then $\{F_k\}$ is a decreasing sequence and $\mu(F_k) \leqslant 2^{1-k}$ and $\nu(F_k) \geqslant \nu(E_k) \geqslant \epsilon_0$ for all $k \in \mathbb{N}$. Now setting $F = \bigcap_{k=1}^{\infty}$ we have $\mu(F) = 0$ and using the continuity from above of ν (recall that ν is finite) we obtain $\nu(F) = \lim_{k \to \infty} \nu(F_k) \geqslant \epsilon_0$, which contradicts the fact that $\nu \ll \mu$. ■

If μ is a positive measure and f is an extended μ-integrable function, the signed measure ν defined by $\nu(E) = \int_E f d\mu$ is clearly absolutely continuous with respect to μ, and it is finite iff $f \in L^1(\mu)$. We have then the following consequence of the previous theorem.

Corollary 4.3.5. *if $f \in L^1(\mu)$, for every $\epsilon > 0$, there exists $\delta > 0$ such that $\left|\int_E f d\mu\right| < \epsilon$ whenever $\mu(E) < \delta$.*

Proof. Apply Theorem 4.3.4 to Ref and Imf. ■

We will use the notation
$$d\nu = f d\mu$$
to express the relationship $\nu(E) = \int_E f d\mu$, and sometimes, we will refer to $f d\mu$ as a signed measure.

Before proving the main result of this section, we will make a technical lemma.

Lemma 4.3.6. *Suppose that ν and μ are finite positive measures on $(X, \mathcal{M})$. Either $\nu \perp \mu$ or there exists $\epsilon > 0$ such that $\mu(E) > 0$ and $\nu \geqslant \epsilon\mu$ on E (that is, E is a positive set for $\nu - \epsilon\mu$).*

Proof. Given $\epsilon > 0$, choose $n_0 \in \mathbb{N}$ such that $n_0^{-1} < \epsilon$. For each $n \geqslant n_0$, $\nu - n^{-1}\mu$ is a signed measure on $(X, \mathcal{M})$. Let $X = P_n \cup N_n$ be a Hahn decomposition for $\nu - n^{-1}\mu$ for each $n \geqslant n_0$. Let $P = \bigcup_{n=n_0}^{\infty} P_n$ and $N = \bigcap_{n=n_0}^{\infty} N_n = P^c$. Then N is a negative set for $\nu - n^{-1}\mu$ for all $n \geqslant n_0$, that is, $0 \leqslant \nu(N) \leqslant n^{-1}\mu(N)$ for all $n \geqslant n_0$, and since μ is finite, this implies that $\nu(N) = 0$. If $\mu(P) = 0$ then $\mu \perp \nu$. If $\mu(P) > 0$, then $\mu(P_n) > 0$ for some $n \geqslant n_0$, and P_n is a positive set for $\nu - n^{-1}\mu$. Thus $\nu - \epsilon\mu \geqslant \nu - n^{-1}\mu \geqslant 0$ in $E := P_n$, and the result is proved. ■

Theorem 4.3.7 (The Lebesgue-Radon-Nikodym Theorem). *Let ν be a σ-finite signed measure and μ a σ-finite positive measure on $(X, \mathcal{M})$. There exist unique σ-finite signed measures λ, ρ on $(X, \mathcal{M})$ such that*

$$\lambda \perp \mu, \quad \rho \ll \mu \quad \text{and} \quad \nu = \lambda + \rho.$$

Moreover, there is an extended μ-integrable function $f \colon X \to \mathbb{R}$ such that $d\rho = f d\mu$ and any two such functions are equal μ-a.e.

Proof. We will split the proof in three cases.

Case 1: ν and μ are finite positive measures.

In this case, define

$$\mathcal{F} = \left\{ f \in L^+(X) \colon \int_E f d\mu \leqslant \nu(E) \text{ for all } E \in \mathcal{M} \right\}.$$

Since ν is a positive measure, $f \equiv 0 \in \mathcal{F}$, and thus $\mathcal{F}$ is nonempty. Also if $f, g \in \mathcal{F}$ then $h = \max\{f, g\} \in \mathcal{F}$. In fact if $A = \{x \in X \colon f(x) > g(x)\}$ we have

$$\int_E h d\mu = \int_{E \cap A} f d\mu + \int_{E \setminus A} g d\mu \leqslant \nu(E \cap A) + \nu(E \setminus A) = \nu(E).$$

Let $a = \sup\{\int f d\mu \colon f \in \mathcal{F}\}$, since $a \leqslant \nu(X) < \infty$, we can choose a sequence $\{f_n\} \subset \mathcal{F}$ such that $\int f_n d\mu \to a$. Define $g_n = \max\{f_1, \cdots, f_n\}$ and $f = \sup f_n$. Then $g_n \in \mathcal{F}$ for each $n \in \mathbb{N}$, g_n increases pointwise to f and $\int g_n d\mu \geqslant \int f_n d\mu$ for each $n \in \mathbb{N}$. Thus from the MCT we obtain

$$\int f d\mu = \lim_{n \to \infty} \int g_n d\mu \geqslant \lim_{n \to \infty} \int f_n d\mu = a.$$

But another application of the MCT shows that for all $E \in \mathcal{M}$ we have

$$\int_E f d\mu = \lim_{n\to\infty} \int_E g_n d\mu \leqslant \nu(E),$$

and hence $f \in \mathcal{F}$, which implies that $\int f d\mu = a$. Since $a < \infty$, this implies in particular that $f < \infty$ μ-a.e., and we can redefine f, if necessary, so that f is a real-valued function.

Now we claim that the measure $d\lambda = d\nu - f d\mu$ (which is positive since $f \in \mathcal{F}$) is singular with respect to μ. If that is not the case, from Lemma 4.3.6, there exist $E \in \mathcal{M}$ and $\epsilon > 0$ such that $\mu(E) > 0$ and $\lambda \geqslant \epsilon\mu$ on E. But then $\epsilon\chi_E d\mu \leqslant d\lambda = d\nu - f d\mu$, that is, $(f + \epsilon\chi_E)d\mu \leqslant d\nu$, so $f + \epsilon\chi_E \in \mathcal{F}$ and $\int(f + \epsilon\chi_E)d\mu = a + \epsilon\mu(E) > a$, which contradicts the definition of a.

Thus the existence of λ, f and $d\rho = f d\mu$ is proved. Now for the uniqueness, if $d\nu = d\tilde{\lambda} + \tilde{f} d\mu$, we have $d\lambda - d\tilde{\lambda} = (\tilde{f} - f)d\mu$. But $\lambda - \tilde{\lambda} \perp \mu$ (see Exercise 9), while $(\tilde{f} - f)d\mu \ll d\mu$, hence $d\lambda - d\tilde{\lambda} = (\tilde{f} - f)d\mu = 0$, so that $\lambda = \tilde{\lambda}$ and $f = \tilde{f}$ μ-a.e. (by Proposition 3.5.7), and concludes the case of finite positive measures.

<u>Case 2:</u> Suppose that μ and ν are σ-finite positive measures.

In this case we can write $X = \bigcup_{j=1}^{\infty} A_j$ with $\{A_j\} \subset \mathcal{M}$ a disjoint sequence with such that $\mu(A_j), \nu(A_j) < \infty$ for all $j \in \mathbb{N}$. Define $\mu_j(E) = \mu(E \cap A_j)$ and $\nu_j(E) = \nu(E \cap E_j)$ for all $E \in \mathcal{M}$. Then applying Case 1, we have $d\nu_j = d\lambda_j + f_j d\mu_j$ for all $j \in \mathbb{N}$, with $\lambda_j \perp \mu_j$. Since $\mu_j(A_j^c) = \nu_j(A_j^c) = 0$ we have $\lambda_j(A_j^c) = \nu_j(A_j^c) - \int_{A_j^c} f_j d\mu_j = 0$, and we can assume that $f_j = 0$ on A_j^c. Define $\lambda = \sum_{j=1}^{\infty} \lambda_j$ and $f = \sum_{j=1}^{\infty} f_j$. Then

$$d\nu = d\lambda + f d\mu \quad \text{ and } \lambda \perp \mu,$$

(see Exercise 9), and $d\lambda$ and $f d\mu$ are σ-finite, as desired. The uniqueness follows as before.

<u>Case 3:</u> ν is a σ-finite signed measure.

Apply Case 2 to ν^+ and ν^- and subtract the results. The uniqueness follows again as in Case 1. ■

The decomposition $\nu = \lambda + \rho$ where $\lambda \perp \mu$ and $\rho \ll \mu$ is called the **Lebesgue decomposition** of ν with respect to μ. In the case where $\nu \ll \mu$, Theorem 4.3.7 says that $d\nu = f d\mu$ for some f. This result is usually known as the **Radon-Nikodym theorem**, and f is called the **Radon-Nikodym derivative** of ν with respect to μ. We denote it by $\frac{d\nu}{d\mu}$:

$$d\nu = \frac{d\nu}{d\mu} d\mu.$$

Strictly speaking, $\frac{d\nu}{d\mu}$ is the class of functions equal to f. The formulas suggested by the differential notation are generally correct. For example, it is simple to see that

$$\frac{d(\nu_1+\nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}.$$

We also have the chain rule:

PROPOSITION 4.3.8. *Suppose that ν is a σ-finite signed measure and μ, λ are σ-finite positive measures on $(X, \mathcal{M})$ such that $\nu \ll \mu$ and $\mu \ll \lambda$.*

(a) *If $g \in L^1(\nu)$, then $g(\frac{d\nu}{d\mu}) \in L^1(\mu)$ and*

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

(b) *We have $\nu \ll \lambda$ and*

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \qquad \lambda\text{-a.e..}$$

Proof. **(a).** By considering ν^+ and ν^- separately, we may assume that $\nu \geqslant 0$. The equation $\int g d\nu = \int g(\frac{d\nu}{d\mu}) d\mu$ is true when $g = \chi_E$, since by definition of $\frac{d\nu}{d\mu}$ we have

$$\int \chi_E d\nu = \nu(E) = \int_E \frac{d\nu}{d\mu} d\mu.$$

Hence it hold by linearity for simple functions, by the MCT for nonnegative measurable functions, and by linearity again for $L^1(\nu)$ functions.

(b) Replacing ν, μ by μ, λ in (a), and taking $g = \chi_E(\frac{d\nu}{d\mu})$ we obtain

$$\nu(E) = \int g d\mu = \int g \frac{d\mu}{d\lambda} d\lambda = \int_E \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} d\lambda,$$

for all $E \in \mathcal{M}$, whence $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$ λ-a.e. by Proposition 3.5.7. ∎

COROLLARY 4.3.9. *If $\mu \ll \lambda$ and $\lambda \ll \mu$ then $(\frac{d\lambda}{d\mu})(\frac{d\mu}{d\lambda}) = 1$ a.e. (with respect to either λ or μ).*

EXAMPLE 4.3.10 (Nonexample). *Let m the Lebesgue measure on $\mathbb{R}$ and ν the point mass at zero on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, that is, for $E \in \mathcal{B}_{\mathbb{R}}$ we have*

$$\nu(E) = \begin{cases} 1 & \text{if } 0 \in E. \\ 0 & \text{if } 0 \notin E. \end{cases}$$

Then $\nu \perp \mu$, since $\mathbb{R} = \{0\} \cup (\mathbb{R} \setminus \{0\})$, $m(\{0\}) = 0$ and $\nu(\mathbb{R} \setminus \{0\}) = 0$.

The nonexistent Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ is popularly known as the Dirac δ-function.

We conclude this section with a simple, but useful, result.

Proposition 4.3.11. *If $\mu_1, \cdots, \mu_n$ are measures on $(X, \mathcal{M})$, there is a measure μ such that $\mu_j \ll \mu$ for all j, namely, $\mu = \sum_{j=1}^{n} \mu_j$.*

Proof. The proof is straightforward. ∎

4.4 | Solved exercises from [1, Page 92]

Exercise 8. $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Solution. This is Proposition 4.3.2.

Exercise 9. Suppose $\{\nu_j\}$ is a sequence of positive measures and μ a positive measure. If $\nu_j \perp \mu$ for all j then $\sum_{j=1}^{\infty} \nu_j \perp \mu$, and if $\nu_j \ll \mu$ for all j then $\sum_{j=1}^{\infty} \nu_j \ll \mu$.

Solution. Assume that $X = E_j \cup F_j$ with $E_j, F_j \in \mathcal{M}$, $E_j \cap F_j = \varnothing$, $\nu_j(F_j) = 0$ and $\mu(E_j) = 0$ for all j. Set $E = \bigcup_{j=1}^{\infty} E_j$ and $F = \bigcap_{j=1}^{\infty} F_j = E^c$. Then for $\nu = \sum_{j=1}^{\infty} \nu_j$ we have $\nu_j(F) = 0$ for all j and

$$\mu(E) \leqslant \sum_{j=1}^{\infty} \mu(E_j) = 0 \quad \text{and} \quad \nu(F) = \sum_{j=1}^{\infty} \nu_j(F) = 0,$$

and $E, F \in \mathcal{M}$, $E \cap F = \varnothing$ and $X = E \cup F$, hence $\nu \perp \mu$.

Now if $\nu_j \ll \mu$ for all j and $E \in \mathcal{M}$ is such $\mu(E) = 0$, then $\nu_j(E) = 0$ for all j and hence $\nu(E) = 0$, thus $\nu \ll \mu$.

Exercise 10. Theorem 4.3.4 mail fail when ν is not finite. Consider $d\nu(x) = \frac{dx}{x}$ and $d\mu(x) = dx$ on $(0, 1)$, or ν the counting measure and $\mu(E) = \sum_{n \in E} 2^{-n}$ on $\mathbb{N}$.

Solution. We know that $\nu \ll \mu$ in the first case, since if $\mu(E) = 0$ then $\nu(E) = \int_E \frac{dx}{x} = 0$. But $\nu((0, \delta)) = \infty$ and $\mu((0, \delta)) = \delta$ for all $\delta > 0$.

As for the other case, clearly if $\mu(E) = 0$ we must have $E = \varnothing$ and hence $\nu(E) = 0$, thus $\nu \ll \mu$. But for $E_k = \{k, k+1, \cdots\}$ we have $\mu(E_k) = 2^{1-k} \to 0$ as $k \to \infty$ and $\nu(E_k) = \infty$ for all k.

Thus in both cases the conclusion of Theorem 4.3.4 is false.

Exercise 11. Let μ be a positive measure. A collection of functions $\{f_\alpha\}_{\alpha\in A} \subset L^1(\mu)$ is called **uniformly integrable** if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\int_E f_\alpha d\mu| < \epsilon$ for all $\alpha \in A$ whenever $\mu(E) < \delta$.

(a) Any finite subset of $L^1(\mu)$ is uniformly integrable.

(b) If $\{f_n\}$ is a sequence in $L^1(\mu)$ that converges to f in $L^1(\mu)$, then $\{f_n\}$ is uniformly integrable.

Solution to (a). The result is true for a single function by Corollary 4.3.5. Thus for each ϵ, taking the minimum of the δ's for each f in the finite subset we have the result.

Solution to (b). For $\epsilon > 0$, choose $n_0 \in \mathbb{N}$ such that

$$\int |f_n - f| < \frac{\epsilon}{2} \qquad \text{for all } n \geqslant n_0.$$

Now for $\{f_1, \cdots, f_{n_0-1}\}$ we know from item (a) that exists $\delta > 0$ such that

$$\left|\int_E f_n d\mu\right| < \epsilon \qquad \text{for all } 1 \leqslant n \leqslant n_0 - 1 \qquad \text{and } \left|\int_E f d\mu\right| < \frac{\epsilon}{2},$$

for all $E \in \mathcal{M}$ with $\mu(E) < \delta$.

Hence for $n \geqslant n_0$ we have

$$\left|\int_E f_n d\mu\right| \leqslant \left|\int_E f d\mu\right| + \int |f_n - f| < \epsilon,$$

and concludes the result.

Exercise 12. For $j = 1, 2$ let μ_j, ν_j be σ-finite positive measures on $(X_j, \mathcal{M}_j)$ such that $\nu_j \ll \mu_j$. Then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2).$$

Solution. Set $f_j = \frac{d\nu_j}{d\mu_j}$ for $j = 1, 2$. If $A \times B$ is a rectangle in $\mathcal{M}_1 \otimes \mathcal{M}_2$ we have

$$\begin{aligned}\nu_1 \times \nu_2(A \times B) = \nu_1(A)\nu_2(B) &= \int_A f_1 d\mu_1 \int_B f_2 d\mu_2 \\ &= \int_{A\times B} f_1 f_2 d(\mu_1 \times \mu_2),\end{aligned}$$

where the last equality follows from Tonelli's Theorem. By additivity of the integral, this results also holds for finite disjoint union of rectangles. By uniqueness of the extension of the product measure, we have then

$$\nu_1 \times \nu_2(E) = \int_E f_1 f_2 d(\mu_1 \times \mu_2).$$

This shows that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and the uniqueness of the Radon-Nikodym derivative shows us that

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)} = f_1 f_2 = \frac{d\nu_1}{d\mu_1}\frac{d\nu_2}{d\mu_2}.$$

Exercise 13. Let $X = [0,1]$, $\mathcal{M} = \mathcal{B}_{[0,1]}$, $m =$ Lebesgue measure, and μ the counting measure on $\mathcal{M}$.

(a) $m \ll \mu$ but $dm \neq f d\mu$ for any f.

(b) μ has no Lebesgue decomposition with respect to m.

Solution to (a). Clearly if $\mu(E) = 0$ we have $E = \varnothing$ and hence $m(E) = 0$. Assume that for some $f \in L^+([0,1])$ we have $dm = f d\mu$. For any $x \in [0,1]$ we have

$$0 = m(\{x\}) = \int_{\{x\}} f d\mu = f(x),$$

thus $f \equiv 0$ in $[0,1]$, which implies that $dm = f d\mu = 0$ in $[0,1]$, which is a contradiction.

Solution to (b). Assume that $d\mu = d\lambda + f dm$ for some $f \in L^+([0,1])$, with $\lambda \perp m$. Then $[0,1] = A \cup B$, $A, B \in \mathcal{M}$, $A \cap B = \varnothing$ with $m(A) = 0$ and $\lambda(B) = 0$.

But given $x \in [0,1]$ we have $1 = \mu(\{x\}) = \lambda(\{x\}) + \int_{\{x\}} f dm = \lambda(\{x\})$, hence $\lambda = \mu$. This implies that $\mu(B) = 0$ and hence $B = \varnothing$. But this gives $A = [0,1]$ and the fact that $m(A) = 0$ gives us a contradiction.

Hence μ has no Lebesgue decomposition with respect to m.

Exercise 14. If ν is an arbitrary signed measure and μ is a σ-finite positive measure on $(X, \mathcal{M})$ such that $\nu \ll \mu$, there exists and extended μ-integrable function $f \colon X \to [-\infty, \infty]$ such that $d\nu = f d\mu$. Hints:

(a) If suffices to assume that μ is finite and ν is positive.

(b) With these assumptions, there exists $E \in \mathcal{M}$ that is σ-finite for ν such that $\mu(E) \geqslant \mu(F)$ for all sets F that are σ-finite for ν.

(c) The Radon-Nikodym theorem applies on E. If $F \cap E = \varnothing$ then either $\nu(F) = \mu(F) = 0$ or $\mu(F) > 0$ and $\nu(F) = \infty$.

Solution to (a). First note that since $\nu \ll \mu$ then $\nu^+ \ll \mu$ and $\nu^- \ll \mu$, hence we can apply this result for ν^+, ν^- and subtract the result. Hence, we can assume that ν is positive.

Now since μ is σ-finite, we can write $X = \bigcup_{j=1}^{\infty} A_j$ with $\{A_j\}$ a disjoint sequence in $\mathcal{M}$ and $\mu(A_j) < \infty$ for all j. Define $\nu_j(E) = \nu(E \cap A_j)$ and $\mu_j(E) = \mu(E \cap A_j)$ for all $E \in \mathcal{M}$ and each $j \in \mathbb{N}$. Thus $\nu_j \ll \mu_j$ and μ_j is finite for each j, and if f_j is such $d\nu_j = f_j d\mu_j$, considering $f = \sum_{j=1}^{\infty} f_j$ we have $d\nu = f d\mu$.

Hence it suffices to assume μ finite and ν positive.

Solution to (b). Let $\alpha = \sup\{\mu(F) \colon F \text{ is } \sigma\text{-finite for } \nu\}$. Since $\varnothing$ is σ-finite for ν and ν is finite, α is well defined and $0 \leqslant \alpha < \infty$. By definition of α, there exists a sequence $\{F_n\}$ of σ-finite sets for ν such that

$$\alpha - \frac{1}{n} < \mu(F_n) \leqslant \alpha.$$

Take $E = \bigcup_{n=1}^{\infty} F_n$. Then E is σ-finite for ν, since it is a countable union of σ-finite sets for ν and $\mu(E) \geqslant \mu(F_n) > \alpha - \frac{1}{n}$ for all $n \in \mathbb{N}$, hence $\mu(E) = \alpha$, and concludes the proof.

