WHAT IS...

a Dessin d'Enfant?

Leonardo Zapponi

A *dessin d'enfant* ("child's drawing") is a connected graph with two extra bits of structure:

- at each vertex is given a cyclic ordering of the edges meeting it;
- each vertex is assigned one of two colors, conventionally black and white, and the two ends of every edge are colored differently.

These structures were introduced, at least in the context about to be described, by Alexandre Grothendieck in about 1984. There is an amazing relationship between these dessins and deep arithmetical questions.

Dessins and Complex Geometry

Dessins arise naturally from finite coverings $X \mapsto \mathbb{P}^1(\mathbb{C})$ by a Riemann surface X unramified outside the points 0, 1, ∞. Here $\mathbb{P}^1(\mathbb{C})$ is just the Riemann sphere $\mathbb{C} \cup \{\infty\}$. To such a covering a dessin is associated in the following way: the black nodes are the inverse images of 0, the white ones the inverse images of 1, and the edges of the dessin are the components of the inverse image of the line segment $(0,1)$. The cyclic order arises from local monodromy around the vertices—i.e., winding around the local sheets of the covering containing a common point. Thus, we get not only a dessin but along with it an embedding into a Riemann surface. We also get a cellular decomposition of the surface. The faces of this decomposition are the connected components of the inverse image of the complement of $[0,1]$.

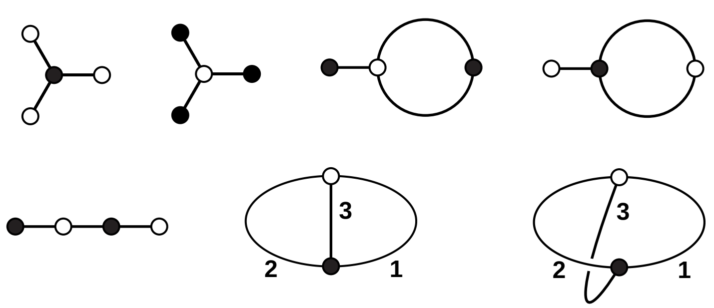

Figure 1. The dessins with three edges. The cyclic ordering at each vertex is indicated geometrically. The last two are distinct because of different cyclic orders at the bottom vertex—$(1,3,2)$ **against** $(1,2,3)$.

Monodromy associates to each path in the fundamental group of $\mathbb{P}^* = \mathbb{P}^1 - \{0,1,\infty\}$ with respect to $1/2$ a permutation of the edges in the dessin: a closed path starting and ending at $1/2$ will lift to a path in the covering starting at one edge of the dessin and ending at another.

This idea allows one to see, conversely, a simple way to construct a covering from a dessin. The fundamental group of $\mathbb{P}^*$ with respect to $1/2$ is a free group on two generators σ_0 and σ_1, loops around 0 and 1. Associated to each of these is a permutation of the edges of a dessin. The one associated to σ_0 rotates the edges around each black node in accord with the cyclic ordering at that node, and similarly σ_1 rotates around the white nodes. This extends to a permutation representation of the whole free group. This group acts transitively on the edges, since the dessin is connected, and the isotropy subgroup of any edge is therefore a subgroup of index equal to the number of edges, hence is associated to a finite covering of $\mathbb{P}^*$. Different isotropy subgroups are conjugate. But the finite coverings of $\mathbb{P}^*$ are also the coverings of $\mathbb{P}^1(\mathbb{C})$ unramified except at 0, 1, or ∞. Thus the dessin determines such a covering.

Grothendieck wrote of this relationship: "This discovery, which is technically so simple, made a very strong impression on me, and it represents a decisive turning point in the course of my reflections, a shift in particular of my centre of interest in mathematics, which suddenly found itself strongly focussed. I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact. This is surely because of the very familiar, non-technical nature of the objects considered, of which any child's drawing scrawled on a bit of

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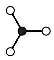
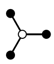
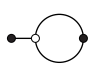
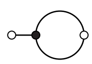
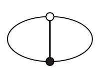
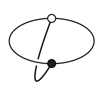
paper (at least if the drawing is made without lifting the pencil) gives a perfectly explicit example. To such a dessin we find associated subtle arithmetic invariants, which are completely turned topsy-turvy as soon as we add one more stroke."

