Fractional Differential Equations and Blowing Up Solutions

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Abstract

In this work we discuss issues regarding solutions of fractional differential equations (see [1, 2, 3, 5] for more details concerning this subject). Concretely, consider the following Cauchy problem

$$\begin{cases} cD_t^\alpha u(t) = f(t, u(t)), \;\; t > 0 \\ u(0) = u_0 \in X, \end{cases} \tag{1}$$

where X is a Banach space, $\alpha \in (0,1)$, cD_t^α is Caputo's fractional derivative and $f : \mathbb{R}^+ \times X \to X$ is a suitable function (see [4] for details of this fractional derivative).

A consistent notion of local and global solution of the above fractional Cauchy problem can be described by the following sentences:

i) A continuous function $u : [0,\infty) \to X$ is a global solution of (1) if

$$u \in C^\alpha([0,\tau], X) := \{u \in C([0,\tau], X) : cD_t^\alpha u \in C([0,\tau], X)\} \tag{2}$$

for every $\tau > 0$ and satisfies the equations of (1).

ii) If there exists $0 < \tau < \infty$ such that a continuous function $u : [0,\tau] \to X$ belongs to $C^\alpha([0,\tau], X)$ and satisfies (1) for $t \in [0,\tau]$, we say that u is a local solution to problem (1) in the interval $[0,\tau]$.

Bearing these definitions in mind, it is possible to analyze the existence of local solutions for problem (1) by relating them with fixed points of an integral equation, as described in the following lemma.

Lemma 1 *Assume that $\alpha \in (0,1)$, $f : [0,\infty) \times X \to X$ is continuous and $u_0 \in X$.*

i) Let $u : [0,\infty) \to X$ be a continuous function. Then u is a global solution of (1) if, and only if, u satisfies

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s))\, ds, \tag{3}$$

for every $t \geq 0$.

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ii) Consider $\tau > 0$ and $u : [0, \tau] \to X$ a continuous function. Then u is a local solution of (1) in $[0, \tau]$ if, and only if, u satisfies (3) for every $t \in [0, \tau]$.

By proving a classical theorem of uniqueness and existence, its possible to study the continuation to a maximal interval of existence and in some cases, obtains a dichotomy property concerning the "longtime behavior" of the solution. More clearly, we can prove the following result.

Theorem 2 *Let $\alpha \in (0, 1)$, X be a Banach space and $f : [0, \infty) \times X \to X$ a continuous function which is locally Lipschitz and maps bounded sets onto bounded sets. Then problem (1) has a global solution in the interval $[0, \infty)$ or there exists a value $\omega \in (0, \infty)$ such that problem (1) has a local solution $u : [0, \omega) \to X$ that does not admit a continuation and yet satisfies*

$$\limsup_{t \to \omega^-} \|u(t)\|_X = \infty.$$

Thus, it is interesting to show that in the absence of some hypothesis of Theorem 2 we can construct a maximal local solution that is also bounded, what in other words ensures that the adopted hypotheses are sufficiently sharp.

Theorem 3 (Sharpness of "Blow Up" Conditions) *Consider $\alpha \in (0, 1)$. Then there exists $f_\alpha : \mathbb{R}^+ \times X \to X$ continuous and locally Lipschitz which does not map every bounded set into bounded set, such that problem*

$$\begin{cases} cD_t^\alpha u(t) = f_\alpha(t, u(t)), & t > 0 \\ u(0) = v_1 \in X, \end{cases}$$

where cD_t^α is the Caputo's fractional derivative and v_1 is the first element of a specific Schauder basis, possesses a bounded maximal solution $u : [0, 1) \to X$.

References

[1] B. De Andrade, A. N. Carvalho, P. M. Carvalho-Neto, P. Marín-Rubio. *Semilinear fractional differential equations: global solutions, critical nonlinearities and comparison results*, Topol. Method. Nonl. An. **45**, (2015) 439–469.

[2] P. M. Carvalho-Neto. *Fractional Differential Equations: a novel study of local and global solutions in Banach spaces*, Ph.D. thesis, Universidade de São Paulo, São Carlos, 2013.

[3] P. M. Carvalho-Neto, R. Felhberg Junior, *Conditions to the absence of blow up solutions to fractional differential equations* - submitted.

[4] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[5] J-G. Peng and K-X. Li. *A novel characteristic of solution operator for the fractional abstract Cauchy problem*, J. Math. Anal. Appl., **385**, (2012) 786–796.