Solution to (c). Applying the Radon-Nikodym Theorem on E, there exists a extended μ-integrable function $g \colon E \to [0, \infty]$ such that $d\nu = g d\mu$ on E.

Now let $F \in E^c$. Then either $\mu(F) = 0$, which implies that $\nu(F) = 0$ since $\nu \ll \mu$, or $\mu(F) > 0$. In the latter case, $\nu(F) = \infty$, since F cannot be σ-finite for ν, for otherwise $\mu(E \cup F) = \mu(E) + \mu(F) = \alpha + \mu(F) > \alpha$, which contradicts the construction of E, since $E \cup F$ is σ-finite for ν.

Thus we define $f \colon X \to [0, \infty]$ by setting $f(x) = g(x)$ for all $x \in E$ and $f(x) = \infty$ for all $x \in E^c$. If $G \in \mathcal{M}$ we have

- if $\mu(G \cap E^c) = 0$:

$$\nu(G) = \nu(G \cap E) + \underbrace{\nu(G \cap E^c)}_{=0} = \nu(G \cap E) = \int_{G \cap E} g d\mu = \int_{G \cap E} f d\mu$$
$$= \int_{G \cap E} f d\mu + \underbrace{\int_{G \cap E^c} f d\mu}_{=0} = \int_G f d\mu.$$

- if $\mu(G \cap E^c) > 0$:

$$\nu(G) = \infty = \nu(G \cap E) + \underbrace{\nu(G \cap E^c)}_{=\infty} = \int_{G \cap E} f d\mu + \underbrace{\int_{G \cap E^c} f d\mu}_{=\infty} = \int_G f d\mu.$$

Thus $d\nu = f d\mu$ on $\mathcal{M}$ and the result is proven.

Exercise 15. A measure μ on $(X, \mathcal{M})$ is called **decomposable** if there is a family $\mathcal{F} \subset \mathcal{M}$ with the following properties:

(i) $\mu(F) < \infty$ for all $F \in \mathcal{F}$;

(ii) the members of $\mathcal{F}$ are disjoint and their union is X;

(iii) if $\mu(E) < \infty$ then $\mu(E) = \sum_{F \in \mathcal{F}} \mu(E \cap F)$;

(iv) if $E \subset X$ and $E \cap F \in \mathcal{M}$ for all $F \in \mathcal{F}$ then $E \in \mathcal{M}$.

(a) Every σ-finite measure is decomposable.

(b) If μ is decomposable and ν is any signed measure on $(X, \mathcal{M})$ such that $\nu \ll \mu$ there exists a measurable $f \colon X \to [-\infty, \infty]$ such that $\nu(E) = \int_E f d\mu$ for any E that is σ-finite for μ, and $|f| < \infty$ on any $F \in \mathcal{F}$ that is σ-finite for ν (Use Exercise 14 if ν is not σ-finite).

Solution to (a). Let $X = \bigcup_{j=1}^{\infty} A_j$ where $\{A_j\}$ is a disjoint sequence in $\mathcal{M}$ with $\mu(A_j) < \infty$ for all $j \in \mathbb{N}$. Then the family $\mathcal{F} = \{A_j \colon j \in \mathbb{N}\}$ satisfies (i)-(iv).

Solution to (b). Working with ν^+ and ν^-, it suffices to assume that ν is a positive measure. Since μ is decomposable, let $\mathcal{F}$ be a family satisfying (i)-(iv). Defining $\nu_F(E) = \nu(E \cap F)$ and $\mu_F(E) = \mu(E \cap F)$, we have $\nu_F \ll \mu_F$ and μ_F is finite. Hence we can apply Exercise 14 to ν_F and μ_F, and obtain an extended μ_F-integrable function $g_F \colon X \to [0, \infty]$ such that $d\nu_F = g_F d\mu_F$. Since $\mu_F(G) = 0$ for all $G \in \mathcal{M}$ with $G \subset F^c$, we can assume that $g_F = 0$ on F^c. If F is σ-finite for ν, then the function g_F can be obtained use the Radon-Nikodym Theorem, and it ensures us that we can take $g_F < \infty$ in F.

Define $f \colon X \to [0, \infty]$ by $f = \sum_{F \in \mathcal{F}} g_F$. We have $f|_F = g_F$ for all $F \in \mathcal{F}$, and hence for $a \geqslant 0$ and $F \in \mathcal{F}$:

$$f^{-1}([a, \infty]) \cap F = (g_F)^{-1}([a, \infty]) \in \mathcal{M},$$

and using property (iv) we obtain $f^{-1}([a, \infty]) \in \mathcal{M}$, which proves that f is measurable. Also $f < \infty$ in any $F \in \mathcal{F}$ that is σ-finite for ν.

Let $E \in \mathcal{M}$ is σ-finite for μ. For each $F \in \mathcal{F}$ we have

$$\nu(E \cap F) = \nu_F(E) = \int_E g_F d\mu_F = \int_{E \cap F} f d\mu,$$

since $\mu_F = 0$ on F^c.

We claim that $\mu(E \cap F) = 0$ for all $F \in \mathcal{F}$ except possibly for a countable amount of F's. In fact, since E is σ-finite for μ, let $E = \bigcup_{j=1}^{\infty} E_j$ with $\mu(E_j) < \infty$ and $\{E_j\}$ a disjoint sequence in $\mathcal{M}$. From (iii), $\mu(E_j) = \sum_{F \in \mathcal{F}} \mu(E_j \cap F) < \infty$, and hence the collection $\mathcal{F}_j = \{F \in \mathcal{F} \colon \mu(E_j \cap F) > 0\}$ is countable. Therefore $\mu(E \cap F) = 0$ for all $F \notin \tilde{\mathcal{F}} := \bigcup_{j=1}^{\infty} \mathcal{F}_j$.

Thus $\nu(E \cap F) = 0$ for all $F \notin \tilde{\mathcal{F}}$ and hence

$$\begin{aligned}
\nu(E) &= \sum_{F \in \tilde{\mathcal{F}}} \nu(E \cap F) = \sum_{F \in \tilde{\mathcal{F}}} \int_{E \cap F} f d\mu \\
&= \int_{E \cap (\cup_{F \in \tilde{\mathcal{F}}} F)} f d\mu + \underbrace{\int_{E \cap (\cup_{F \notin \tilde{\mathcal{F}}} F)} f d\mu}_{=0} = \int_E f d\mu.
\end{aligned}$$

Exercise 16. Suppose that μ, ν are positive σ-finite measures on $(X, \mathcal{M})$ with $\nu \ll \mu$ and let $\lambda = \mu + \nu$. If $f = \dfrac{d\nu}{d\lambda}$ then $0 \leqslant f < 1$ μ-a.e. and $\dfrac{d\nu}{d\mu} = \dfrac{f}{1-f}$.

Solution. We have $\mu \ll \lambda$, $\nu \ll \lambda$ and $\lambda \ll \mu$. Hence all the Radon-Nikodym derivatives $\frac{d\mu}{d\lambda}$, $\frac{d\nu}{d\lambda}$ and $\frac{d\lambda}{d\mu}$ are defined. Moreover we have

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda}\frac{d\lambda}{d\mu} = f\frac{d\lambda}{d\mu} \quad \text{and} \quad \frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu} = 1 \ \mu\text{-a.e.}$$

Since the measures are σ-finite, we can assume that they are finite (decompose X into a countable disjoint union of μ and ν finite measure sets). Thus, if $F = \{x \in X \colon f(x) \geqslant 1\}$ is such that $0 < \mu(F) < \infty$ we have

$$\lambda(F) = \mu(F) + \nu(F) = \mu(F) + \int_F f d\lambda \geqslant \mu(F) + \lambda(F),$$

and we obtain a contradiction, hence $0 \leqslant f < 1$ μ-a.e.

Finally note that for $E \in \mathcal{M}$ we have

$$\mu(E) = \lambda(E) - \int_E f d\lambda = \int_E (1-f) d\lambda,$$

and thus $\dfrac{d\mu}{d\lambda} = 1 - f$, which shows that $\dfrac{d\lambda}{d\mu} = \frac{1}{1-f}$, and hence

$$\frac{d\nu}{d\mu} = \frac{f}{1-f}.$$

Exercise 17. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $\mathcal{N}$ a sub-algebra of $\mathcal{M}$, and $\nu = \mu|_{\mathcal{N}}$. If $f \in L^1(\mu)$, there exists $g \in L^1(\nu)$ (thus g is $\mathcal{N}$-measurable) such that $\int_E f d\mu = \int_E g d\nu$ for all $E \in \mathcal{N}$; if $\tilde{g}$ is another such function then $g = \tilde{g}$ ν-a.e. (in probability theory, g is called **conditional expectation** of f on $\mathcal{N}$).

Solution. Define λ on $\mathcal{N}$ by $d\lambda = f d\nu$. Since $\nu = \mu|_{\mathcal{N}}$, we also have $d\lambda = f d\mu$ on $\mathcal{N}$. Since $f \in L^1(\mu)$, λ is finite. Furthermore, if $E \in \mathcal{N}$ is such $\nu(E) = 0$ then $\lambda(E) = 0$, that is, $\lambda \ll \nu$. We can thus use the Radon-Nikodym Theorem to obtain a unique (ν-a.e.) $\mathcal{N}$-measurable function $g \colon X \to [0, \infty]$ such that

$$d\lambda = g d\nu.$$

Since λ is finite, $g \in L^1(\nu)$. Also for $E \in \mathcal{N}$ we have

$$\int_E f d\mu = \lambda(E) = \int_E g d\nu.$$

If $\tilde{g}$ is such a function, then $\int_E \tilde{g} d\nu = \int_E f d\mu = \lambda(E)$ and hence $d\lambda = \tilde{g} d\nu$, and thus $\tilde{g} = g$ ν-a.e. from the uniqueness of the Radon-Nidodym derivative.

4.5 | Complex measures

Definition 4.5.1. *A* **complex measure** *on a measurable space* $(X, \mathcal{M})$ *is a map* $\nu \colon \mathcal{M} \to \mathbb{C}$ *such that:*

(a) $\nu(\varnothing) = 0$;

(b) *if* $\{E_j\} \subset \mathcal{M}$ *is a disjoint sequence, then*

$$\nu\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \sum_{j=1}^{\infty} \nu(E_j),$$

where the series converges absolutely.

In particular, a complex measure does not assume infinite values. Hence a positive measure is a complex measure only if it is finite. A simple example of complex measure is $d\nu = f d\mu$, where μ is a positive measure and $f \in L^1(\mu)$.

Given a complex measure ν, we will write ν_r and ν_i for the real and imaginary parts of ν. Thus ν_r and ν_i are signed measures that do not assume both the values $\pm\infty$. Thus both ν_r and ν_i are finite, and therefore the range of ν is a bounded subset of $\mathbb{C}$.

All the notions we have developed so far for signed measures have simple generalizations to the case of complex measures, namely:

Definition 4.5.2. *For a complex measure ν we define:*

(a) $L^1(\nu) = L^1(\nu_r) \cap L^1(\nu_i)$ *and for* $f \in L^1(\nu)$ *we define*

$$\int f d\nu = \int f d\nu_r + i \int f d\nu_i.$$

(b) *If μ and ν are complex measures we say that $\nu \perp \mu$ if $\nu_r \perp \mu_r$, $\nu_i \perp \mu_r$, $\nu_r \perp \mu_i$ and $\nu_i \perp \mu_i$.*

(c) *If ν is a complex measure and λ is a positive measure, we say that $\nu \ll \lambda$ if $\nu_r \ll \lambda$ and $\nu_i \ll \lambda$.*

Also, all the results of the previous sections can be generalized to complex measure, one only has to apply each one of them to the real and imaginary parts separately. In particular we have:

Theorem 4.5.3 (The Lebesgue-Radon-Nikodym Theorem for Complex Measures)**.** *If ν is a complex measure and μ is a σ-finite positive measure on $(X, \mathcal{M})$, there exist a complex measure λ and an $f \in L^1(\mu)$ such that $\lambda \perp \mu$ and $d\nu = d\lambda + f d\mu$. If also $\tilde{\lambda} \perp \mu$ and $d\nu = d\tilde{\lambda} + \tilde{f} d\mu$, then $\lambda = \tilde{\lambda}$ and $f = \tilde{f}$ μ-a.e.*

As before, if $\nu \ll \mu$, we denote f of the previous theorem as $\frac{d\nu}{d\mu}$.

The **total variation** of a complex measure ν is the positive measure $|\nu|$ determined by the property that if $d\nu = f d\mu$, where μ is a positive measure, then $d|\nu| = |f| d\mu$.

First of all, we need to see that this is well defined. First, given a complex measure ν, defining $\mu = |\nu_r| + |\nu_i|$, we have $\nu \ll \mu$, and Theorem 4.5.3 gives us $f \in L^1(\mu)$ such that $d\nu = f d\mu$. Thus such positive measure μ and $f \in L^1(\mu)$ always exist.

Now assume that $d\nu = f_1 d\mu_1 = f_2 d\mu_2$. Let $\rho = \mu_1 + \mu_2$. Thus $\mu_1 \ll \rho$ and $\mu_2 \ll \rho$, and by Proposition 4.3.8 we have

$$f_1 \frac{d\mu_1}{d\rho} d\rho = d\nu = f_2 \frac{d\mu_2}{d\rho} d\rho,$$

so that $f_1\frac{d\mu_1}{d\rho} = f_2\frac{d\mu_2}{d\rho}$ ρ-a.e. Since $\frac{d\mu_1}{d\rho}$ and $\frac{d\mu_1}{d\rho}$ are nonnegative, we have

$$|f_1|\frac{d\mu_1}{d\rho} = \left|f_1\frac{d\mu_1}{d\rho}\right| = \left|f_2\frac{d\mu_2}{d\rho}\right| = |f_2|\frac{d\mu_2}{d\rho},$$

and thus

$$|f_1|d\mu_1 = |f_1|\frac{d\mu_1}{d\rho}d\rho = |f_2|\frac{d\mu_2}{d\rho}d\rho = |f_2|d\mu_2.$$

Hence the definition of $|\nu|$ is independent of the choice of μ and f. Thus definition also agrees with the previous definition of ν when ν is a signed measure, since in this case $d\nu = (\chi_P - \chi_N)d|\nu|$ where $X = P \cup N$ is a Hahn decomposition for ν and $|\chi_P - \chi_N| = 1$.

Proposition 4.5.4. *Let ν be a complex measure on $(X, \mathcal{M})$. We have*

(a) $|\nu(E)| \leqslant |\nu|(E)$ *for all* $E \in \mathcal{M}$.

(b) $\nu \ll |\nu|$ *and*

$$\left|\frac{d\nu}{d|\nu|}\right| = 1 \quad |\nu| - a.e.$$

(c) $L^1(\nu) = L^1(|\nu|)$ *and if* $f \in L^1(\nu)$ *then*

$$\left|\int f d\nu\right| \leqslant \int |f| d|\nu|.$$

Proof. Let $d\nu = f d\mu$ as in the definition of $|\nu|$. Then for $E \in \mathcal{M}$ we have

$$|\nu(E)| = \left|\int_E f d\mu\right| \leqslant \int_E |f| d\mu = |\nu|(E),$$

which proves (a). From (a) it follows directly that $\nu \ll |\nu|$. Since

$$f d\mu = d\nu = \frac{d\nu}{d|\nu|}d|\nu| = \frac{d\nu}{d|\nu|}f d\mu,$$

and hence $f\frac{d\nu}{d|\nu|} = f$ $|\nu|$-a.e. But either $|\nu| = 0$ (which implies that $\nu = 0$) or $|f| > 0$ $|\nu|$-a.e., hence $\frac{d\nu}{d|\nu|} = 1$ $|\nu|$-a.e., which proves (b).

The proof of (c) is left as an exercise (see Exercise 18). ■

Proposition 4.5.5. *If ν_1, ν_2 are complex measures on $(X, \mathcal{M})$, then $|\nu_1 + \nu_2| \leqslant |\nu_1| + |\nu_2|$.*

Proof. Using item (b) of the previous proposition, we have $\nu_1 \ll |\nu_1|$ and $\nu_2 \ll |\nu_2|$. Then for $\mu = |\nu_1| + |\nu_2|$ we have $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, and we can write $d\nu_1 = f_1 d\mu$ and $d\nu_2 = f_2 d\mu$.

Then $d(\nu_1+\nu_2)=(f_1+f_2)d\mu$ and

$$d|\nu_1+\nu_2| = |f_1+f_2|d\mu \leqslant (\ |f_1|+|f_2|\ )d\mu = d|\nu_1|+d|\nu_2|,$$

and concludes the result. ■

4.6| Solved exercises from [1, Page 94]

Exercise 18. Prove Proposition 4.5.4, item (c).

Solução: First, since $\nu \ll |\nu|$, we have by definition $\nu_r \ll |\nu|$ and $\nu_i \ll |\nu|$, and we can consider $d\nu_r = f_r d|\nu|$ and $d\nu_i = f_i d|\nu|$. Hence $d\nu = f d|\nu|$ where $f = f_r + i f_i$, and $|f| = 1$ $|\nu|$-a.e. since $d|\nu| = |f|d|\nu|$.

Using Proposition 4.3.8, we have for $g \in L^1(\nu) = L^1(\nu_r)\cap L^1(\nu_i)$ that $gf_r \in L^1(|\nu|)$, $gf_i \in L^1(|\nu|)$, and

$$\int g d\nu = \int g d\nu_r + i\int g d\nu_i = \int g f_r d|\nu| + i\int g f_i d|\nu| = \int g f d|\nu|. \tag{$\star$}$$

Hence $|g| = |gf| = |gf_r + igf_i| \leqslant |gf_r| + |gf_i| \in L^1(|\nu|)$, thus $g \in L^1(|\nu|)$.

For the other inclusion, if $g \in L^1(|\nu|)$ then since $|f_r|, |f_i| \leqslant 1$ $|\nu|$-a.e. we have $g|f_r|, g|f_i| \in L^1(|\nu|)$ and since $d|\nu_r| = |f_r|d|\nu|$, $d|\nu_i| = |f_i|d|\nu|$, we have $g \in L^1(|\nu_r|)$ and $g \in L^1(|\nu_i|)$. From Exercise 3(a), this implies that $g \in L^1(\nu_r)\cap L^1(\nu_i) = L^1(\nu)$.

From ($\star$) we have

$$\left|\int g d\nu\right| = \left|\int g f d|\nu|\right| \leqslant \int |gf| d|\nu| = \int |g| d|\nu|,$$

since $|f| = 1$ $|\nu|$-a.e.

Exercise 19. If μ, ν are complex measures and λ is a positive measure, then $\nu \perp \mu$ iff $|\nu| \perp |\mu|$ and $\nu \ll \lambda$ iff $|\nu| \ll \lambda$.

Solution. From the definition of $\nu \perp \mu$ we have:

(1) $\nu_r \perp \mu_r$ and $X = E_1 \cup F_1$, $E_1 \cap F_1 = \varnothing$, E_1 μ_r-null and F_1 ν_r-null.

(2) $\nu_r \perp \mu_i$ and $X = E_2 \cup F_2$, $E_2 \cap F_2 = \varnothing$, E_2 μ_i-null and F_2 ν_r-null.

(3) $\nu_i \perp \mu_r$ and $X = E_3 \cup F_3$, $E_3 \cap F_3 = \varnothing$, E_3 μ_r-null and F_3 ν_i-null.

(4) $\nu_i \perp \mu_i$ and $X = E_4 \cup F_4$, $E_4 \cap F_4 = \varnothing$, E_4 μ_i-null and F_4 ν_i-null.

We write $d\nu_r = f_r d|\nu|$, $d\nu_i = f_i d|\nu|$, $d\mu_r = g_r d|\mu|$ and $d\mu_i = g_i d|\mu|$. From (1), (2), (3) and (4) we have $f_r = 0$ $|\nu|$-a.e. on $F_1 \cup F_2$, $f_i = 0$ $|\nu|$-a.e. on $F_3 \cup F_4$, $g_r = 0$ $|\mu|$-a.e. on $E_1 \cup E_3$ and $g_i = 0$ $|\mu|$-a.e. on $E_2 \cup E_4$.

Defining $f = f_r + if_i$ and $g = g_r + ig_i$ we have $d\nu = fd|\nu|$ and $d\mu = gd|\nu|$, hence $f = 0$ $|\nu|$-a.e on $\tilde{F} = (F_1 \cup F_2) \cap (F_3 \cup F_4)$, $|f| = 1$ $|\nu|$-a.e. on $X \setminus \tilde{F}$, $g = 0$ $|\mu|$-a.e. on $E := (E_1 \cup E_3) \cap (E_2 \cup E_4)$ and $|g| = 1$ $|\mu|$-a.e. on $F := X \setminus E \subset \tilde{F}$. Hence $|\nu|(F) = 0$, $|\mu|(E) = 0$, $X = E \cup F$ and $E \cap F = \varnothing$, hence $|\nu| \perp |\mu|$.