Arithmetic and Algebraic Geometry

Any finite cover of $\mathbb{P}^1(\mathbb{C})$ has an algebraic structure defined over $\mathbb{C}$. That is to say, the Riemann surface and the projection are both defined by polynomials in $\mathbb{C}$. In the following table we give the explicit list for the dessins we have already seen.

Dessin	$\hat{X}$	Equation for the cover
	$\mathbb{P}^1(\mathbb{C})$	$\beta_1(x) = x^3$
	$\mathbb{P}^1(\mathbb{C})$	$\beta_2(x) = 1 - \beta_1(x) = 1 - x^3$
	$\mathbb{P}^1(\mathbb{C})$	$\beta_3(x) = \dfrac{(4-x)(1+2x)^2}{27x}$
	$\mathbb{P}^1(\mathbb{C})$	$\beta_4(x) = 1 - \beta_3(x) = \dfrac{4(x-1)^3}{27x}$
	$\mathbb{P}^1(\mathbb{C})$	$\beta_5(x) = \dfrac{x^3+3x^2}{4}$
	$\mathbb{P}^1(\mathbb{C})$	$\beta_6(x) = \dfrac{x^3}{x^3-1}$
	$y^2 = x^3 + 1$	$\beta_7(x,y) = \frac{1}{2}(1+y)$

We can see easily now that there are three faces for the next to last dessin, which sits in a sphere, but only one for the last, which is embedded in a torus (Figure 2).

In Figure 3 these faces can be read off directly from the dessin as the connected components of a thickened dessin.

But now we enter into the realm of arithmetic algebraic geometry with this pleasant observation: *Any dessin arises from a finite covering of $\mathbb{P}^1$ that can be defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers.* This is essentially a consequence of Weil's descent theory.

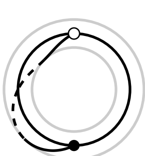


Figure 2. The last dessin embedded in a torus.

Figure 3. Three faces on the left, one on the right.

Belyĭ's Theorem

Everything so far is elementary, and yet…the manner in which "squishy" combinatorial objects (clay) turn out to possess canonical rigid structures (crystal) remains astonishing. At this point we have seen that dessins correspond to certain finite coverings of $\mathbb{P}^1$ defined over $\overline{\mathbb{Q}}$, but we do not know what algebraic curves arise in this way. Grothendieck was amazed by this famous and remarkably simple theorem due to G. V. Belyĭ, first announced in Helsinki in 1978: *Every algebraic curve defined over $\overline{\mathbb{Q}}$ can be represented as a covering of $\mathbb{P}^1$ ramified over at most three points.* In other words, *every algebraic curve defined over $\overline{\mathbb{Q}}$ contains an embedded dessin.*

Galois Action

Dessins correspond to covers of $\mathbb{P}^1$ defined over $\overline{\mathbb{Q}}$. The covers are permuted by the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, so this group also acts on the set of dessins, and one consequence of Belyĭ's theorem is that the action is faithful. The deepest open question in the theory of dessins is this: *Can the Galois orbits of dessins be distinguished by combinatorial or topological invariants?* That is, is there an effective way to tell whether two dessins belong to the same Galois orbit? There are several obvious invariants, such as genus, valency lists, etc., but it is known that they are insufficient to answer this question. Other more delicate invariants have been discovered, but whether a complete list exists—and, if so, whether finite or infinite—remains a mystery.

References

[1] G. V. Belyĭ, Another proof of three points theorem, preprint MPI-1997-46, 1997. Available at `http://www.mpim-bonn.mpg.de`. A version of this has recently appeared in print: *Mat. Sb.* **193** (2002), 21–24.

[2] Leila Schneps, ed., *The Grothendieck Theory of Dessins d'Enfants*, London Math. Soc. Lecture Note Ser., vol. 200, Cambridge Univ. Press, 1994.

[3] Leila Schneps and Pierre Lochak, eds., *Geometric Galois Actions*, London Math. Soc. Lecture Note Ser., vol. 242, Cambridge Univ. Press, 1997. This contains Grothendieck's *Esquisse d'un Programme* (as well as the English translation from which the quote was taken), originally written in 1984.