For the converse, let $X = E \cup F$ with $E \cap F = \varnothing$, $|\nu|(F) = 0$ and $|\mu|(E) = 0$. Let also $d\nu = fd|\nu|$ and $d\mu = gd|\mu|$. We have $f = 0$ $|\nu|$-a.e on F and $|f| = 1$ $|\nu|$-a.e. on E, and $g = 0$ $|\mu|$-a.e. on E and $|g| = 1$ $|\mu|$-a.e. on F. If $f_r = \mathrm{Re} f$, $f_i = \mathrm{Im} f$, $g_r = \mathrm{Re} g$ and $g_i = \mathrm{Im} g$, then $d\nu_r = f_r d|\nu|$, $d\nu_i = f_i d|\nu|$, $d\mu_r = g_r d|\mu|$ and $d\mu_i = g_i d|\mu|$. Also E is μ_r, μ_i-null and F is ν_r, ν_i null, and $\nu_s \perp \mu_q$ for $s, q = r, i$, thus $\nu \perp \mu$.

Now if $\nu \ll \lambda$ we have $\nu_r \ll \lambda$ and $\nu_i \ll \lambda$. Then if $\rho = |\nu_r| + |\nu_i|$, $d\nu_r = f_r d\rho$ and $d\nu_i = f_i d\rho$, for $E \in \mathcal{M}$ is such that $\lambda(E) = 0$ we obtain E ν_r, ν_i null and thus $f_r = f_i = 0$ ρ-a.e. in E. Thus

$$|\nu|(E) = \int_E |f_r + if_i| d\rho = 0,$$

and thus $|\nu| \ll \lambda$. The converse follows from $|\nu(E)| \leqslant |\nu|(E)$, since if $\lambda(E) = 0$ we have $|\nu|(E) = 0$, which implies that $|\nu(E)| = 0$, and hence $\nu_r(E) = \nu_i(E) = 0$, that is, $\nu_r \ll \lambda$ and $\nu_i \ll \lambda$, hence $\nu \ll \lambda$.

EXERCISE 20. If ν is a complex measure on $(X, \mathcal{M})$ and $\nu(X) = |\nu|(X)$ then $\nu = |\nu|$.

Solution. For $E \in \mathcal{M}$ we have

$$\nu(E) + \nu(E^c) = \nu(X) = |\nu|(X) = |\nu|(E) + |\nu|(E^c),$$

and thus

$$\nu(E^c) - |\nu|(E^c) = |\nu|(E) - \nu(E).$$

Since $\nu_r(E^c) \leqslant |\nu_r(E^c)| \leqslant |\nu(E^c)| \leqslant |\nu|(E^c)|$ we have

$$\mathrm{Re}(\nu(E^c) - |\nu|(E^c)) = \nu_r(E^c) - |\nu|(E^c) \leqslant 0 \text{ and } \mathrm{Re}(|\nu|(E) - \nu(E)) = |\nu|(E) - \nu_r(E) \geqslant 0.$$

Hence $\mathrm{Re}(|\nu|(E) - \nu(E)) = |\nu|(E) - \nu_r(E) = 0$, which implies that $|\nu|(E) = \nu_r(E)$, for each $E \in \mathcal{M}$, and in particular $\nu_r(E) \geqslant 0$ for all $E \in \mathcal{M}$. Now we have

$$\nu_r(E)^2 = |\nu|(E)^2 \geqslant |\nu(E)|^2 = \nu_r(E)^2 + \nu_i(E)^2,$$

and hence $\nu_i(E) = 0$ for all $E \in \mathcal{M}$. Therefore

$$|\nu|(E) = \nu_r(E) = \nu_r(E) + i\nu_i(E) = \nu(E),$$

for all $E \in \mathcal{M}$, that is, $|\nu| = \nu$.

Exercise 21. Let ν be a complex measure on $(X, \mathcal{M})$. If $E \in \mathcal{M}$, define

$$\mu_1(E) = \sup\left\{\sum_{j=1}^{n} |\nu(E_j)| \colon n \in \mathbb{N},\ E_1, \cdots, E_n \text{ disjoint},\ E = \bigcup_{j=1}^{n} E_j\right\},$$

$$\mu_2(E) = \sup\left\{\sum_{j=1}^{\infty} |\nu(E_j)| \colon E_1,\ E_2, \cdots \text{ disjoint},\ E = \bigcup_{j=1}^{\infty} E_j\right\},$$

$$\mu_3(E) = \sup\left\{\left|\int_E f d\nu\right| : |f| \leqslant 1\right\}.$$

Then $\mu_1 = \mu_2 = \mu_3 = |\nu|$ (First show that $\mu_1 \leqslant \mu_2 \leqslant \mu_3$. To see that $\mu_3 = |\nu|$, let $f = \overline{d\nu/d|\nu|}$ and apply Proposition 4.5.4. To see that $\mu_3 \leqslant \mu_1$, approximate f by simple functions).

Solution: Note that if $\{E_j\}_{j=1}^n$ is a disjoint sequence with $E = \bigcup_{j=1}^n E_j$, defining $E_j = \varnothing$ for $j > n$, $\{E_j\}$ is a disjoint sequence, $E = \bigcup_{j=1}^\infty E_j$ and

$$\sum_{j=1}^{n} |\nu(E_j)| = \sum_{j=1}^{\infty} |\nu(E_j)| \leqslant \mu_2(E),$$

taking the supremum on the left hand side of this inequality we have $\mu_1(E) \leqslant \mu_2(E)$.

Now if $\{E_j\}$ is a disjoint sequence with $E = \bigcup_{j=1}^\infty E_j$, define $g = \sum_{j=1}^\infty \overline{\text{sgn } \nu(E_j)} \chi_{E_j}$. Then g is measurable and $|g| \leqslant 1$, moreover

$$\sum_{j=1}^{\infty} |\nu(E_j)| = \sum_{j=1}^{\infty} \overline{\text{sgn } \nu(E_j)} \nu(E_j) = \sum_{j=1}^{\infty} \int \overline{\text{sgn } \nu(E_j)} \chi_{E_j} d\nu = \int_E g d\nu = \left|\int_E g d\nu\right| \leqslant \mu_3(E),$$

and taking the supremum on the left hand side of this inequality we have $\mu_2(E) \leqslant \mu_3(E)$.

Now clearly for $|f| \leqslant 1$ we have

$$\left|\int_E f d\nu\right| \leqslant \int_E |f| d|\nu| \leqslant |\nu|(E),$$

and taking the supremum on the left hand side of this inequality we have $\mu_3(E) \leqslant |\nu|(E)$. Taking $f = \overline{d\nu/d|\nu|}$ we know that $|f| = 1$ $|\nu|$-a.e., and we can redefine f in a $|\nu|$-null set, if necessary, so that $|f| = 1$ everywhere. Thus using Proposition 4.5.4 we have

$$|\nu|(E) = \int_E d|\nu| = \int_E |f|^2 d|\nu| = \int_E f\overline{f} d|\nu| = \int_E f d\nu \overset{(\star)}{=} \left|\int_E f d\nu\right| \leqslant \mu_3(E),$$

where in $(\star)$ we used the fact that the left hand side is equal to $|\nu|(E)$ (first equality of the equation) and hence is it a positive real number.

Now to show that $\mu_3 \leqslant \mu_1$, fix f with $|f| \leqslant 1$ and $E \in \mathcal{M}$. We know that there exists a sequence of simple measurable complex functions $\{s_n\}$ such that $s_n \to f$ uniformly on X (since f is bounded), and $0 \leqslant |s_1| \leqslant |s_2| \leqslant \cdots \leqslant |f|$ (see Proposition 3.1.22). Hence, given ϵ there exists n_0 such that

$$\sup_{x \in X} |s_{n_0}(x) - f(x)| < \frac{\epsilon}{|\nu|(X)},$$

and we recall that $|\nu|(X) < \infty$, since ν is a complex measure. We have $s_{n_0} = \sum_{j=1}^n a_j \chi_{A_j}$, and $s_{n_0}\chi_E = \sum_{j=1}^n a_j \chi_{E_j}$, where $E_j = A_j \cap E$, with $\{A_j\}_{j=1}^n$ disjoint and $X = \bigcup_{j=1}^n A_j$, therefore $\{E_j\}_{j=1}^n$ is disjoint and $E = \bigcup_{j=1}^n E_j$. Also, since $|s_{n_0}| \leqslant |f| \leqslant 1$, we have $|a_j| \leqslant 1$ for all $j = 1, \cdots, n$. Hence

$$\begin{aligned}
\left|\int_E f d\nu\right| &\leqslant \left|\int_E (f - s_{n_0}) d\nu\right| + \left|\int_E s_{n_0} d\nu\right| \\
&\leqslant \epsilon + \left|\int s_{n_0}\chi_E d\nu\right| = \epsilon + \left|\sum_{j=1}^n a_j \nu(E_j)\right| \\
&\leqslant \epsilon + \sum_{j=1}^n |\nu(E_j)| \leqslant \epsilon + \mu_1(E).
\end{aligned}$$

Taking the supremum for $|f| \leqslant 1$ on the left hand side we have $\mu_3(E) \leqslant \epsilon + \mu_1(E)$, and since this holds for each $\epsilon > 0$, we have $\mu_3(E) \leqslant \mu_1(E)$, and we complete the proof.

CHAPTER 5

L^P SPACES

The theory of L^p spaces comes to generalize the idea of L^1 functions, are they have extreme importance in the study of differential equations.

5.1 | Basic theory of L^p spaces

We will fix now a measure space $(X, \mathcal{M}, \mu)$, where μ is a positive measure. Unless clearly stated otherwise, we will be speaking of this fixed space.

Definition 5.1.1. *Let $f\colon X \to \mathbb{C}$ be a measurable function on X and $0 < p < \infty$. We define*

$$\|f\|_p = \left[\int |f|^p d\mu\right]^{1/p},$$

(where $\|f\|_p$ could be ∞), and also

$$L^p(X, \mathcal{M}, \mu) = \{f\colon X \to \mathbb{C}\colon f \text{ is measurable and } \|f\|_p < \infty\}.$$

We will abbreviate $L^p(X, \mathcal{M}, \mu)$ by $L^p(\mu)$ (when X and $\mathcal{M}$ are understood) or $L^p(X)$ (when $\mathcal{M}$ and μ are understood), or simply by L^p (when $(X, \mathcal{M}, \mu)$ is understood). As well as for L^1, we will see L^p as a space of classes of functions that are equal μ-almost everywhere, and we will use the notation $f \in L^p$ to mean that f is equal μ-a.e. to a function in L^p.

If A is a nonempty set, we define $\ell^p(A)$ to be $L^p(A, \mathcal{P}(A), \mu)$, where μ is the counting measure. We will denote $\ell^p(\mathbb{N})$ simply by ℓ^p.

It is clear that L^p is a complex vector space (or real, if we consider only real-valued functions), since for $f, g \in L^p$ we have

$$|f+g|^p \leqslant (2\max\{|f|,|g|\})^p \leqslant 2^p(|f|^p+|g|^p),$$

so that $f+g \in L^p$.

We clearly have $\|f\|_p = 0$ iff $f = 0$ μ-a.e. and $\|cf\|_p = |c|\|f\|_p$. Hence, to verify that $\|\cdot\|_p$ is in fact a norm in L^p, it remains to show the triangle inequality.

Before proving the triangle inequality, we will see that it fails for $0 < p < 1$.

Example 5.1.2. *Suppose $a, b > 0$ and $0 < p < 1$. For $t > 0$ we have $t^{p-1} > (a+t)^{p-1}$ and integrating both sides with respect to t, from 0 to b we have*

$$a^p + b^p > (a+b)^p. \tag{5.1.1}$$

Thus if E, F are disjoint sets of positive finite measure on X, setting $a = \mu(E)^{1/p}$, $b = \mu(F)^{1/p}$, $f = \chi_E$ and $g = \chi_F$ we have $|f+g|^p = f+g$ and

$$\begin{aligned}\|f+g\|_p = \left[\int |f+g|^p\right]^{1/p} = \left[\int (f+g)\right]^{1/p} = (\mu(E)+\mu(F))^{1/p}\\ = (a^p+b^p)^{1/p} > a+b = \|f\|_p + \|g\|_p\end{aligned}$$

Now we want to prove that the triangle inequality is in fact true for $p \geqslant 1$. To do that, we first need to prove the most important inequality for L^p spaces, the **Hölder inequality**. We will prove this inequality after proving the following simple lemma.

Lemma 5.1.3. *If $a, b \geqslant 0$ and $0 < \lambda < 1$ then*

$$a^\lambda b^{1-\lambda} \leqslant \lambda a + (1-\lambda)b,$$

with equality iff $a = b$.

Proof. The result is trivial for $b = 0$. For $b > 0$, diving both sides by b we have

$$\left(\frac{a}{b}\right)^\lambda \leqslant \lambda\frac{a}{b} + (1-\lambda),$$

and setting $t = a/b$, all we have to show is that $t^\lambda \leqslant \lambda t + (1-\lambda)$ for all $t > 0$ with equality iff $t = 1$.

To that, define $h(0,\infty) \to \mathbb{R}$ by $h(t) = t^\lambda - \lambda t$. We have $h'(t) = \lambda t^{\lambda-1} - \lambda$, hence $h'(t) = 0$ iff $t = 1$, and $h'(t) > 0$ for $0 < t < 1$ and $h'(t) < 0$ for $t > 1$. Therefore h is strictly increasing for $0 < t < 1$ and strictly decreasing for $t > 1$, attaining its maximum value at $t = 1$, namely $1 - \lambda$, and the result is proven. ■

THEOREM 5.1.4 (Hölder Inequality). *Suppose $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ (that is, $q = \frac{p}{p-1}$). If f and g are measurable functions on X, then*

$$\|fg\|_1 \leqslant \|f\|_p \|g\|_q. \tag{5.1.2}$$

In particular, if $f \in L^p$ and $g \in L^q$, then $fg \in L^1$, and in this case equality holds in (5.1.2) iff $\alpha|f|^p = \beta|g|^q$ μ-a.e. for some constants $\alpha, \beta \geqslant 0$ not both null.

Proof. If $\|f\|_p = 0$ or $\|g\|_q = 0$ then $f = 0$ a.e. or $g = 0$ a.e., and hence $fg = 0$ a.e., and the inequality is trivial. If $\|f\|_p = \infty$ or $\|g\|_q = \infty$, then the result is also trivial.

Now we note that if the result is true for f and g, and $a, b \in \mathbb{C}$, then the result is also true for af and bg, since both sides of the inequality is scaled by $|ab|$.

Thus, it suffices to prove the result for $\|f\|_p = \|g\|_q = 1$ and the equality holds iff $|f|^p = |g|^q$ a.e. In this case, we will apply Lemma 5.1.3 with $a = |f(x)|^p$, $b = |g(x)|^q$ and $\lambda = p^{-1}$ to obtain

$$|f(x)g(x)| \leqslant \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q, \tag{5.1.3}$$

and integrating both sides we obtain

$$\|fg\|_1 \leqslant \frac{1}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q = \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_p \|g\|_q.$$

Equality holds iff it holds a.e. in (5.1.3), which is true precisely when $|f|^p = |g|^p$ a.e. ■

The condition $\frac{1}{p} + \frac{1}{q} = 1$ appears frequently in L^p theory. If $1 < p < \infty$, the number q such that $\frac{1}{p} + \frac{1}{q} = 1$ (that is, $q = \frac{p}{p-1}$) is called the **conjugate exponent** to p. Clearly $1 < q < \infty$.

THEOREM 5.1.5 (Minkowski's Inequality). *If $1 \leqslant p < \infty$ and $f, g \in L^p$, then*

$$\|f + g\|_p \leqslant \|f\|_p + \|g\|_p.$$

Proof. The result is obvious if $p = 1$ or if $f + g = 0$ a.e. Otherwise, since

$$|f + g|^p = |f + g||f + g|^{p-1} \leqslant (|f| + |g|)|f + g|^{p-1}, \tag{5.1.4}$$

using the Hölder inequality, and noting that $(p - 1)q = p$ when q is the conjugate exponent to p, we have

$$\begin{aligned}\int |f+g|^p &\leqslant \|f\|_p \||f+g|^{p-1}\|_q + \|g\|_p \||f+g|^{p-1}\|_q \\ &= (\|f\|_p + \|g\|_p)\left(\int |f+g|^p\right)^{1/q},\end{aligned} \tag{5.1.5}$$

and hence

$$\|f+g\|_p = \left(\int |f+g|^p\right)^{1-1/q} \leqslant \|f\|_p + \|g\|_p.$$

■

With this result, we see that $(L^p, \|\cdot\|_p)$ is a normed vector space. But more is true:

Theorem 5.1.6. *For $1 \leqslant p < \infty$, L^p is a Banach space, that is, it is a normed vector space which is complete with the metric defined by $d(f,g) = \|f-g\|_p$.*

Proof. We will show that every absolutely convergent series is convergent in L^p, that is, consider $\{f_k\} \subset L^p$ such that $\sum_{k=1}^\infty \|f_k\|_p = B < \infty$. Let

$$G(x) = \sum_{k=1}^\infty |f_k(x)| \quad \text{and} \quad G_n(x) = \sum_{k=1}^n |f_k(x)| \quad \text{for all } n \in \mathbb{N} \text{ and } x \in X.$$

We have $\|G_n\|_p \leqslant \sum_{k=1}^n \|f_k\|_p \leqslant B$ for all $n \in \mathbb{N}$. Moreover $G_n(x)$ increases to $G(x)$ for each $x \in X$, and hence $G_n^p(x)$ increases to $G^p(x)$ for each $x \in X$. Thus we can use the MCT to conclude that

$$\int G^p = \lim_{n\to\infty} \int G_n^p = \lim_{n\to\infty} \|G_n\|_p^p \leqslant B^p < \infty,$$

which shows us that $G \in L^p$, and in particular, $G(x) < \infty$ a.e., and we obtain that the series $\sum_{k=1}^\infty f_k(x)$ converges a.e. Defining

$$F(x) = \limsup_{n\to\infty} \sum_{k=1}^n f_k(x),$$

we have F measurable and $F(x) = \sum_{k=1}^\infty f_k(x) \in \mathbb{C}$ a.e. Moreover, we have $|F| \leqslant G$ and hence $F \in L^p$. Also $|F - \sum_{k=1}^n f_k| \leqslant (2G)^p \in L^1$, so by the DCT we obtain

$$\left\| F - \sum_{k=1}^n f_k \right\|_p^p = \int \left| F - \sum_{k=1}^n f_k \right|^p \to 0,$$

thus the series $\sum_{k=1}^\infty f_k$ converges in the L^p norm, and concludes the result. ■

Proposition 5.1.7. *For $1 \leqslant p < \infty$, the set of simple functons $s = \sum_{j=1}^n a_j \chi_{E_j}$, where $\mu(E_j) < \infty$ for all j, is dense in L^p.*

Proof. If $s = \sum_{j=1}^n a_j \chi_{E_j}$ with $\mu(E_j) < \infty$ for all j, with the E_j disjoint and $a_j \neq 0$, we have $|s|^p = \sum_{j=1}^n |a_j|^p \chi_{E_j}$ and hence

$$\int s_p = \sum_{j=1}^n |a_j|^p \mu(E_j) < \infty,$$

which shows that $s \in L^p$. Now if $f \in L^p$, f is measurable and we can choose a sequence of simples functions $\{s_n\}$ such that $|s_n| \leqslant |f|$ and $s_n \to f$ pointwise, using Proposition 3.1.22. Then $s_n \in L^p$, since $|s_n|^p \leqslant |f|^p$ and $|s_n - f|^p \leqslant 2^p |f|^p \in L^1$, and the DCT gives us $\|s_n - f\|_p \to 0$ as $n \to \infty$. Moreover if $f_n = \sum a_j \chi_{E_j}$ with $\{E_j\}$ disjoint and a_j nonzero, we must have $\mu(E_j) < \infty$ for all j, since $\sum |a_j|^p \mu(E_j) = \int |f_n|^p < \infty$, and the proof is complete. ■

To complete the definitions of L^p spaces, we introduce a space corresponding to the limiting value $p = \infty$. If f is a measurable function on X, for each $a \geqslant 0$ we define $\Lambda_a = \{x \in X \colon |f(x)| > a\}$, $J = \{a \geqslant 0 \colon \mu(\Lambda_a) = 0\}$ and also

$$\|f\|_\infty = \inf J.$$

with the convention that $\inf \varnothing = \infty$.

We observe that the infimum is actually attained: in fact, let $\alpha = \|f\|_\infty$. If $\alpha = \infty$, there is nothing to prove. Now assume that $\alpha < \infty$. There exists a sequence $a_n \in \Lambda$ with $\alpha \leqslant a_n < \alpha + n^{-1}$. We have then

$$\Lambda_\alpha = \bigcup_{n=1}^\infty \Lambda_{a_n},$$

and from the monotonicity of μ we obtain

$$\mu(\Lambda_\alpha) \leqslant \sum_{n=1}^\infty \mu(\Lambda_{a_n}) = 0,$$

hence $\alpha \in J$ and the infimum is attained.

The number $\|f\|_\infty$ is called the **essential supremum** of $|f|$ and it sometimes written

$$\|f\|_\infty = \operatorname{esssup}_{x \in X} |f(x)|.$$

Proposition 5.1.8. *For a measurable function $f \colon X \to \mathbb{C}$ there exists a set $E \in \mathcal{M}$ such that $\mu(E^c) = 0$ and $|f(x)| \leqslant \|f\|_\infty$ for all $x \in E$.*

Proof. Define $E = \{x \in X \colon |f(x)| \leqslant \|f\|_\infty\}$, which is in $\mathcal{M}$, since f is measurable. Now since the infimum $\|f\|_\infty$ is attained in its definition, we have

$$\mu(E^c) = \mu(\{x \in X \colon |f(x)| > \|f\|_\infty\}) = 0,$$

and concludes the proof. ∎

We now define

$$L^\infty = L^\infty(X, \mathcal{M}, \mu) = \{f \colon X \to \mathbb{C} \colon f \text{ is measurable and } \|f\|_\infty < \infty\},$$

with the usual convention that two functions that are equal a.e. (and hence have the same esssup) define the same element in L^∞.

Proposition 5.1.9. *$f \in L^\infty$ iff there is a bounded measurable function g such that $f = g$ a.e.*

Proof. Let $E \in \mathcal{M}$ be as in Proposition 5.1.8 and define $g = f\chi_E$. Thus g is measurable and $|g(x)| \leqslant \|f\|_\infty < \infty$ for all $x \in X$. ∎

Remark 5.1.10. *Note that once $(X, \mathcal{M})$ is fixed, $L^\infty(X, \mathcal{M}, \mu)$ depends on μ only as μ determines which sets have zero measure. If $\mu \ll \nu$ and $\nu \ll \mu$, then clearly $L^\infty(\mu) = L^\infty(\nu)$.*

When μ is not semifinite, it is appropriate to consider a slightly different definition of L^∞, and this will be explored in Exercises 23-25.

The results we have proved for L^p with $1 \leqslant p < \infty$ can be extended to the case of $p = \infty$, as follows:

Theorem 5.1.11. *We have*

(a) *If f, g are measurable functions on X, then $\|fg\|_1 \leqslant \|f\|_1 \|g\|_\infty$. If $f \in L^1$ and $g \in L^\infty$, then $\|fg\|_1 = \|f\|_1\|g\|_\infty$ iff $|g(x)| = \|g\|_\infty$ a.e. on the set where $f(x) \neq 0$.*

(b) *$\|\cdot\|_\infty$ is a norm on L^∞.*

(c) *$\|f_n - f\|_\infty \to 0$ iff there exists $E \in \mathcal{M}$ such that $\mu(E^c) = 0$ and $f_n \to f$ uniformly on E.*

(d) *L^∞ is a Banach space.*

(e) *The simple functions are dense in L^∞.*

Proof. **(a).** The first inequality is trivial. Now if $f \in L^1$ and $g \in L^\infty$, if $|g(x)| = \|g\|_\infty$ a.e. on the set where $f(x) \neq 0$ then $\|fg\|_1 = \|f\|_1\|g\|_\infty$. Now assume that $\|fg\|_1 = \|f\|_1\|g\|_\infty$. We have $|f(x)g(x)| \leqslant |f(x)|\|g\|_\infty$ a.e. and then the hypothesis implies that

$$\int (|f|\|g\|_\infty - |fg|) = 0,$$

hence $|f|\|g\|_\infty - |fg| = 0$ a.e., which in turn implies that $\|g\|_\infty = |g|$ a.e. on the set where $f(x) \neq 0$.

(b). If $f, g \in L^\infty$, then $|f + g| \leqslant \|f\|_\infty + \|g\|_\infty$ a.e., and hence $f + g \in L^\infty$ with $\|f + g\|_\infty \leqslant \|f\|_\infty + \|g\|_\infty$. Clearly if $\|f\|_\infty = 0$ then $f = 0$ a.e. and $\|cf\|_\infty = |c|\|f\|_\infty$, and thus $\|\cdot\|$ is a norm in L^∞.

(c). If $f_n \to f$ in L^∞, let $E = \bigcap_{n=1}^\infty \{x \in X\colon |f_n(x) - f(x)| \leqslant \|f_n - f\|_\infty\}$. Thus

$$E^c = \bigcup_{n=1}^\infty \{x \in X\colon |f_n(x) - f(x)| \geqslant \|f_n - f\|_\infty\},$$

and since $\mu(\{x \in X\colon |f_n(x) - f(x)| \geqslant \|f_n - f\|_\infty\}) = 0$ for all n, we have $\mu(E^c) = 0$. Moreover,

$$\sup_{x\in E} |f_n(x) - f(x)| \leqslant \|f_n - f\|_\infty \to 0,$$

that is, $f_n \to f$ uniformly in E. The converse is trivial (note that the conclusion of the converse is that $f \in L^\infty$ and $f_n \to f$ in L^∞) .

(d). If $\{f_n\}$ is a Cauchy sequence in L^∞, given $\epsilon > 0$ there exists n_0 such that

$$\|f_n - f_m\|_\infty < \epsilon \quad \text{for all } n, m \geqslant n_0.$$

Set $E = \bigcap_{n,m=1}^\infty \{x \in X\colon |f_n(x) - f_m(x)| \leqslant \|f_n - f_m\|_\infty\}$. Thus

$$E^c = \bigcup_{n,m=1}^\infty \{x \in X\colon |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\},$$

and hence $\mu(E^c) = 0$, since all the sets on the right are μ-null.

Thus for all $x \in E$ we have $|f_n(x) - f_m(x)| < \epsilon$ for $n, m \geqslant n_0$. Hence we can define $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in E$. Setting $f = 0$ on $X \setminus E$ we have f measurable, and for $n \geqslant n_0$ and all $x \in E$

$$|f_n(x) - f(x)| \leqslant \epsilon,$$

thus $f_n \to f$ uniformly on E. Hence from item (b) we have $f \in L^\infty$ and $f_n \to f$ in L^∞, which proves that L^∞ is a Banach space.

(e). Clearly if $s = \sum_{j=1}^{n} a_j \chi_{E_j}$ is a simple function, we have $\sup_{x\in X} |s(x)| \leqslant \max_{j=1,\cdots,n} |a_j| = \|s\|_\infty < \infty$. Hence $s \in L^\infty$. Also if f is a measurable function in L^∞, there exists a sequence $\{s_n\}$ of simple measurable functions and a set $E \in \mathcal{M}$ with $\mu(E^c) = 0$ such that $|s_n| \leqslant |f| \leqslant \|f\|_\infty$ for all n and $x \in E$ and $s_n \to f$ uniformly on E (see Proposition 5.1.9).

Thus given $\epsilon > 0$, we can choose n_0 such that $|s_n(x) - f(x)| < \epsilon$ for all $n \geqslant n_0$ and $x \in E$. Hence $\|s_n - f\|_\infty \leqslant \epsilon$ for all $n \geqslant n_0$, and thus $s_n \to f$ in L^∞, which proves that the simple functions are dense in L^∞. ■

In view of item (a) of this theorem and the formal equality $1^{-1} + \infty^{-1} = 1$, it is natural to consider 1 and ∞ **conjugate exponents** of each other, and we do just that from now on.

Item (c) of this result shows that $\|\cdot\|_\infty$ is closely related (but not identical with) the uniform norm $\|\cdot\|_u$ given by $\|f\|_u = \sup_{x\in X} |f(x)|$. We have, however, the next result:

Proposition 5.1.12. *Let μ be any Borel measure in a topological space X that assigns positive values to every nonempty open set. Then if $f\colon X \to \mathbb{C}$ is continuous we have $\|f\|_\infty = \|f\|_u$.*

Proof. Since the set $\{x \in X\colon |f(x)| > \|f\|_\infty\}$ is open in X, it either is empty or it has positive measure. But from the definition of $\|f\|_\infty$, it must have zero measure, and hence $|f(x)| \leqslant \|f\|_\infty$ for all $x \in \mathbb{R}$. Thus $\|f\|_u \leqslant \|f\|_\infty$.

On the other hand, if $\|f\|_u < \|f\|_\infty$ there exists $M \geqslant 0$ such that $|f(x)| \leqslant M < \|f\|_\infty$ for all $x \in \mathbb{R}$. But then $\{x \in X\colon |f(x)| > M\} = \varnothing$, and contradicts the definition of $\|f\|_\infty$. Thus $\|f\|_u = \|f\|_\infty$. ■

Using this result, when restricting ourselves to continuous functions and such Borel measures, we may use the notations $\|f\|_u$ and $\|f\|_\infty$ without distinction, and we may regar the space of bounded continuous functions as a (closed!) subspace of L^∞.

We have, in general, $L^q \subsetneq L^p$ for all $p \neq q$. To see what is the issue, it is useful to have this following example in mind.

Example 5.1.13. *Consider m the Lebesgue measure on $X = (0, \infty)$ and set $f_a(x) = x^{-a}$.*

1. $f_a \chi_{(0,1)} \in L^p$ *iff* $p < a^{-1}$.

2. $f_a \chi_{(1,\infty)} \in L^p$ *iff* $p > a^{-1}$.

Thus that are two apparent reasons for why a function f fails to be in L^p. Either $|f|^p$ blows up too fast near a given point, or $|f|^p$ fails to decay sufficiently rapidly at infinity. In the first situation, the behavior of $|f|^p$ becomes worse as p increases, while the second becomes better. In other words, if $p < q$, function in L^p can be locally more singular than functions in L^q, whereas functions in L^q can be globally more spread out than functions in L^p. These somewhat imprecise ideas are a rather accurate guide to the general situation, which we present in the next results.

Proposition 5.1.14. *If $0 < p < q < r \leqslant \infty$, then $L^q \subset L^p + L^r$, that is, each $f \in L^q$ is the sum of a function in L^p and a function in L^r.*

Proof. If $f \in L^q$, let $E = \{x \in X : |f(x)| > 1\}$ and set $g = f\chi_E$ and $h = f\chi_{E^c}$. Then $|g|^p = |f|^p\chi_E \leqslant |f|^q\chi_E \leqslant |f|^q$, hence $g \in L^p$, and $|h|^r = |f|^r\chi_{E^c} \leqslant |f|^q\chi_{E^c} \leqslant |f|^q$, hence $h \in L^r$ (for $q < r < \infty$). For $r = \infty$, we have $\|h\|_\infty \leqslant 1$, and the proof is complete. ■

Proposition 5.1.15. *If $0 < p < q < r \leqslant \infty$, then $L^p \cap L^r \subset L^q$ and $\|f\|_q \leqslant \|f\|_p^\lambda \|f\|_r^{1-\lambda}$, where $\lambda \in (0,1)$ is defined by*

$$\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r},$$

that is, $\lambda = \dfrac{p}{q} \cdot \dfrac{r-q}{r-p}$.

Proof. If $r = \infty$, we have $\lambda = p/q$ and $|f|^q = |f|^p|f|^{q-p} \leqslant |f|^p\|f\|_\infty^{q-p}$, so

$$\|f\|_q \leqslant \|f\|_p^{p/q}\|f\|_\infty^{(q-p)/q} = \|f\|_p^\lambda\|f\|_\infty^{1-\lambda}.$$

Now we consider the case $r < \infty$. We have $|f|^q = |f|^{\lambda q}|f|^{(1-\lambda)q}$ and we will use Hölder's inequality with the pair of conjugate exponents $p/\lambda q$ and $r/(1-\lambda)q$ to obtain

$$\begin{aligned}\int |f|^q &= \int |f|^{\lambda q}|f|^{(1-\lambda)q} \leqslant \||f|^{\lambda q}\|_{p/\lambda q}\|f^{(1-\lambda)q}\|_{r/(1-\lambda)q} \\ &= \left[\int |f|^p\right]^{\lambda q/p}\left[\int |f|^r\right]^{(1-\lambda)q/r} = \|f\|_p^{\lambda q}\|f\|_r^{(1-\lambda)q},\end{aligned}$$

and taking the q-th root on both sides we conclude the proof. ■

Proposition 5.1.16. *If A is any set and $0 < p < q \leqslant \infty$, then $\ell^p(A) \subset \ell^q(A)$ and $\|f\|_q \leqslant \|f\|_p$.*

Proof. If $q = \infty$, it is clear that

$$\|f\|_\infty^p = \left(\sup_{\alpha\in A}|f(\alpha)|\right)^p = \sup_{\alpha\in A}|f(\alpha)|^p \leqslant \sum_{\alpha\in A}|f(\alpha)|^p = \|f\|_p^p,$$

so that $\|f\|_\infty \leqslant \|f\|_p$.

For $q < \infty$, we use Proposition 5.1.15, with $r = \infty$ and $\lambda = p/q$ to obtain

$$\|f\|_q \leqslant \|f\|_p^\lambda \|f\|_\infty^{1-\lambda} \leqslant \|f\|_p^\lambda \|f\|_p^{1-\lambda} = \|f\|_p.$$

■

PROPOSITION 5.1.17. *If $\mu(X) < \infty$ and $0 < p < q \leqslant \infty$, then $L^q(\mu) \subset L^p(\mu)$ and*

$$\|f\|_p \leqslant \|f\|_q \mu(X)^{\frac{1}{p}-\frac{1}{q}}.$$

Proof. If $q = \infty$, we have

$$\int |f|^p \leqslant \||f|^p\|_\infty \mu(X) = \|f\|_\infty^p \mu(X),$$

and the result follows taking the p-th root. For $q < \infty$ we will use Hölder's inequality with the pair of conjugate exponents q/p and $q/(q-p)$ to obtain

$$\int |f|^p = \int |f|^p \cdot 1 \leqslant \||f|^p\|_{q/p} \|1\|_{q/(q-p)} = \left[\int |f|^q\right]^{p/q} \mu(X)^{(q-p)/q} = \|f\|_q^p \mu(X)^{(q-p)/q},$$

and taking p-th roots, the result is proven. ■

Among the L^p spaces, three have great importance: L^1, which is the landmark of integration theory; L^∞, because of its close relation with uniform convergence; and L^2 which is a Hilbert space with inner product given by

$$(f, g)_2 = \int f\overline{g} d\mu.$$

Unfortunately, L^1 and L^∞ are pathological at some points, one of these is the duality theorem we will present later, see Theorem 5.3.3. So sometimes it is useful to work in the intermediate spaces L^p for $1 < p < \infty$.

5.2| SOLVED EXERCISES FROM [1, PAGE 186]

EXERCISE 1. When does equality hold in Minkowski's inequality? (The answer is different for $p = 1$ and for $1 < p < \infty$. What about $p = \infty$?)

Solution. If $p = 1$, the equality occurs iff $|f + g| = |f| + |g|$ a.e., which happens iff $f\overline{g} \geqslant 0$ a.e. For the last claim, note that

$$\begin{aligned}
& |f+g| = |f| + |g| \\
\Leftrightarrow\quad & |f+g|^2 = (|f| + |g|)^2 \\
\Leftrightarrow\quad & |f+g|^2 = |f|^2 + 2|fg| + |g|^2 \\
\Leftrightarrow\quad & (f+g)(\overline{f+g}) = |f|^2 + 2|fg| + |g|^2 \\
\Leftrightarrow\quad & |f|^2 + 2\mathrm{Re}(f\overline{g}) + |g|^2 = |f|^2 + 2|fg| + |g|^2 \\
\Leftrightarrow\quad & \mathrm{Re}(f\overline{g}) = |fg| = |f\overline{g}|,
\end{aligned}$$

and since $\mathrm{Re}(z) = |z|$ iff $z \geqslant 0$, the claim is proven.

For $1 < p < \infty$, the equality holds iff it holds in (5.1.4) and (5.1.5). In (5.1.4), the equality holds iff $|f + g| = |f| + |g|$, that is, iff $f\overline{g} \geqslant 0$ a.e. In (5.1.5), equality holds if it holds in Hölder's inequality for f and $|f+g|^{p-1}$ (with exponents p and q, respectively), and for g and $|f+g|^{p-1}$ (with exponents p and q, respectively). That is there exists nonnegative constants c_1, c_2, c_3, c_4, (with c_1, c_2 not both null, and c_3, c_4 not both null) such that a.e. we have

$$c_1|f|^p = c_2|f+g|^{(p-1)q} = c_2|f+g|^p \quad \text{and} \quad c_3|g|^p = c_4|f+g|^{(p-1)q} = c_4|f+g|^p.$$

If $c_2 = c_4 = 0$, then $f = g = 0$ a.e. and the equality holds. If $c_4 > 0$ we have

$$c_1|f|^p = c_2|f+g|^p = \frac{c_2}{c_4}c_4|f+g|^p = \frac{c_2}{c_4}c_3|g|^p,$$

and taking p-th roots, we obtain $c_1^{1/p}|f| = \left(\frac{c_2}{c_4}c_3\right)^{1/p}|g|$ a.e. with not both constants zero (analogously we treat the case $c_2 > 0$). Hence equality holds in (5.1.5) iff there exists nonnegative constants α, β, not both zero, such that $\alpha|f| = \beta|g|$ a.e.

Thus equality holds in Minkowski's inequality for $1 < p < \infty$ iff $f\overline{g} \geqslant 0$ and there exists nonnegative constants α, β, not both zero, such that $\alpha|f| = \beta|g|$ a.e.

For $p = \infty$, equality holds if $f\overline{g} \geqslant 0$ a.e., since in this case $|f + g| = |f| + |g|$ a.e. On the other hand, if equality holds, then a.e. we have

$$|f+g| \leqslant |f| + |g| \leqslant \|f\|_\infty + \|g\|_\infty = \|f+g\|_\infty.$$

If $|f + g| < |f| + |g|$ in a positive measure set, then $\|f + g\|_\infty < \|f\|_\infty + \|g\|_\infty$, hence $|f + g| = |f| + |g|$ a.e. and this holds iff $f\overline{g} \geqslant 0$ a.e.

Exercise 2. Prove Theorem 5.1.11.

Solution. This is done in the text.

Exercise 3. If $1 \leqslant p < r \leqslant \infty$, $L^p \cap L^r$ is a Banach space with norm $\|f\| = \|f\|_p + \|f\|_r$, and if $p < q < r$, the inclusion map $L^p \cap L^r \to L^q$ is continuous.

Solution. Clearly $L^p \cap L^r$ is a vector subspace (both of L^p and L^r), and $\|\cdot\|$ is a norm in $L^p \cap L^r$. Now if $\{f_n\}$ is a Cauchy sequence in $L^p \cap L^r$, from the definition of $\|\cdot\|$ it follows directly that $\{f_n\}$ is also a Cauchy sequence in L^p and L^r. Hence there exists functions $g \in L^p$ and $h \in L^r$ such that $f_n \to g$ in L^p and $f_n \to h$ in L^r.

Assume $r < \infty$. Using the result of Exercise 9 below, since $f_n \to g$ in L^p (and $1 \leqslant p < \infty$), $f_n \to g$ in measure and hence there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to g$ a.e. Then since $f_{n_k} \to h$ in L^r, using again Exercise 9, there exists a subsequence $\{f_{n_{k_j}}\}$ of $\{f_{n_k}\}$ such that $f_{n_{k_j}} \to h$ a.e. But this implies that $g = h$ a.e. and hence $g \in L^p \cap L^r$ and $f_n \to g$ in $L^p \cap L^r$.

If $r = \infty$, since $f_n \to h$ in L^∞, there exists a set $E \in \mathcal{M}$ with $\mu(E^c) = 0$ such that $f_n \to h$ uniformly in E. But then $f_n \to g$ in $L^p(E)$ and using Exercise 9, up to a subsequence, $f_n \to g$ a.e. in E. Hence $g = h$ a.e. in E, and since $\mu(E^c) = 0$, $g = h$ a.e. Hence $g \in L^p \cap L^\infty$ and $f_n \to g$ in $L^p \cap L^\infty$.

Therefore $L^p \cap L^r$ with the norm $\|\cdot\|$ is a Banach space.

If remains to prove the second claim. From Proposition 5.1.15 we have $L^p \cap L^r \subset L^q$ and

$$\|f\|_q \leqslant \|f\|_p^\lambda \|f\|_r^{1-\lambda},$$

where $\lambda \in (0,1)$ is such $\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}$. Using Lemma 5.1.3 we obtain

$$\|f\|_q \leqslant \lambda \|f\|_p + (1-\lambda)\|f\|_r \leqslant \|f\|_p + \|f\|_r = \|f\|,$$

and the inclusion $L^p \cap L^r \to L^q$ is continuous.

Exercise 4. If $1 \leqslant p < r \leqslant \infty$, $L^p + L^r$ is a Banach space with norm $\|f\| = \inf\{\|g\|_p + \|h\|_r \colon f = g + h\}$, and if $p < q < r$, the inclusion map $L^q \to L^p + L^r$ is continuous.

Solution. Since $L^p + L^r$ is the sum of the vector subspaces of the vector space of all measurable functions $f \colon X \to \mathbb{C}$, it is a vector space. Now we prove that $\|\cdot\|$ is a norm in $L^p + L^r$. First note that if $\|f\| = 0$ then there exist sequence $\{g_n\} \subset L^p$ and $\{h_n\} \subset L^r$ such that $\|g_n\|_p + \|h_n\|_r \to 0$ and $f = g_n + h_n$ for all $n \in \mathbb{N}$. Thus $g_n \to 0$ in L^p and $h_n \to 0$ in L^r.

Repeating an analogous argument from Exercise 3, we can extract subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and $\{h_{n_k}\}$ of $\{h_n\}$ such that $g_{n_k} \to 0$ and $h_{n_k} \to 0$ a.e. Hence

$$f = g_{n_k} + h_{n_k} \to 0 \text{ a.e.}$$

and thus $f = 0$ a.e.

Let $c \in \mathbb{C}$. if $c = 0$ then $\|cf\| = \|0\| = 0 = 0\|f\|$. If $c \neq 0$, then $f = g + h$ for $g \in L^p$ and $h \in L^r$ iff $cf = cg + ch$, and hence $\|cf\| = |c|\|f\|$.

For the triangle inequality, let $f_1, f_2 \in L^p + L^r$. If $f_1 = g_1 + h_1$ and $f_2 = g_2 + h_2$ then

$$f_1 + f_2 = (g_1 + g_2) + (h_1 + h_2),$$

and hence

$$\|f_1 + f_2\| \leqslant \|g_1 + g_2\|_p + \|h_1 + h_2\|_r \leqslant \|g_1\|_p + \|h_1\|r + \|g_2\|_p + \|h_2\|_r,$$

and take the infimum over all possible representations of f_1 and f_2 we obtain

$$\|f_1 + f_2\| \leqslant \|f_1\| + \|f_2\|.$$

Thus $\|\cdot\|$ is a norm in $L^p + L^r$. To prove that $L^p + L^r$ is a Banach space, let $\{f_n\}$ be a sequence in $L^p + L^r$ such that $\sum_{n=1}^{\infty} \|f_n\| < \infty$. From the definition of $\|\cdot\|$, for each $n \in \mathbb{N}$, there exists $g_n \in L^p$ and $h_n \in L^r$ such that $f_n = g_n + h_n$ and $\|g_n\|_p + \|h_n\|_r < \|f_n\| + 2^{-n}$.

Thus $\sum_{n=1}^{\infty}(\|g_n\|_p + \|h_n\|_r) \leqslant \sum_{n=1}^{\infty}(\|f_n\| + 2^{-n}) < \infty$, hence $\sum_{n=1}^{\infty} \|g_n\|_q < \infty$ and $\sum_{n=1}^{\infty} \|h_n\|_r < \infty$, and there exists $g \in L^p$ and $h \in L^r$ such that $\sum_{n=1}^{\infty} g_n = g$ in L^p and $\sum_{n=1}^{\infty} h_n$ in L^r. Let $f = g + h$. We have

$$\left\|f - \sum_{k=1}^{n} f_k\right\| \leqslant \left\|g - \sum_{k=1}^{n} g_k\right\|_q + \left\|h - \sum_{k=1}^{n} h_k\right\|_r \to 0 \text{ as } n \to \infty,$$

hence $\sum_{n=1}^{\infty} f_n = f$ in $L^p + L^r$, which proves that $L^p + L^r$ is a Banach space.

Now if $p < q < r$, Proposition 5.1.14 shows that $L^q \subset L^p + L^r$. Let $f \in L^q$ with $\|f\|_q = 1$. Set $E = \{x \in X \colon |f(x)| > 1\}$ and define $g = \chi_E$ and $h = \chi_{E^c}$. Then if $r < \infty$ we have

$$|g|^p = |f|^p \chi_E \leqslant |f|^q \chi_E \leqslant |f|^q \quad \text{ and } |h|^r = |f|^r \chi_{E^c} \leqslant |f|^q \chi_{E^c} \leqslant |f|^q,$$

and thus $\|g\|_p^p \leqslant \|f\|_q^q = 1$ and $\|h\|_r^r \leqslant \|f\|_q^q = 1$. This shows that $g \in L^p$, $h \in L^r$ and

$f = g + h$, therefore

$$\|f\| \leqslant \|g\|_p + \|h\|_r \leqslant 1 + 1 = 2.$$

If $r = \infty$, then $\|h\|_\infty \leqslant 1$ (since $|f|\chi_{E^c} \leqslant 1$ for all $x \in X$), and the same conclusion holds, that is, $\|f\| \leqslant 2$.

If $f \in L^q$ is any function with $\|f\|_q > 0$, then $\tilde{f} = f/\|f\|_q \in L^q$ and $\|\tilde{f}\|_q = 1$, then

$$\|f\| = \|f\|_q \left\| \frac{f}{\|f\|_q} \right\| = \|f\|_q \|\tilde{f}\| \leqslant 2\|f\|_q,$$

and since this inequality is true for $f = 0$, we have $\|f\| \leqslant 2\|f\|_q$ for all $f \in L^q$, and proves that the inclusion $L^q \to L^p + L^r$ is continuous.

Exercise 5. Suppose $0 < p < q < \infty$. Then $L^p \not\subset L^q$ iff X contains sets of arbitrarily small positive measure, and $L^q \not\subset L^p$ iff X contains sets of arbitrarily large finite measure (for the 'if' implication: in the first case there is a disjoint sequence $\{E_n\}$ with $0 < \mu(E_n) < 2^{-n}$, and in the second case there is a disjoint sequence $\{E_n\}$ with $1 \leqslant \mu(E_n) < \infty$. Consider $f = \sum a_n \chi_{E_n}$ for suitable constants a_n). What about the case $q = \infty$?

Solution. Suppose X contains sets of arbitrarily small positive measure. Then we can choose $F_1 \in \mathcal{M}$ with $0 < \mu(F_1) < 4^{-1}$. Choose $F_2 \in \mathcal{M}$ such that $0 < \mu(F_2) < 4^{-1}\mu(F_1)$. Inductively, we construct a sequence $\{F_n\} \subset \mathcal{M}$ with $0 < \mu(F_n) < 4^{-1}\mu(F_{n-1})$ for $n \geqslant 1$ (and $0 < \mu(F_1) < 2^{-1}$). In particular $0 < \mu(F_n) < 4^{-n}$.

Now we consider $E_n = F_n \setminus \bigcup_{k=n+1}^\infty F_k$. Thus $F_n \subset E_n \cup \left(\bigcup_{k=n+1} F_k\right)$ and hence

$$\mu(F_n) \leqslant \mu(E_n) + \sum_{k=n+1}^\infty \mu(F_k) \leqslant \mu(E_n) + \mu(F_n) \sum_{k=1}^\infty 4^{-k} = \mu(E_n) + \frac{1}{3}\mu(F_n),$$

and thus $0 < \frac{2}{3}\mu(F_n) < \mu(E_n)$. Since $E_n \subset F_n$ we have $0 < \mu(E_n) \leqslant \mu(F_n) < 4^{-n} < 2^{-n}$. Furthermore, the sequence $\{E_n\}$ is disjoint, since for $n < m$ we have

$$E_n \cap E_m = F_n \cap \left(\bigcap_{k=n+1}^\infty F_k^c\right) \cap F_m \left(\bigcap_{k=m+1}^\infty F_k^c\right) \subset F_m^c \cap F_m = \varnothing,$$

since F_m^c appears in $\bigcap_{k=n+1}^\infty F_k^c$ because $n < m$. Hence there exists a disjoint sequence $\{E_n\} \subset \mathcal{M}$ such that $0 < \mu(E_n) < 2^{-n}$.

Now consider $f = \sum (n\mu(E_n))^{-1/q} \chi_{E_n}$, which is well defined, since $\{E_n\}$ is disjoint. We have

$$\int |f|^q = \sum \frac{1}{n} = \infty,$$

hence $f \notin L^q$. However

$$\int |f|^p = \sum n^{-p/q}\mu(E_n)^{1-p/q} \leqslant \sum 2^{-n(1-p/q)} < \infty,$$

since $1 - p/q > 0$, and thus $f \in L^p$. Therefore $L^p \not\subset L^q$.

For the converse, assume that $L^p \not\subset L^q$, that is, there exists $f \in L^p$ such that $f \notin L^q$. Define $E_n = \{x \in X\colon |f(x)| > n\}$. Using Proposition 5.1.15, if $f \in L^\infty$, we would have $f \in L^q$, which is a contradiction. Hence $\|f\|_\infty = \infty$ and thus $\mu(E_n) > 0$ for all $n \in \mathbb{N}$. Also we have

$$\int |f|^p \geqslant \int_{E_n} |f|^p \geqslant n^p \mu(E_n),$$

and hence $0 < \mu(E_n) < \|f\|_p^p n^{-p} \to 0$ as $n \to \infty$, since $\|f\|_p < \infty$.

Now for the other case, suppose first that X contains sets of arbitrarily large finite measure. Let $F_1 \in \mathcal{M}$ with $1 < \mu(F_1) < \infty$. For each $n \in \mathbb{N}$ we construct $F_{n+1} \in \mathcal{M}$ such that $1 + \sum_{k=1}^{n} \mu(F_n) < \mu(F_{n+1}) < \infty$. Consider $E_1 = F_1$ and $E_n = F_n \setminus \bigcup_{k=1}^{n-1} F_k$. Then $\{E_n\}$ is a disjoint sequence in $\mathcal{M}$ and

$$\mu(E_n) \geqslant \mu(F_n) - \sum_{k=1}^{n-1} F_k > 1.$$

Let $f = \sum (n\mu(E_n))^{-1/p}\chi_{E_n}$. Then

$$\int |f|^p = \sum \frac{1}{n} = \infty,$$

thus $f \notin L^p$ and

$$\int |f|^q = \sum n^{-q/p}\mu(E_n)^{1-q/p} \leqslant \sum n^{-q/p} < \infty,$$

since $\mu(E_n) > 1$ and $q/p > 1$. Hence $f \in L^q$.

Conversely assume that $L^q \not\subset L^p$ and let $f \in L^q$ such that $f \notin L^p$. Let $E_n = \{x \in X\colon |f(x)| > 1/n\}$. Since

$$\int |f|^q \geqslant \int_{E_n} |f|^q \geqslant n^{-q}\mu(E_n),$$

it follows that $\mu(E_n) \leqslant n^q \|f\|_q^q < \infty$ for all $n \in \mathbb{N}$. Furthermore, $\{E_n\}$ is an increasing sequence, and hence, with $E = \bigcup_{n=1}^{\infty} E_n$ we have $\mu(E) = \lim_{n\to\infty} \mu(E_n)$. But $E = \{x \in X\colon f(x) \neq 0\}$. Thus if $\mu(E) < \infty$, applying Proposition 5.1.17 to $(E, \mathcal{M}_E, \mu|_E)$ we would have

$$\|f\|_p = \|f|_E\|_p \leqslant \|f|_E\|_q \mu(E)^{1/p-1/q} = \|f\|_q \mu(E)^{1/p-1/q} < \infty,$$

and this implies that $f \in L^p$, which is a contradiction. Hence $\mu(E) = \infty$, and thus $\lim_{n\to\infty} \mu(E_n) = \infty$.

Now for the case $q = \infty$. The first assertion remains true. In fact, assume that X contains sets of arbitrarily small positive measure, and as above consider a disjoint sequence $\{E_n\} \subset \mathcal{M}$ with $0 < \mu(E_n) < 2^{-n}$. Define $f = \sum n\chi_{E_n}$. Then $f \notin L^\infty$ but

$$\int |f|^p = \sum n^p \mu(E_n) \leqslant \sum n^p 2^{-n} < \infty,$$

and hence $f \in L^p$. Conversely, let $f \in L^p$ such that $f \notin L^\infty$, and define $E_n = \{x \in X : |f(x)| > n\}$. Then since $f \notin L^\infty$ we have $\mu(E_n) > 0$. Also

$$\int |f|^p \geqslant \int_{E_n} |f|^p \geqslant n^p \mu(E_n),$$

and hence $\mu(E_n) \leqslant n^{-p}\|f\|_p^p \to 0$ as $n \to \infty$.

For the second, if X contains sets of arbitrarily large finite measure, as before, we can consider a disjoint sequence $\{E_n\} \subset \mathcal{M}$ with $1 < \mu(E_n) < \infty$ for all $n \in \mathbb{N}$. Let $f = \chi_{E_n}$. Then $f \in L^\infty$, but

$$\int |f|^p = \sum \mu(E_n) \geqslant \sum 1 = \infty,$$

and thus $f \notin L^p$.

The converse in this case is not true. Consider X a nonempty set, $\mathcal{M} = \{\varnothing, X\}$, $\mu(\varnothing) = 0$ and $\mu(X) = \infty$. Hence $f \equiv 1$ is in L^∞ but not in L^p (since $\int |f|^p = \mu(X) = \infty$), and therefore $L^\infty \not\subset L^p$ but X does not contain sets of arbitrarily large but finite measures.

Exercise 6. Suppose $0 < p_0 < p_1 \leqslant \infty$. Find examples of functions f on $(0, \infty)$ (with Lebesgue measure), such that $f \in L^p$ iff

(a) $p_0 < p < p_1$,

(b) $p_0 \leqslant p \leqslant p_1$,

(c) $p = p_0$.

(Consider functions of the form $f(x) = x^{-a}|\log x|^b$).

Solution to (a). If $p_1 < \infty$ define

$$f(x) = x^{-1/p_1}\chi_{(0,1)}(x) + x^{-1/p_0}\chi_{[1,\infty)}(x).$$

Then

$$\int |f|^p = \int_0^1 x^{-p/p_1} dx + \int_1^\infty x^{-p/p_0} dx,$$

and $f \in L^p$ iff both integrals are finite. The first is finite when $p < p_1$ and the second when $p > p_0$. Thus $f \in L^p$ iff $p_0 < p < p_1$.

If $p_1 = \infty$, consider $f(x) = x^{-1/p_0} |\log(x-1)| \chi_{[1,2)} + x^{-1/p_0} \chi_{[2,\infty)}$. Then

$$\int |f|^p = \int_1^2 x^{-p/p_0} |\log(x-1)|^p dx + \int_2^\infty x^{-p/p_0} dx,$$

which is finite iff both integrals are finite. The second is finite iff $p > p_0$. The first if finite for all $p > 0$, and to see that note first that for all $\alpha > 0$ we have

$$\lim_{x\to 1^+} (x-1)^\alpha |\log(x-1)| = \lim_{u\to 0^+} u^\alpha \log(1/u) \overset{\text{L'Hôpital Rule}}{=} \lim_{u\to 0^+} \frac{u^\alpha}{\alpha} = 0.$$

Now choose $\alpha > 0$ such that $\alpha p < 1$. From the limit above, there exists $0 < \delta < 1$ such that $|\log(x-1)| < (x-1)^{-\alpha}$ for $x \in (1, 1+\delta)$. Then we obtain

$$\begin{aligned}
\int_1^2 x^{-p/p_0} |\log(x-1)|^p dx &\leqslant \int_1^2 |\log(x-1)|^p dx \\
&= \int_1^{1+\delta} |\log(x-1)|^p dx + \int_{1+\delta}^2 |\log(x-1)|^p dx \\
&\leqslant \int_1^{1+\delta} (x-1)^{-p\alpha} dx + |\log(\delta)|^p (1-\delta) < \infty,
\end{aligned}$$

since $\alpha p < 1$. Thus $f \in L^p$ for all $p > p_0$, but since $\lim_{x\to 1^+} f(x) = \infty$, $f \notin L^\infty$. Thus $f \in L^p$ iff $p_0 < p < \infty$.

Solution to (b). Assume $p_1 < \infty$. Take

$$f(x) = (x \log^2(1/x))^{-1/p_1} \chi_{(0,1/2)}(x) + (x \log^2(1/x))^{-1/p_0} \chi_{[2,\infty)}(x),$$

and we have

$$\int |f|^p = \int_0^{1/2} (x \log^2(1/x))^{-1/p_1} dx + \int_2^\infty (x \log^2(1/x))^{-1/p_0} dx,$$

and thus $f \in L^p$ iff both integrals are finite.

We analyze the first integral, and the second is analogous. For $p = p_1$, we have

$$\int_0^{1/2} (x \log^2(1/x))^{-p/p_1} dx = \int_0^{1/2} \frac{1}{x \log^2(x)} dx,$$

but using the substitution $u = \log(x)$ we have

$$\int \frac{1}{x\log^2(x)}dx = \int \frac{1}{u^2}du = -\frac{1}{u} + c = -\frac{1}{\log(x)} + c,$$

and from the Fundamental Theorem of Calculus we obtain

$$\int_0^{1/2} \frac{1}{x\log^2(x)}dx = \frac{1}{\log(2)} < \infty.$$

Before continuing, we note that for $\alpha, \beta > 0$ we have $\lim_{x\to 0^+} x^\alpha(\log^2(x))^{-\beta} = 0$ and hence there exists a constant $c > 0$ such that $(\log^2(x))^{-\beta} \leqslant cx^{-\alpha}$ for all $x \in (0, 1/2)$.

Now for $p < p_1$ we choose $\lambda > 0$ such that $p/p_1 + \lambda < 1$, and applying this last remark with $\beta = p/p_1$ and $\alpha = \lambda$ we obtain

$$\int_0^{1/2} (x\log^2(1/x))^{-p/p_1}dx \leqslant \int_0^{1/2} x^{-p/p_1}x^{-\alpha}dx = \int_0^{1/2} x^{-(p/p_1+\lambda)}dx < \infty,$$

since $p/p_1 + \lambda < 1$.

For the remaining case ($p > p_1$) we need to note that for $\alpha, \beta > 0$ we have $\lim_{x\to 0^+} x^\alpha(\log^2(x))^\beta = 0$, and hence there exists a constant $c > 0$ such that $x^\alpha(\log^2(x))^\beta \leqslant c$ for all $x \in (0, 1/2)$ and thus

$$(\log^2(x))^{-\beta} \geqslant \frac{x^\alpha}{c} \qquad \text{for all } x \in (0, 1/2).$$

Thus for $p > p_1$, choose $\lambda > 0$ such that $p/p_1 - \lambda > 1$. Applying this previous estimate with $\alpha = \lambda$ and $\beta = -p/p_1$, we obtain

$$\int_0^{1/2} (x\log^2(1/x))^{-p/p_1}dx \geqslant c^{-1}\int_0^{1/2} x^{-p/p_1}x^\lambda dx = c^{-1}\int_0^{1/2} x^{-(p/p_1-\lambda)}dx = \infty,$$

since $p/p_1 - \lambda > 1$. Thus the first integral is finite iff $p \leqslant p_1$. Analogously for the second integral we obtain finiteness iff $p \geqslant p_0$, and thus $f \in L^p$ iff $p_0 \leqslant p \leqslant p_1$.

If $p_1 = \infty$ take $f(x) = (x\log^2(x))^{-1/p_0}\chi_{[2,\infty)}$. Since $\|f\|_\infty \leqslant (2\log^2(2))^{-1/p_0} < \infty$, we have $f \in L^\infty$, and hence $f \in L^p$ iff $p_0 \leqslant p \leqslant \infty$.

Solution to (c). Take $p_1 = p_0$ in the case $p_1 < \infty$ in item (b). From the computations we already have done, $f \in L^p$ iff $p_0 \leqslant p \leqslant p_1 = p_0$, that is, iff $p = p_0$.

Exercise 7. If $f \in L^p \cap L^\infty$ for some $p < \infty$, so that $f \in L^q$ for all $p > q$, then $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$.

Solution. If $\|f\|_\infty = 0$ then $f = 0$ a.e. and the equality is trivial. Assume that $\|f\|_\infty > 0$

and choose $0 < \epsilon < \|f\|_\infty$. Set $A = \{x \in X : |f(x)| > \|f\|_\infty - \epsilon\}$. Clearly $\mu(A) > 0$. We have

$$\|f\|_p^p = \int |f|^p \geqslant \int_A |f|^p \geqslant \mu(A)(\|f\|_\infty - \epsilon)^p,$$

and thus $\mu(A) \leqslant (\|f\|_\infty - \epsilon)^{-p}\|f\|_p^p < \infty$. Using the same computations with q instead of p we obtain

$$\|f\|_q^q \geqslant \mu(A)(\|f\|_\infty - \epsilon)^q,$$

and hence $\liminf_{q\to\infty} \|f\|_q \geqslant \|f\|_\infty - \epsilon$. Since $\epsilon > 0$ is arbitrary, we obtain $\liminf_{q\to\infty} \|f\|_q \geqslant \|f\|_\infty$.

For the other inequality, using Proposition 5.1.15 for $q > p$ we obtain $\|f\|_q \leqslant \|f\|_p^{p/q}\|f\|_\infty^{1-p/q}$, and making $q \to \infty$ have

$$\limsup_{q\to\infty} \|f\|_q \leqslant \|f\|_\infty,$$

and the proof is complete.

Exercise 8. This exercises makes use of Jensen's inequality, which is a topic not seen in this course.

Exercise 9. Suppose $1 \leqslant p < \infty$. If $\|f_n - f\|_p \to 0$, then $f_n \to f$ in measure, and hence some subsequence converges to f a.e. On the other hand, if $f_n \to f$ in measure and $|f_n| \leqslant g \in L^p$ for all $n \in \mathbb{N}$, then $\|f_n - f\|_p \to 0$.

Solution. Let $E_{n,\epsilon} = \{x \in X : |f_n(x) - f(x)| \geqslant \epsilon\}$. Then

$$\int |f_n - f|^p \geqslant \int_{E_{n,\epsilon}} |f_n - f|^p \geqslant \epsilon^p \mu(E_{n,\epsilon}),$$

and hence $\mu(E_{n,\epsilon}) \leqslant \epsilon^{-p}\|f_n - f\|_p^p \to 0$ as $n \to \infty$, and thus $f_n \to f$ in measure. Using Theorem 3.7.5, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$ a.e.

Now assume that $f_n \to f$ in measure and $|f_n| \leqslant g \in L^p$. Since

$$\begin{aligned}\{x \in X : ||f_n(x) - f(x)|^p - 0| \geqslant \epsilon\} &= \{x \in X : |f_n(x) - f(x)|^p \geqslant \epsilon\} \\ &= \{x \in X : |f_n(x) - f(x)| \geqslant \epsilon^{1/p}\},\end{aligned}$$

we obtain $|f_n - f|^p \to 0$ in measure. Using the subsequence $\{f_{n_k}\}$ that converges a.e. to f, we obtain $|f| \leqslant g$ a.e., and hence $|f_n - f|^p \leqslant (|f_n| + |f|)^p \leqslant 2^p g^p \in L^1$ and using Exercise 34 we have $|f_n - f|^p \to 0$ in L^1, that is

$$\|f_n - f\|_p^p = \int |f_n - f|^p \to 0,$$

hence $\|f_n - f\|_p \to 0$.

Exercise 10. Suppose $1 \leqslant p < \infty$. If $f_n, f \in L^p$ and $f_n \to f$ a.e., then $\|f_n - f\|_p \to 0$ iff $\|f_n\|_p \to \|f\|_p$. (Use Exercise 20 in Section 3.6).

Solution. Using the triangle inequality for the norm $\|\cdot\|_p$ (for $p \geqslant 1$) we have

$$\|f_n\|_p \leqslant \|f_n - f\|_p + \|f\|_p \quad \text{and} \quad \|f\|_p \leqslant \|f_n - f\|_p + \|f_n\|_p,$$

and hence $|\|f_n\|_p - \|f\|_p| \leqslant \|f_n - f\|_p$, and thus if $\|f_n - f\|_p \to 0$ then $\|f_n\|_p \to \|f\|_p$.

For the converse, note that since $f_n \to f$ a.e. we have $|f_n - f|^p \to 0$ a.e. Now $|f_n - f|^p \leqslant 2^p(|f_n|^p + |f|^p) := g_n$. We have $g_n \to 2^{p+1}|f|^p := g$ a.e. and

$$\int g_n = 2^p\Big(\int |f_n|^p + \int |f|^p\Big) = 2^p(\|f_n\|_p^p + \|f\|_p^p) \to 2^{p+1}\|f\|_p^p = \int g,$$

since $\|f_n\|_p \to \|f\|_p$. Thus using Exercise 20 of Section 3.6 we obtain $\int |f_n - f|^p \to 0$, that is, $\|f_n - f\|_p^p \to 0$, and thus $\|f_n - f\|_p \to 0$.

Exercise 11. If f is a measurable function on X, define the **essential range** R_f of f to be the set of all $z \in \mathbb{C}$ such that $\{x \in X : |f(x) - z| < \epsilon\}$ has a positive measure for all $\epsilon > 0$.

(a) R_f is closed.

(b) If $f \in L^\infty$, then R_f is compact and $\|f\|_\infty = \max\{|z| : z \in R_f\}$.

Solution to (a). Let $\{z_n\} \subset R_f$ be such that $z_n \to z$ for some $z \in \mathbb{C}$. If Given $\epsilon > 0$ choose n_0 such that $|z_n - z| < \epsilon/2$ for $n \geqslant n_0$. Then for $x \in \{x \in X : |f(x) - z_n| < \epsilon/2\}$ we have

$$|f(x) - z| \leqslant |f(x) - z_n| + |z_n - z| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and thus $\{x \in X : |f(x) - z_n| < \epsilon/2\} \subset \{x \in X : |f(x) - z| < \epsilon\}$ for $n \geqslant n_0$. Since $\{x \in X : |f(x) - z_n| < \epsilon/2\}$ has positive measure then $\{x \in X : |f(x) - z| < \epsilon\}$ has positive measure, and therefore $z \in R_f$ and R_f is closed.

Solution to (b). Assume $f \in L^\infty$. Let $z \in \mathbb{C}$ such that $|z| > \|f\|_\infty$ and choose $\epsilon > 0$

such that $|z| > \|f\|_\infty + \epsilon$. If $x \in \{x \in X : |f(x) - z| < \epsilon\}$ we have

$$\|f\|_\infty + \epsilon - |f(x)| < |z| - |f(x)| \leqslant |f(x) - z| < \epsilon,$$

and hence $\|f\|_\infty < |f(x)|$. This means that $\{x \in X : |f(x) - z| < \epsilon\} \subset \{x \in X : |f(x)| > \|f\|_\infty\}$, and thus, by definition of $\|f\|_\infty$, we obtain

$$\mu(\{x \in X : |f(x) - z| < \epsilon\}) \leqslant \mu(\{x \in X : |f(x)| > \|f\|_\infty\}) = 0,$$

which implies that $z \notin R_f$. Hence $R_f \subset \{z \in \mathbb{C} : |z| \leqslant \|f\|_\infty\}$. Thus R_f is bounded, and since it is closed (by item (a)), it is compact in $\mathbb{C}$.

From what we just have proved, we obtain $\max\{|z| : z \in R_f\} \leqslant \|f\|_\infty$. If $\alpha := \max\{|z| : z \in R_f\} < \|f\|_\infty$, we know that the sphere $S = \{z \in \mathbb{C} : |z| = \|f\|_\infty\}$ does not intersect R_f. Hence for each $z \in S$, there exists $\epsilon_z > 0$ such that $\{x \in X : |f(x) - z| < \epsilon_z\}$ has zero measure. The collection of open balls $\{B_{\epsilon_z}(z)\}_{z \in S}$ (here $B_r(w) = \{z \in \mathbb{C} : |z - w| < r\}$) constitutes a open cover of S, and since S is compact, there exists $z_1, \cdots, z_n$ such that $S \subset \bigcup_{i=1}^n B_{\epsilon_{z_i}}(z_i)$. Take $0 < \eta < \min\limits_{i=1,\cdots,n} \epsilon_{z_i}$ and also $0 < \eta < \|f\|_\infty - \alpha$. If $\|f\|_\infty - \eta < |f(x)| \leqslant \|f\|_\infty$, that is, $f(x)$ is closer than η to S, it must be in one of the balls $B_{\epsilon_{z_i}}(z_i)$, and therefore we have

$$\{x \in X : \|f\|_\infty - \eta < |f(x)| \leqslant \|f\|_\infty\} \subset \bigcup_{i=1}^n \{x \in X : |f(x) - z_i| < \epsilon_{z_i}\},$$

and thus $\{x \in X : \|f\|_\infty - \eta < |f(x)| \leqslant \|f\|_\infty\}$ has zero measure. Also

$$\{x \in X : |f(x)| > \|f\|_\infty - \eta\} = \{x \in X : \|f\|_\infty - \eta < |f(x)| \leqslant \|f\|_\infty\} \cup \{x \in X : |f(x)| > \|f\|_\infty\}$$

has zero measure and contradicts the definition of $\|f\|_\infty$.

Exercise 12. If $p \neq 2$, the L^p norm does not arise from an inner product on L^p, except in trivial cases when $\dim(L^p) \leqslant 1$ (show that the parallelogram law fails).

Solution. If Y is a normed vector space and $\dim(Y) \leqslant 1$ and $Y = \operatorname{span}\{y_0\}$ ($y_0 = 0$ when $\dim(Y) = 0$), then $(\alpha y_0, \beta y_0) = \alpha\overline{\beta}\|y_0\|^2$ is an inner product on Y and $\|\alpha y_0\|^2 = |\alpha|^2\|y_0\|^2 = (\alpha y_0, \alpha y_0)$.

Now assume that $\dim(L^p) > 1$ and $p \neq 2$. In order to show that the parallelogram law fails, we will need to show the existence of sets $E_1, E_2 \in \mathcal{M}$ such that $0 < \mu(E_1),\ \mu(E_2) < \infty$ and $E_1 \cap E_2 = \varnothing$. To obtain such sets we will prove following two claims:

Claim 1. There exists a set $E_1 \in \mathcal{M}$ with $0 < \mu(E_1) < \infty$.

In fact, if that is not the case then for $f \in L^p$, since $\{x \in X : f(x) \neq 0\}$ is σ-finite, we must have $\mu(\{x \in X : f(x) \neq 0\}) = 0$, which implies that $f = 0$ in L^p, and thus $\dim(L^p) = 0$, which is a contradiction.

Claim 2. There exists a set $E \in \mathcal{M}$ with $0 < \mu(E) < \infty$ and $\mu(E \Delta E_1) > 0$.

In fact, if that is not the case then let $f \in L^p$. Fix $x_0 \in E_1$, and let $A := \{x \in X : f(x) - f(x_0)\chi_{E_1}(x) \neq 0\}$. Thus this set is σ-finite since $f - f(x_0)\chi_{E_1}$ in is L^p. Hence $A = \bigcup_{n=1}^{\infty} A_n$, with $\{A_n\} \subset \mathcal{M}$ is disjoint and $\mu(A_n) < \infty$. Thus either $\mu(A_n) = 0$ or $\mu(A_n \Delta E_1) = 0$. We have

$$A \setminus E_1 = \bigcup_{n=1}^{\infty} (A_n \setminus E) \quad \text{and} \quad E_1 \setminus A = \bigcap_{n=1}^{\infty} E_1 \setminus A_n,$$

We have $\mu(A \setminus E_1) = 0$. If $\mu(A_n) = 0$ for all $n \in \mathbb{N}$, then $\mu(A) = 0$ and $f = f(x_0)\chi_{E_1}$ a.e. Otherwise $\mu(A_n \Delta E_1) = 0$ for some $n \in \mathbb{N}$, and in this case we also have $\mu(E_1 \setminus A) = 0$, and then $\mu(A \Delta E_1) = 0$. Therefore $A \subset (A \setminus E_1) \cup (A \cap E_1)$, but $\mu(A \cap E_1) \leqslant \mu(E_1) < \infty$ and since $(A \cap E_1)\Delta E_1 = E_1 \setminus A$, we have

$$\mu(A) = \mu(A \setminus E_1) + \mu(A \cap E_1) \leqslant \mu(A \setminus E_1) + \mu(E_1 \setminus A) = \mu(A \Delta E_1) = 0,$$

and hence $f = f(x_0)\chi_{E_1}$ a.e. Thus $f = f(x_0)\chi_{E_1}$ in L^p. Thus we have shown that each f in L^p is a constant multiple of χ_{E_1}, which means that $\dim(L^p) = 1$. This contradicts our hypothesis and proves Claim 2.

Setting E_1 as in Claim 1, E as in Claim 2 and $E_2 = E \Delta E_1$ we have $0 < \mu(E_2) < \infty$ and $E_1 \cap E_2 = \varnothing$, as needed.

Assume $1 \leqslant p < \infty$ and define $f_1 = \mu(E_1)^{-1/p}\chi_{E_1}$ and $f_2 = \mu(E_2)^{-1/p}\chi_{E_2}$. Then $f_1, f_2 \in L^p$ and $\|f_1\|_p = \|f_2\|_p = 1$. If the parallelogram law holds, we have

$$\|f_1 + f_2\|_p^2 + \|f_1 - f_2\|_p^2 = 2(\|f_1\|_p^2 + \|f_2\|_p^2) = 4.$$

But we have

$$\begin{aligned}\|f_1 + f_2\|_p^2 + \|f_1 - f_2\|_p^2 &= \Big(\int |f_1 + f_2|^p\Big)^{2/p} + \Big(\int |f_1 - f_2|^p\Big)^{2/p} \\ &= 2\Big(\int \mu(E_1)^{-1}\chi_{E_1} + \mu(E_2)^{-1}\chi_{E_2}\Big)^{2/p} = 2^{1+2/p},\end{aligned}$$

and thus the parallelogram law holds iff $1 + 2/p = 2$, that is, iff $p = 2$.

For $p = \infty$, set $f_1 = \chi_{E_1}$ and $f_2 = \chi_{E_2}$.

$$\|f_1 + f_2\|_\infty^2 + \|f_1 - f_2\|_\infty^2 = 2 \neq 4 = 2(\|f_1\|_\infty^2 + \|f_2\|_\infty^2),$$

and the parallelogram law fails.

Exercise 13. $L^p(\mathbb{R}^n, m)$ is separable for $1 \leqslant p < \infty$. However, $L^\infty(\mathbb{R}^n, m)$ is not separable. (There is an uncountable set $\mathcal{F} \subset L^\infty$ such that $\|f - g\|_\infty = 1$ for all $f, g \in \mathcal{F}$ with $f \neq g$).

Solution. Let $\mathcal{G}$ be the set of all simple functions of the form $s = \sum_{j=1}^m c_j \chi_{R_j}$, where R_j is a rectangle $R_j = \prod_{i=1,\cdots,n}(a_{i,j}, b_{i,j})$, $a_{i,j} < b_{i,j}$ and $c_j, a_{i,j}, b_{i,j}$ are rational numbers for all $j = 1, \cdots, m$ and $i = 1, \cdots, n$. Then $\mathcal{G} \subset L^p(\mathbb{R}^n, m)$ is countable, and we will show that is dense in $L^p(\mathbb{R}^n, m)$.

By Proposition 5.1.7, it only remains to show that given $\epsilon > 0$ (and we assume $0 < \epsilon < 1$) and a simple function $\phi = \sum_{j=1}^k d_j \chi_{E_j}$ with $m(E_j) < \infty$ for each $j = 1, \cdots, k$ there exists a function $s \in \mathcal{G}$ such that $\|s - \phi\|_p < \epsilon$. But we know from Theorem 3.11.1 item (c) that for each $E_j \in \mathcal{M}$ with $m(E_j) < \infty$ and $\epsilon > 0$ there exists a finite collection of disjoint rectangles $\{R_{i,j}\}_{i=1}^\ell$ whose sides are open intervals such that $m(E_j \Delta \bigcup_{i=1}^\ell R_{i,j}) < \epsilon$. By shrinking or enlarging a little each side of each $R_{i,j}$ we can assume that then endpoint are rational numbers. Also, for each $j = 1, \cdots, k$ we can choose w_j rational such that $|d_j - w_j| < \epsilon$. Thus for $s = \sum_{j,i=1}^{k,\ell} w_j \chi_{R_{i,j}}$ we have $s \in \mathcal{G}$ and

$$\begin{aligned} \|s - \phi\|_p &\leqslant \sum_{j,i=1}^{k,\ell} \|(w_j - d_j)\chi_{R_{i,j}}\|_p + \sum_{j=1}^k \|d_j(\chi_{E_j} - \chi_{\bigcup_{i=1}^\ell R_{i,j}})\|_p \\ &\leqslant \max_{j=1,\cdots,k} |d_j - w_j| \sum_{j,i=1}^{k,\ell} m(R_{i,j})^{1/p} + \max_{j=1,\cdots,k} m\Big(E\Delta \bigcup_{i=1}^\ell R_{i,j}\Big)^{1/p} \sum_{j=1}^k |d_j| \end{aligned} \tag{5.2.1}$$

Now since $\max_{j=1,\cdots,k} |d_j - w_j| < \epsilon$, $\max_{j=1,\cdots,k} m\Big(E\Delta \bigcup_{i=1}^\ell R_{i,j}\Big)^{1/p} < \epsilon^{1/p} < \epsilon$ and

$$m\Big(\bigcup_{i=1}^\ell R_{i,j}\Big) \leqslant m\Big(E_j \cup \bigcup_{i=1}^\ell R_{i,j}\Big) \overset{(\star)}{\leqslant} m(E_j) + m\Big(E_j \Delta \bigcup_{i=1}^\ell R_{i,j}\Big) < m(E_j) + \epsilon < m(E_j) + 1,$$

where in $(\star)$ we have used $A \cup B = (A \cap B) \cup (A \Delta B) \subset A \cup A\Delta B$. Therefore we have from (5.2.1) that

$$\|s - \phi\|_p \leqslant C\epsilon,$$

where $C = \max\{k + \sum_{j=1}^k m(E_j), \sum_{j=1}^k |d_j|\}$. Thus $\mathcal{G}$ is dense in $L^p(\mathbb{R}^n, m)$ and $L^p(\mathbb{R}^n, m)$

is separable.

Now for $L^\infty(\mathbb{R}^n, m)$. Let $\{f_n\}$ be a countable sequence in $L^\infty(\mathbb{R}^n, m)$. Define a sequence A_n by

$$A_1 = \{x \in \mathbb{R}^n \colon \|x\| \leqslant 1\} \text{ and } A_n = \{x \in \mathbb{R}^n \colon n < \|x\| \leqslant n+1\} \text{ for } n \in \mathbb{N}.$$

Define also a sequence a_n by

$$a_n = \begin{cases} -1 & \text{if } f_n \geqslant 0 \text{ a.e. on } A_n \\ 1 & \text{otherwise.} \end{cases}$$

and finally define $g = \sum_{n=1}^\infty a_n \chi_{A_n}$. Since $|g| = \sum_{n=1}^\infty |a_n| \chi_{A_n} = \sum_{n=1}^\infty \chi_{A_n}$ we have $g \in L^\infty(\mathbb{R}^n, m)$ and $\|g\|_\infty = 1$. Now for each $n \in \mathbb{N}$, we have $|g - f_n| \geqslant 1$ a.e. on A_n and hence $\|g - f_n\|_\infty \geqslant 1$, so no countable subset of $L^\infty(\mathbb{R}^n, m)$ can be dense. Thus $L^\infty(\mathbb{R}^n, m)$ is not separable.

Another possible proof (using the hint) is to consider the set $\mathcal{F}$ consisting of functions f such that $f = 0$ or 1 in each A_n. Thus $\mathcal{F} \subset L^\infty(\mathbb{R}^n, m)$ is equivalent (bijective) to the set of all sequences $\{a_n\}$ such that $a_n = 0$ or 1, hence $\mathcal{F}$ is uncountable. Moreover, if $f, g \in \mathcal{F}$ and $f \neq g$, then $\|f - g\|_\infty = 1$. Hence no countable subset of $L^\infty(\mathbb{R}^n, m)$ can be dense in $L^\infty(\mathbb{R}^n, m)$.

EXERCISE 14. If $g \in L^\infty$, the operator defined by $Tf = fg$ is bounded in L^p for $1 \leqslant p \leqslant \infty$. Its operator norm is at most $\|g\|_\infty$, with equality if μ is semifinite.

Solution. For $1 \leqslant p < \infty$ we have

$$\int |Tf|^p = \int |fg|^p \leqslant \|g\|_\infty^p \int |f|^p = \|g\|_\infty^p \|f\|_p^p,$$

hence $\|Tf\|_p \leqslant \|g\|_\infty \|f\|_p$. For $p = \infty$ we have

$$|Tf(x)| = |f(x)g(x)| \leqslant \|g\|_\infty |f(x)| \leqslant \|g\|_\infty \|f\|_\infty \text{ a.e.}$$

and thus $\|Tf\|_\infty \leqslant \|g\|_\infty \|f\|_\infty$. Thus Tf is bounded on L^p and its operator norm is at most $\|g\|_\infty$.

If $g = 0$ a.e., then the equality is trivial. Now assume $g \neq 0$ a.e., μ semifinite and $1 \leqslant p < \infty$. Choose $\epsilon > 0$, set $A = \{x \in X \colon |g(x)| \geqslant \|g\|_\infty - \epsilon\}$ and choose $B \subset A$ with

$0 < \mu(B) < \infty$. Define $f = \mu(B)^{-1/p}\chi_B$. We have

$$\|f\|_p = \left(\mu(B)^{-1}\int_B 1\right)^{1/p} = 1,$$

and

$$\|Tf\|_p = \left(\mu(B)^{-1}\int_B |g|^p\right)^{1/p} \geqslant \|g\|_\infty - \epsilon,$$

since $p \geqslant 1$. Thus $\|T\| \geqslant \|g\|_\infty - \epsilon$. Since $\epsilon > 0$ is arbitrary, we obtain $\|T\| \geqslant \|g\|_\infty$, and thus $\|T\| = \|g\|_\infty$.

For $p = \infty$, set $f = \chi_B$. Hence $\|f\|_\infty = 1$ and

$$|Tf(x)| = |f(x)g(x)| = |\chi_B(x)g(x)| \geqslant \|g\|_\infty - \epsilon,$$

and thus $\|T\| \geqslant \|Tf\|_\infty \geqslant \|g\|_\infty - \epsilon$, and as before we conclude that $\|T\| = \|g\|_\infty$.

Exercise 15. **The Vitali Convergence Theorem.** Suppose $1 \leqslant p < \infty$ and $\{f_n\} \subset L^p$. In order for $\{f_n\}$ to be Cauchy in the L^p norm is it necessary and sufficient for the following three conditions to hold:

(i) $\{f_n\}$ is Cauchy in measure;

(ii) the sequence $\{|f_n|^p\}$ is uniformly integrable (see Exercise 11 in Section 4.4);

(iii) for every $\epsilon > 0$ there exists $E \subset X$ such that $\mu(E) < \infty$ and $\int_{E^c} |f_n|^p < \epsilon$ for all $n \in \mathbb{N}$.

(To prove the sufficiency: given $\epsilon > 0$, let E be as in (iii), and let $A_{mn} = \{x \in E \colon |f_m(x) - f_n(x)| \geqslant \epsilon\}$. Then the integrals of $|f_n - f_m|^p$ over $E \setminus A_{mn}$, $A_{m,n}$ and E^c are small when m and n are large - for three different reasons).

Solution. For the necessity: assume that $\{f_n\}$ is a Cauchy sequence in L^p. Then $f_n \to f$ in L^p for some $f \in L^p$. Using Exercise 9 we have $f_n \to f$ in measure and hence $\{f_n\}$ is Cauchy in measure. Since $\{|f_n|^p\}$ is in L^1 and converges to $|f|^p$ in L^1, using Exercise 11 of Section 4.4, the sequence $\{|f_n|^p\}$ is uniformly integrable. To prove that $\{f_n\}$ satisfies (iii), given $\epsilon > 0$ choose $n_0 \in \mathbb{N}$ such that $\|f_n - f\|_p < \epsilon^{1/p}$ for $n \geqslant n_0$. Define $g = \max\{f, f_1, \cdots, f_{n_0-1}\}$. Thus $g \in L^p$ and for $\eta > 0$ we set

$$E_\eta = \{x \in X \colon |g(x)| > \eta\}.$$

Then

$$\int |g|^p \geqslant \int_{E_\eta} |g|^p \geqslant \eta^p \mu(E_\eta),$$

and hence $\mu(E_\eta) \leqslant \eta^{-p}\|g\|_p^p < \infty$ for each $\eta > 0$. Moreover, since $|g|^p \chi_{E_\eta^c} \to 0$ a.e. as $\eta \to 0^+$, using the DCT we obtain $\int_{E_\eta^c} |g|^p \to 0$ as $\eta \to 0^+$, and we choose $\eta > 0$ small so that $\int_{E_\eta^c} |g|^p < \epsilon$ and set $E = E_\eta$. Hence

$$\int_{E^c} |f_n|^p \leqslant \int_{E^c} |g|^p < \epsilon \quad \text{ for } n = 1, \cdots, n_0 - 1.$$

Now for $n \geqslant n_0$ we have

$$\int_{E^c} |f_n|^p \leqslant 2^p \int_{E^c} |f|^p + 2^p \int_{E^c} |f_n - f|^p \leqslant 2^p \int_{E^c} |g|^p + 2^p \int |f_n - f|^p < 2^{p+1}\epsilon,$$

and hence $\mu(E) < \infty$ and $\int_{E^c} |f_n|^p < 2^{p+1}\epsilon$ for all $n \in \mathbb{N}$.

Now for the converse assume that $\{f_n\}$ satisfies conditions (i), (ii) and (iii). Given $\epsilon > 0$, using (iii) let $E \in \mathcal{M}$ be such that $0 < \mu(E) < \infty$ and $\int_{E^c} |f_n|^p < \epsilon$ for all $n \in \mathbb{N}$.

Define $A_{mn} = \{x \in E \colon |f_m(x) - f_n(x)| \geqslant \epsilon\}$. Then we have

$$\int_{E^c} |f_m - f_n|^p \leqslant 2^p \int_{E^c} |f_m|^p + 2^p \int_{E^c} |f_n|^p < 2^{p+1}\epsilon. \tag{5.2.2}$$

Using (ii), there exists $\delta > 0$ such that $\int_A |f_n|^p < \epsilon$ for all $A \in \mathcal{M}$ with $\mu(A) < \delta$ and for all $n \in \mathbb{N}$.

Since $\{f_n\}$ is Cauchy in measure there exists $n_0 \in \mathbb{N}$ such that $\mu(A_{mn}) < \delta$ for $m, n \geqslant n_0$, and hence

$$\int_{A_{mn}} |f_m - f_n|^p \leqslant 2^p \int_{A_{mn}} |f_m|^p + 2^p \int_{A_{mn}} |f_n|^p < 2^{p+1}\epsilon, \tag{5.2.3}$$

for all $n, m \geqslant n_0$.

We have $|f_m - f_n|^p \chi_{E \setminus A_{mn}} \leqslant \epsilon^p \chi_E \in L^1$ (since $\mu(E) < \infty$) and also $|f_m - f_n|^p \chi_{E \setminus A_{mn}} \to 0$ in measure as $m, n \to \infty$. Thus using Exercise 34 of Section 3.8 we obtain

$$\int_{E \setminus A_{mn}} |f_m - f_n|^p \to 0,$$

and hence there exists $n_1 \geqslant n_0$ such that

$$\int_{E \setminus A_{mn}} |f_m - f_n|^p \leqslant \epsilon, \tag{5.2.4}$$

for $m, n \geqslant n_1$. Thus for $m, n \geqslant n_1$, combining (5.2.2), (5.2.3) and (5.2.4) we obtain

$$\int |f_m - f_n|^p = \int_{E^c} |f_m - f_n|^p + \int_{A_{mn}} |f_m - f_n|^p + \int_{E \setminus A_{mn}} |f_m - f_n|^p < (2^{p+2} + 1)\epsilon,$$

which proves that $\{f_n\}$ is a Cauchy sequence in L^p.

Exercise 16. If $0 < p < 1$, the formula $\rho(f,g) = \int |f-g|^p$ defines a metric on L^p that makes L^p into a complete topological vector space (the proof of Theorem 5.1.6 still works for $p < 1$ if $\|f\|_p$ is replaced by $\int |f|^p$, as it uses only the triangle inequality and not the homogeneity of the norm).

Solution. To prove that d is a metric, it only remains to prove the triangle inequality. From (5.1.1) we have

$$|f-h|^p \leqslant (|f-g|+|g-h|)^p \leqslant |f-g|^p + |g-h|^p,$$

and hence $\rho(f,h) \leqslant \rho(f,g) + \rho(g,h)$ for all $f,g,h \in L^p$. Hence ρ is a metric. The proof of Theorem 5.1.6 remains unchanged for $0 < p < 1$ replacing $\|f\|_p$ for $\int |f|^p$, and thus L^p is a complete topological vector space.

5.3| The dual of L^p

Suppose that p and q are conjugate exponents, then Hölder's inequality shows that each $g \in L^q$ defines a bounded linear functional ϕ_g on L^p by

$$\phi_g(f) = \int fg,$$

and the operator norm of ϕ_g is at most $\|g\|_q$.

If $p = 2$ and we are thinking of L^2 as a HIlbert space, it is more appropriate to define $\phi_g(f) = \int f\overline{g}$. The same convention can be used for $p \neq 2$ without changing the results that will be presented below in an essential way.

In fact, the map $g \mapsto \phi_g$ is almost always an isometry from L^q to $(L^p)^*$, the dual space of L^p.

Proposition 5.3.1. *Suppose that p and q are conjugate exponents and $1 \leqslant q < \infty$. If $g \in L^q$, then*

$$\|g\|_q = \|\phi_g\| = \sup\left\{\left|\int fg\right| : \|f\|_p = 1\right\}.$$

If μ is semifinte, this result holds also for $q = \infty$.

Proof. From Hölder's inequality we have $\|\phi_g\| \leqslant \|g\|_q$ and the equality is trivial if $g = 0$ a.e. If $g \neq 0$ a.e. and $1 < q < \infty$, define

$$f = \frac{|g|^{q-1}}{\|g\|_q^{q-1}} \overline{\operatorname{sgn}\, g}.$$

Then

$$\|f\|_p^p = \frac{1}{\|g\|_q^{p(q-1)}} \int |g|^{(q-1)p} = \frac{\|g\|_q^q}{\|g\|_q^q} = 1,$$

recalling that $(q-1)p = q$, since p and q are conjugate exponents. Thus we have

$$\|\phi_g\| \geqslant \phi_g(f) = \int fg = \int \frac{|g|^{q-1}}{\|g\|_q^{q-1}} \overline{\operatorname{sgn}\, g}\; g = \frac{1}{\|g\|_q^{q-1}} \int |g|^q = \frac{\|g\|_q^q}{\|g\|_q^{q-1}} = \|g\|_q,$$

hence $\|\phi_g\| = \|g\|_q$.

If $q = 1$, define $f = \overline{\operatorname{sgn}\, g}$. We have $\|f\|_\infty = 1$ and $\int fg = \|g\|_1$.

Now assume that μ is semifinite and let $q = \infty$. For $\epsilon > 0$ consider $A = \{x \in X \colon |g(x)| > \|g\|_\infty - \epsilon\}$. Then $\mu(A) > 0$ and since μ is semifinite, there exists $B \subset A$ with $0 < \mu(B) < \infty$. Let

$$f = \frac{1}{\mu(B)} \chi_B \overline{\operatorname{sgn}\, g}.$$

Then $\|f\|_1 = 1$ and

$$\|\phi_g\| \geqslant \int fg = \frac{1}{\mu(B)} \int_B |g| \geqslant \|g\|_\infty - \epsilon,$$

and since $\epsilon > 0$ is arbitrary, we obtain $\|\phi_g\| = \|g\|_\infty$. ■

Conversely, if $f \mapsto \int fg$ is a bounded linear function of L^p, then $g \in L^q$ in almost all cases. In fact, we have the following stronger result.

Theorem 5.3.2. *Let p and q be conjugate exponents. Suppose that g is a measurable function on X such that $fg \in L^1$ for all f in the space Σ of simple functions that vanish outside a set of finite measure, and the quantity*

$$M_q(g) = \sup \left\{ \left| \int fg \right| : f \in \Sigma \text{ and } \|f\|_p = 1 \right\}$$

is finite. Also, suppose either that $S_g = \{x \in X \colon g(x) \neq 0\}$ is σ-finite or that μ is semifinite. Then $g \in L^q$ and $M_q(g) = \|g\|_q$.

Proof. First we note that if f is any function in Σ we have

$$\left| \int fg \right| \leqslant M_q(g) \|f\|_p.$$

In fact, if $f = 0$ a.e., then this inequality is trivial. If $f \neq 0$ a.e., then $\tilde{f} = \frac{f}{\|f\|_p}$ is also in Σ and $\|\tilde{f}\|_p = 1$, hence

$$\frac{1}{\|f\|_p}\left|\int fg\right| = \left|\int \tilde{f}g\right| \leqslant M_q(g),$$

which concludes this first claim.

No we remark that if h is a bounded measurable function that vanishes outside a set E of finite measure and $\|h\|_p = 1$, then $\left|\int hg\right| \leqslant M_q(g)$. In fact, using Proposition 3.1.22 there exists a sequence $\{f_n\}$ of simple functions such that $f_n \to h$ uniformly on X (since h is bounded) and $|f_n| \leqslant |h|$ for all $n \in \mathbb{N}$, in particular f_n vanishes outside E and $f_n \in L^p$ for all $n \in \mathbb{N}$. We have $f_n^p \to h^p$ uniformly in X and $|f_n|^p \leqslant |h|^p \in L^1$ for all $n \in \mathbb{N}$ and using once the DCT we have $\|f_n\|_p \to \|h\|_p = 1$.

Also $f_n g \to hg$ and $|f_n g| \leqslant |h||g| \leqslant \|h\|_\infty g\chi_E$, since h vanishes outside E. Since $g\chi_E \in L^1$ by hypothesis, we can use again the DCT to obtain

$$\left|\int hg\right| = \left|\lim_{n\to\infty}\int f_n g\right| = \lim_{n\to\infty}\left|\int f_n g\right| \leqslant \lim_{n\to\infty} M_q(g)\|f_n\|_p = M_q(g).$$

Now we can begin the proof of the result. Consider first $q < \infty$. We may assume that S_g is σ-finite, as this condition automatically holds when μ is semifinite (see Exercise 17). Let $\{E_n\}$ be an increasing sequence of sets of finite measure such that $S_g = \bigcup_{n=1}^\infty E_n$. Let $\{s_n\}$ be a sequence of simple functions such that $s_n \to g$ pointwise and $|s_n| \leqslant |g|$, and let $g_n = s_n\chi_{E_n}$. Then $g_n \to g$ pointwise, $|g_n| \leqslant |g|$ and g_n vanishes outside E_n. Let

$$f_n = \frac{|g_n|^{q-1}\overline{\text{sgn } g}}{\|g_n\|_q^{q-1}}.$$

Then, as in the proof of Proposition 5.3.1 we have $\|f_n\|_p = 1$, and by Fatou's Lemma we obtain

$$\|g\|_q \leqslant \liminf \|g_n\|_q = \liminf \int |f_n g_n| \leqslant \liminf \int |f_n g| = \liminf \int f_n g \leqslant M_q(g),$$

and for the last estimate we used our second claim. Using Hölder's inequality we obtain $M_q(g) \leqslant \|g\|_q$, and the proof is complete for the case $q < \infty$.

Now suppose $q = \infty$. Given $\epsilon > 0$, let $A = \{x \in X \colon |g(x)| \geqslant M_\infty(g) + \epsilon\}$. If $\mu(A) > 0$, we can choose $B \subset A$ with $0 < \mu(B) < \infty$ (either because μ is semifinite or because $A \subset S_g$). Setting $f = \mu(B)^{-1}\chi_B\overline{\text{sgn } g}$, we have $\|f\|_1 = 1$ and $\int fg = \mu(B)^{-1}\int_B |g| \geqslant M_\infty(g) + \epsilon$, which contradicts our second claim from the beginning of the proof. Hence $\mu(A) = 0$ and hence $\|g\|_\infty \leqslant M_\infty(g) + \epsilon$ for each $\epsilon > 0$, and the proof is complete. ■

The last and deepest part of the description of $(L^p)^*$ is the fact that the map $g \mapsto \phi_g$ is, in almost all cases, a surjection.

THEOREM 5.3.3. *Let p and q be conjugate exponents. If $1 < p < \infty$, for each $\phi \in (L^p)^*$ there exists $g \in L^q$ such that $\phi(f) = \int fg$ for all $f \in L^p$, and hence L^q is isometrically isomorphic to $(L^p)^*$. The same conclusion holds for $p = 1$ provided μ is σ-finite.*

Proof. First suppose that μ is finite. Thus all simple functions are in L^p. If $\phi \in (L^p)^*$ and E is a measurable set, let $\nu(E) = \phi(\chi_E)$. For any disjoint sequence $\{E_j\}$, if $E = \bigcup_{j=1}^{\infty} E_j$ we have $\chi_E = \sum_{j=1}^{\infty} \chi_{E_j}$ where the series converges in the L^p norm, since

$$\left\|\chi_E - \sum_{j=1}^{\infty} \chi_{E_j}\right\|_p = \| \sum_{j=n+1}^{\infty} \chi_{E_j}\|_p = \mu\Big(\bigcup_{j=n+1}^{\infty} E_j\Big)^{1/p} \to 0 \text{ as } n \to \infty,$$

and here the assumption $p < \infty$ is crucial. Hence, since ϕ is linear and continuous,

$$\nu(E) = \phi(\chi_E) = \phi\Big(\sum_{j=1}^{\infty} \chi_{E_j}\Big) = \sum_{j=1}^{\infty} \phi(\chi_{E_j}) = \sum_{j=1}^{\infty} \nu(E_j),$$

so that ν is a complex measure. Also, if $\mu(E) = 0$ then $\chi_E = 0$ as an element of L^p, so $\nu(E) = 0$, that is, $\nu \ll \mu$. By the Radon-Nikodym theorem there exists $g \in L^1(\mu)$ such that $\phi(\chi_E) = \nu(E) = \int_E g d\mu$ for all E, and hence $\phi(f) = \int fg d\mu$ for all simple functions f. Moreover

$$\left|\int fg\right| = |\phi(f)| \leqslant \|\phi\| \|f\|_p,$$

and Theorem 5.3.2 implies that $g \in L^q$. Hence, since the set of simple functions are dense in L^p, using the DCT we obtain $\phi(f) = \int fg$ for all $f \in L^p$.

Now we suppose that μ is σ-finite. Let $\{E_n\}$ be an increasing sequence of sets such that $0 < \mu(E_n) < \infty$ and $X = \bigcup_{n=1}^{\infty} E_n$, and we will identify $L^p(E_n)$ and $L^q(E_n)$ as subsets of $L^p(X)$ and $L^q(X)$, respectively, consisting of functions that vanish outside E_n. The preceding argument shows that for each $n \in \mathbb{N}$, there exists $g_n \in L^q(E_n)$ such that $\phi(f) = \int fg_n$ for all $f \in L^p(E_n)$, and $\|g_n\|_q = \|\phi|_{L^p(E_n)}\| \leqslant \|\phi\|$. The function g_n is unique modulo alterations on null sets, so $g_n = g_m$ a.e. on E_n for $n < m$, and we can define g a.e. on X by setting $g = g_n$ on E_n, for each $n \in \mathbb{N}$. By the MCT we have $\|g\|_p = \lim \|g_n\|_q \leqslant \|\phi\|$, so $g \in L^q$. Moreover, if $f \in L^p$, then by the DCT, $f\chi_{E_n} \to f$ in L^p and hence $\phi(f) = \lim \phi(f\chi_{E_n}) = \lim \int_{E_n} fg = \int fg$.

Finally, suppose that μ is arbitrary and $p > 1$, so that $q < \infty$. As above, for each σ-finite set $E \subset X$, there exists an a.e.-unique $g_E \in L^q(E)$ such that $\phi(f) = \int fg_E$ for all $f \in L^p(E)$

and $\|g_E\|_q \leqslant \|\phi\|$. If F is σ-finite and $F \supset E$, then $g_F = g_E$ a.e. on E, so $\|g_F\|_q \geqslant \|g_E\|_q$. Defining

$$M = \sup\{\|g_E\|_q \colon E \text{ is a } \sigma\text{-finite set}\},$$

we have $M \leqslant \|\phi\| < \infty$. Choose a sequence $\{E_n\}$ of σ-finite sets such that $\|g_{E_n}\|_q \to M$ and set $F = \bigcup_{n=1}^{\infty} E_n$. Then F is σ-finite and

$$\|g_{E_n}\|_q \leqslant \|g_F\|_q \leqslant M \quad \text{ for all } n \in \mathbb{N},$$

thus making $n \to \infty$ we obtain $\|g_F\|_q = M$. Now if A is a σ-finite set with $A \supset F$, we have $|g_A|^q = |g_F|^q + |g_{A\setminus F}|^q$ a.e. on X (since $g_A = g_F$ a.e. on F and $g_A = g_{A\setminus F}$ a.e. on $A \setminus F$) and hence

$$\int |g_F|^q + \int |g_{A\setminus F}|^q = \int |g_A|^q \leqslant M^q = \int |g_F|^q,$$

and hence $g_{A\setminus F} = 0$ a.e., which in turn implies that $g_A = g_F$ a.e. on X (here the fact that $q < \infty$ is used).

If $f \in L^p$, then $A = F \cup \{x \in X \colon f(x) \neq 0\}$ is σ-finite, so

$$\phi(f) = \int f g_A = \int f g_F,$$

thus we take $g = g_F$ and the proof is complete. ∎

Corollary 5.3.4. *For $1 < p < \infty$, L^p is reflexive.*

Now we conclude this section with some remarks regarding the cases $p = 1$ and $p = \infty$.

Remark 5.3.5. *For any measure μ, the correspondence $g \mapsto \phi_g$ maps L^∞ into $(L^1)^*$, but in general is neither injective nor surjective. Injectivity fails when μ is not semifinite.*

Indeed, if $E \subset X$ is a set of infinite measure with no subset of positive finite measure and $f \in L^1$, then $\{x \in X \colon f(x) \neq 0\}$ is σ-finite and hence it intersects E in a null set. It follows that $\phi_{\chi_E} = 0$ although $\chi_E \neq 0$ in L^∞.

This problem, however, can be fixed, by redefining L^∞ - see Exercises 23 and 24.

The failure of surjectivity is more subtle and we will give an example (see also Exercise 25). Let X be an uncountable set, μ the counting measure on $(X, \mathcal{P}(X))$, $\mathcal{M}$ the σ-algebra of countable or co-countable sets, and $\mu_0 =$ the restriction of μ to $\mathcal{M}$. Every $f \in L^1(\mu)$ must vanish outside a countable set, and hence $L^1(\mu) = L^1(\mu_0)$.

On the other hand, $L^\infty(\mu)$ consists of all bounded functions on X, whereas $L^\infty(\mu_0)$ consists of those bounded functions that are constant except on a countable set. With this in mind, it is easy to see that the dual of $L^1(\mu_0)$ is $L^\infty(\mu)$, and not the smaller space $L^\infty(\mu_0)$.

Remark 5.3.6. *As for the case $p = \infty$, the map $g \mapsto \phi_g$ is always an isometric injection of L^1 into $(L^\infty)^*$ by Proposition 5.3.1, but is it almost never a surjection.*

Indeed, let $X = [0,1]$ with $\mu = m$ the Lebesgue measure. The map $f \mapsto f(0)$ is a bounded linear functional on $C(X)$, which we regard as a subspace of L^∞. Using the Hahn-Banach Theorem, there exists $\phi \in (L^\infty)^*$ such that $\phi(f) = f(0)$ for all $f \in C(X)$. To see that ϕ cannot be given by an integration against an L^1 function, consider the functions $f_n \in C(X)$ defined by $f_n(x) = \max\{1 - nx, 0\}$. Then $\phi(f_n) = f_n(0) = 1$ for all $n \in \mathbb{N}$, but $f_n(x) \to 0$ for all $x > 0$ and $|f_n(x)| \leqslant 1 \in L^1([0,1])$ for all $n \in \mathbb{N}$, so by the DCT we have $\int f_n g \to 0$ for all $g \in L^1$.

5.4| Solved exercises from [1, Page 191]

Exercise 17. With notation as in Theorem 5.3.2, if μ is semifinite, $q < \infty$ and $M_q(g) < \infty$, then $\{x \in X \colon |g(x)| > \epsilon\}$ has finite measure for all $\epsilon > 0$ and hence S_g is σ-finite.

Solution. Assume that for a given $\epsilon > 0$ we have $\mu(\{x \in X \colon |g(x)| > \epsilon\}) = \infty$. Then using Exercise 14 of Section 2.2, given $c > 0$ there exists $F \subset \{x \in X \colon |g(x)| > \epsilon\}$ with $c < \mu(F) < \infty$. Assume $1 < p < \infty$, and define

$$f = \mu(F)^{-1/p} \chi_F \overline{\operatorname{sgn} g}.$$

We have $\|f\|_p = 1$ and f is a bounded measurable function that vanishes outside F. Thus we obtain

$$M_q(g) \geqslant \left| \int fg \right| = \mu(F)^{-1/p} \int_F |g| \geqslant \epsilon \mu(F)^{1-1/p} \geqslant \epsilon c^{1/q},$$

and thus making $c \to \infty$ we obtain $M_q(g) = \infty$, and contradicts the hypothesis.

For $p = \infty$, define $f = \chi_F \overline{\operatorname{sgn} g}$. We have $\|f\|_\infty = 1$ and f is a bounded measurable function that vanishes outside F. As before

$$M_\infty(g) \geqslant \epsilon c,$$

and making $c \to \infty$ we obtain a contradiction. Hence $\{x \in X \colon |g(x)| > \epsilon\}$ has finite measure for all $\epsilon > 0$. Since

$$S_g = \{x \in X \colon g(x) \neq 0\} = \bigcup_{n=1}^{\infty} \{x \in X \colon |g(x)| > 1/n\},$$

we obtain S_g σ-finite.

Exercise 18, 19, 20, 21 and 22. These exercises make use of Functional Analysis topics, which are not seen in this course.

Exercise 23. Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \in \mathcal{M}$ is called **locally null** if $\mu(E \cap F) = 0$ for every $F \in \mathcal{M}$ such that $\mu(F) < \infty$. If $f : X \to \mathbb{C}$ is a measurable function, define

$$\|f\|_* = \inf\{a \geqslant 0 : \{x \in X : |f(x)| > a\} \text{ is locally null}\},$$

and let $\mathcal{L}^\infty = \mathcal{L}^\infty(X, \mathcal{M}, \mu)$ be the space of all measurable f such that $\|f\|_* < \infty$. We consider $f, g \in \mathcal{L}^\infty$ to be identical if $\{x \in X : f(x) \neq g(x)\}$ is locally null.

(a) If E is locally null, then $\mu(E)$ is either 0 or ∞. If μ is semifinite, then every locally null set is null.

(b) $\|\cdot\|_*$ is a norm on $\mathcal{L}^\infty$ that makes $\mathcal{L}^\infty$ into a Banach space. If μ is semifinite then $\mathcal{L}^\infty = L^\infty$.

Solution to (a). If E is locally null and $0 < \mu(E) < \infty$, taking $F = E$ in the definition of locally null set we have $\mu(E) = \mu(E \cap E) = 0$, which is a contradiction. Hence $\mu(E)$ is either 0 or ∞.

Now assume that μ is semifinite and E is a locally null set which is not null. Then $\mu(E) = \infty$ and since μ is semifinite, there exists $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$. But since E is locally null we have $\mu(F) = \mu(E \cap F) = 0$, which is a contradiction. Therefore E must be null.

Solution to (b). Before we proceed, we prove that the infimum is attained. If $\|f\|_* = \infty$, there is nothing to do. Now if $\alpha := \|f\|_* < \infty$, than we can construct a decreasing sequence $\{a_n\}$ such that $a_n \to \alpha$ ($a_n \geqslant \alpha$ for all $n \in \mathbb{N}$) and $\{x \in X : |f(x)| > a_n\}$ is a locally null set. Hence

$$\{x \in X : |f(x)| > \alpha\} = \bigcup_{n=1}^{\infty} \{x \in X : |f(x)| > a_n\}.$$

Thus if $F \in \mathcal{M}$ has $\mu(F) < \infty$ we have

$$\mu(F \cap \{x \in X : |f(x)| > \alpha\}) \leqslant \sum_{n=1}^{\infty} \mu(F \cap \{x \in X : |f(x)| > a_n\}) = 0,$$

and hence $\{x \in X : |f(x)| > \alpha\}$ is also a locally null set, and the infimum is attained.

Secondly, we prove that the relation $f = g$ in $\mathcal{L}^\infty$ if $\{x \in X\colon f(x) \neq g(x)\}$ is in fact an equivalence relation. The reflexivity and symmetry are clear. If $f = g$ and $g = h$ in $\mathcal{L}^\infty$ then

$$\{x \in X\colon f(x) \neq h(x)\} \subset \{x \in X\colon f(x) \neq g(x)\} \cup \{x \in X\colon g(x) \neq h(x)\},$$

and hence $\{x \in X\colon f(x) \neq h(x)\}$ is locally null. Thus $f = h$ in $\mathcal{L}^\infty$, and this relation defines an equivalence relation in $\mathcal{L}^\infty$.

Now we prove that $\|\cdot\|_*$ is a norm on $\mathcal{L}^\infty$.

(i) $\|f\|_* = 0$ iff $f = 0$ in $\mathcal{L}^\infty$.

In fact if $f = 0$ in $\mathcal{L}^\infty$ then $\{x \in X\colon f(x) \neq 0\} = \{x \in X\colon |f(x)| > 0\}$ is null, and therefore is locally null. Hence $\|f\|_* = 0$. Conversely, if $\|f\|_* = 0$ then $\{x \in X\colon |f(x)| > 0\} = \{x \in X\colon f(x) \neq 0\}$ is locally null, and thus $f = 0$ in $\mathcal{L}^\infty$.

(ii) $\|cf\|_* = |c|\|f\|_*$ for each $c \in \mathbb{C}$ and $f \in \mathcal{L}^\infty$.

If $c = 0$ then $cf = 0$ and $\{x \in X\colon |cf(x)| > 0\} = \varnothing$, hence $\|cf\|_* = 0 = |c|\|f\|_*$. Assume $c \neq 0$, then $\{x \in X\colon |cf(x)| > a\} = \{x \in X\colon |f(x)| > a/|c|\}$. Thus

$$\begin{aligned}
\|cf\|_* &= \inf\{a \geqslant 0\colon \{x \in X\colon |cf(x)| > a\} \text{ is locally null}\}\\
&= \inf\{a \geqslant 0\colon \{x \in X\colon |f(x)| > a/|c|\} \text{ is locally null}\}\\
&= \inf\{a|c| \geqslant 0\colon \{x \in X\colon |f(x)| > a\} \text{ is locally null}\}\\
&= |c| \inf\{a \geqslant 0\colon \{x \in X\colon |f(x)| > a\} \text{ is locally null}\}\\
&= |c|\|f\|_*.
\end{aligned}$$

(iii) If $f, g \in \mathcal{L}^\infty$, then $\|f + g\|_* \leqslant \|f\|_* + \|g\|_*$.

Note that

$$\begin{aligned}
&\{x \in X\colon |f(x) + g(x)| > \|f\|_* + \|g\|_*\}\\
&\qquad\qquad \subset \{x \in X\colon |f(x)| > \|f\|_*\} \cup \{x \in X\colon |g(x)| > \|g\|_*\},
\end{aligned}$$

and hence $\{x \in X\colon |f(x) + g(x)| > \|f\|_* + \|g\|_*\}$ is locally null, thus $\|f + g\|_* \leqslant \|f\|_* + \|g\|_*$.

Finally we prove that $(\mathcal{L}^\infty, \|\cdot\|_*)$ is a Banach space. To that end let $\{f_n\}$ be a Cauchy sequence in $\mathcal{L}^\infty$. Define $E = \bigcap_{n,m=1}^\infty \{x \in X\colon |f_n(x) - f_m(x)| \leqslant \|f_n - f_m\|_*\}$. Then

$$E^c = \bigcup_{n,m=1}^{\infty} \{x \in X\colon |f_n(x) - f_m(x)| > \|f_n - f_m\|_*\},$$

and E^c is locally null, since all sets on the right hand side are locally null. Now given $\epsilon > 0$

there exists $n_0 \in \mathbb{N}$ such that $\|f_n - f_m\|_* < \epsilon$ for $n, m \geqslant n_0$. Thus for $x \in E$ we have

$$|f_n(x) - f_m(x)| \leqslant \|f_n - f_m\|_* < \epsilon \quad \text{for all } n, m \geqslant n_0,$$

thus $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{C}$. We can define

$$f(x) = \begin{cases} \lim\limits_{n\to\infty} f_n(x) & \text{for } x \in E, \\ 0 & \text{for } x \in E^c. \end{cases}$$

Thus f is a measurable function and making $m \to \infty$, for $x \in E$ we have

$$|f_n(x) - f(x)| \leqslant \epsilon,$$

and therefore $f_n \to f$ uniformly on E. Hence $\{x \in X : |f_n(x) - f(x)| > \epsilon\} \subset E^c$ for $n \geqslant n_0$, therefore $\|f_n - f\|_* \leqslant \epsilon$ for $n \geqslant n_0$ and hence $f_n \to f$ in $\mathcal{L}^\infty$. Thus $\mathcal{L}^\infty$ is a Banach space.

Using item (a), it is easy to see that, when μ is semifinite, the concepts of locally null sets and null sets are the same. Also, the concept of equality a.e. is equality in $\mathcal{L}^\infty$. Hence $\mathcal{L}^\infty = L^\infty$.

Exercise 24. If $g \in \mathcal{L}^\infty$ (see Exercise 23), then $\|g\|_* = \sup\{\left|\int fg\right| : \|f\|_1 = 1\}$, so the map $g \mapsto \phi_g$ is an isometry from $\mathcal{L}^\infty$ into $(L^1)^*$. Conversely, if $M_\infty(g) < \infty$ as in Theorem 5.3.2, then $g \in \mathcal{L}^\infty$ and $M_\infty(g) = \|g\|_*$.

Solution. Let $g \in \mathcal{L}^\infty$ and $f \in L^1$. We will prove that Hölder's inequality holds in this case. Let $F = \{x \in X : f(x) \neq 0\}$, which is σ-finite since $f \in L^1$, and write $F = \bigcup_{n=1}^\infty F_n$ with $\{F_n\} \subset \mathcal{M}$ a disjoint sequence and $\mu(F_n) < \infty$ for all $n \in \mathbb{N}$.

Hence $\mu(F_n \cap \{x \in X : |g(x)| > \|g\|_*\}) = 0$, since $\{x \in X : |g(x)| > \|g\|_*\}$ is locally null. Thus $|g(x)| \leqslant \|g\|_*$ a.e. on F_n for each $n \in \mathbb{N}$, hence $|g(x)| \leqslant \|g\|_*$ a.e. on F and thus

$$\int |fg| = \int_F |fg| \leqslant \int_F |f| \|g\|_* = \int |f| \|g\|_* = \|f\|_1 \|g\|_*.$$

From this it follows easily that $\alpha := \sup\{\left|\int fg\right| : \|f\|_1 = 1\} \leqslant \|g\|_*$.

Now given $\epsilon > 0$, let $A = \{x \in X : |g(x)| \geqslant \alpha + \epsilon\}$. Assume that A is not locally null, then there exists $F \in \mathcal{M}$ with $\mu(F) < \infty$ with $\mu(A \cap F) > 0$ (and clearly we have $\mu(A \cap F) \leqslant \mu(F) < \infty$ and we can assume that $F \subset A$, for otherwise we take $A \cap F$ instead of F). Setting $f = \mu(F)^{-1}\chi_F\overline{\operatorname{sgn} g}$ we obtain $\|f\|_1 = 1$ and

$$\alpha \geqslant \int fg = \mu(F)^{-1}\int_F |g| \geqslant \alpha + \epsilon,$$

which is a contradiction. Hence A is locally null, which implies that $\|g\|_* \leqslant \alpha + \epsilon$. Since $\epsilon > 0$ is arbitrary we obtain $\|g\|_* \leqslant \alpha$. Therefore $\|g\|_* = \alpha = \sup\{\left|\int fg\right| : \|f\|_1 = 1\}$.

This last part work as well with $M_\infty(g)$ replacing α, and we obtain $\|g\|_* \leqslant M_\infty(g)$, so $g \in \mathcal{L}^\infty$. Hölder's inequality shows that $M_\infty(g) \leqslant \|g\|_*$ and the equality holds.

Exercise 25. Suppose μ is decomposable (see Exercise 15 in Section 4.4). Then every $\phi \in (L^1)^*$ is of the form $\phi(f) = \int fg$ for some $g \in \mathcal{L}^\infty$, and hence $(L^1)^* \cong \mathcal{L}^\infty$ (see Exercises 23 and 24), where $\cong$ means they are isometrically isomorphic. (If $\mathcal{F}$ is a decomposition of μ and $f \in L^1$, there exists $\{E_j\} \subset \mathcal{F}$ such that $f = \sum_{j=1}^\infty f\chi_{E_j}$ where the series converges in L^1.)

Solution. Let $\mathcal{F} \subset \mathcal{M}$ a decomposition of μ (see the properties of $\mathcal{F}$ in Exercise 15 of Section 4.4). In each $F \in \mathcal{F}$, we identify $\mathcal{L}^\infty(F)$ with the subset of $\mathcal{L}^\infty$ composed by function that are zero (in $\mathcal{L}^\infty$) outside F. Since $\mu(F) < \infty$, using item (b) of Exercise 23 we have $\mathcal{L}^\infty(F) = L^\infty(F)$. Also if $h \in L^\infty(F)$ then $\|h\|_{L^\infty(F)} = \|h\|_{*,\mathcal{L}^\infty(F)}$ and

$$\begin{aligned}\{x \in X : |h(x)| > \|h\|_{L^\infty(F)}\} &= \{x \in F : |h(x)| > \|h\|_{L^\infty(F)}\} \cup \{x \in F^c : |h(x)| > \|h\|_{L^\infty(F)}\}\\ &\subset \{x \in F : |h(x)| > \|h\|_{L^\infty(F)}\} \cup \{x \in F^c : h(x) \neq 0\}\\ &= \{x \in F : |h(x)| > \|h\|_{*,\mathcal{L}^\infty(F)}\} \cup \{x \in F^c : h(x) \neq 0\},\end{aligned}$$

thus $\{x \in X : |h(x)| > \|h\|_{L^\infty(F)}\}$ is locally null, then $\|h\|_* \leqslant \|h\|_{L^\infty(F)}$.

Also, we identify $L^1(F)$ with the subset of L^1 composed of functions that are zero a.e. outside F.

From Theorem 5.3.3, for each $F \in \mathcal{F}$ we obtain a function $g_F \in L^\infty(F)$ such that $\|g_F\|_* \leqslant \|g_F\|_\infty \leqslant \|\phi\|$ and $\phi(f) = \int fg_F$ for each $f \in L^1(F)$.

Define $g = g_F$ in each $F \in \mathcal{F}$. For a Borelian $B \subset \mathbb{C}$, we have

$$g^{-1}(B) \cap F = (g_F)^{-1}(B) \in \mathcal{M}$$

since g_F is measurable for each F, and property (iv) of the definition of decomposability, we obtain g measurable. Also

$$\{x \in X : |g(x)| > a\} = \bigcup_{F \in \mathcal{F}} \{x \in F : |g_F(x)| > a\}.$$

Taking $a = \sup_{F\in\mathcal{F}} \|g_F\|_*$, we have $a \leqslant \|\phi\|$ and the above inclusion gives us $\|g\|_* \leqslant a \leqslant \|\phi\|$.

Now given $f \in L^1$, for each $n \in \mathbb{N}$, the set $A_n = \{x \in X \colon |f(x)| > 1/n\}$ has finite measure and using item (iii) of definition of decomposability we obtain $\mu(A_n) = \sum_{F\in\mathcal{F}} \mu(A_n \cap F)$, and since this sum is finite, we obtain $\mu(A_n \cap F) = 0$ for all except a countable number of $F \in \mathcal{F}$. The collection of all these sets (for all $n \in \mathbb{N}$) is also countable subset of $\mathcal{F}$, and we will enumerate them as $\{E_j\}$.

We now will prove that $f = 0$ a.e. in $B = \left(\bigcup_{j=1}^\infty E_j\right)^c = \bigcup_{F\notin\{E_j\}} F$. For $F \notin \{E_j\}$ and $A = \{x \in X \colon f(x) \neq 0\}$ we have

$$A \cap F = \bigcup_{n=1}^\infty A_n \cap F,$$

and hence $\mu(A\cap F) = 0$. Now assume that f is not zero a.e. in B, that is, there exists $C \subset B$ with $\mu(C) > 0$ such that $\mu(A\cap C) > 0$. Since $A\cap C = \bigcup_{n=1}^\infty (A_n \cap C)$ we have $\mu(A_n \cap C) > 0$ for some $n \in \mathbb{N}$. Since $\mu(A_n \cap C) < \infty$ we obtain

$$\mu(A_n \cap C) = \sum_{F\in\mathcal{F}} \mu(A_n \cap C \cap F),$$

and hence $\mu(A_n \cap C \cap F) > 0$ for some $F \in \mathcal{F}$. Since $C \subset B$, this set F cannot be any $\{E_j\}$, but then

$$0 < \mu(A_n \cap C \cap F) \leqslant \mu(A \cap F) = 0,$$

which is a contradiction, hence $f = 0$ a.e. in B. Hence $f = \sum_{j=1}^\infty f\chi_{E_j}$ a.e. and since $\left|f - \sum_{j=1}^n f\chi_{E_j}\right| \leqslant 2|f| \in L^1$, from the DCT we obtain

$$\int \left| f - \sum_{j=1}^n f\chi_{E_j} \right| \to 0 \qquad \text{as } n \to \infty,$$

that is $f = \sum_{j=1}^\infty f\chi_{E_j}$ in L^1, thus

$$\begin{aligned}\phi(f) &= \sum_{j=1}^\infty \phi(f\chi_{E_j}) = \sum_{j=1}^\infty \int_{E_j} fg_{E_j} = \sum_{j=1}^\infty \int fg\chi_{E_j}\\ &= \lim_{n\to\infty} \sum_{j=1}^n \int fg\chi_{E_j} = \lim_{n\to\infty} \int fg\chi_{\bigcup_{j=1}^n E_j},\end{aligned}$$

and since $fg\chi_{\bigcup_{j=1}^n E_j} \to fg$ a.e. and $|fg\chi_{\bigcup_{j=1}^n E_j}| \leqslant |fg| \in L^1$ (using Hölder for $f \in L^1$ an

$g \in \mathcal{L}^\infty$), from the DCT applied in the last equality we obtain

$$\phi(f) = \lim_{n\to\infty} \int fg\chi_{\bigcup_{j=1}^n E_j} = \int fg.$$

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