WHAT IS...

an Operad?

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An operad is an abstraction of a family of composable functions of n variables for various n, useful for the "bookkeeping" and applications of such families. Operads are particularly important and useful in categories with a good notion of "homotopy", where they play a key role in organizing hierarchies of higher homotopies. Operads as such were originally studied as a tool in homotopy theory, but the theory of operads has recently received new inspiration from homological algebra, category theory, algebraic geometry, and mathematical physics, especially string field theory and deformation quantization, as well as new developments in algebraic topology. The name *operad* and the formal definition appear first in the early 1970s in J. Peter May's *The Geometry of Iterated Loop Spaces*, but there is an abundance of prehistory. Particularly noteworthy is the work of Boardman and Vogt.

The most fundamental example of an operad is the *endomorphism operad* $\mathcal{E}nd_X := \{Map(X^n, X)\}_{n\geq 1}$, where for a set or topological space X, $\{Map(X^n, X)\}$ means the set or space of functions or continuous functions from the n-fold product of X with itself to X, together with the operations

$$\circ_i : Map(X^n, X) \times Map(X^m, X) \longrightarrow Map(X^{n+m-1}, X)$$

given, for $1 \leq i \leq n$, by

$$(f \circ_i g)(x_1, \cdots, x_{m+n-1}) = f(x_1, \cdots, x_{i-1}, g(x_i, \cdots, x_{i+m-1}), x_{i+m}, \cdots).$$

In the endomorphism operad $\mathcal{E}nd_X$ there are easily discovered relations involving iterated $\circ_i$-operations and the symmetric group Σ_n actions on the X^ns. For example,

$$(f \circ_i g) \circ_j h = f \circ_j (g \circ_{i-j+1} h) \quad \text{for} \quad j \leq i \leq j+n-1$$

if g is a function of n variables, since only the name of the position for the insertion is changed.

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An *operad* $(\mathcal{O}, \circ_i)$ consists of a collection $\{\mathcal{O}(n)\}_{n\geq 1}$ of objects and maps $\circ_i : \mathcal{O}(m) \times \mathcal{O}(n) \to \mathcal{O}(m+n-1)$ for $m, n \geq 1$ satisfying the relations manifest in the example $\mathcal{E}nd_X$.

May's original definition corresponds to simultaneous insertions into all possible positions of inputs into $f \in Map(X^n, X)$. In most examples, the structures are "manifest" without appeal to the technical definitions; as Frank Adams used to say, to operate the machine, it is not necessary to raise the bonnet (look under the hood).

It helps to see graphic examples of operads. Two kinds that are particularly important are the tree operads and the little cubes (or disks) operads.

Let $\mathcal{T}(n)$ be the set of (nonplanar) trees with 1 root and n leaves labeled (arbitrarily) 1 through n. The collection $\mathcal{T} = \{\mathcal{T}(n)\}_{n\geq 1}$ of sets of trees forms an operad by grafting the root of g to the leaf of f labeled i, as in Figure 1.

The *little n-cubes operad* $C_n = \{C_n(j)\}_{j\geq 1}$, where $C_n(j)$ consists of an ordered collection of j n-cubes linearly embedded in the standard n-dimensional unit cube I^n with disjoint interiors and axes parallel to those of I^n. The operations are given as indicated in Figure 2.

Just as group theory without representations is rather sterile, so operads are best appreciated by their representations, known as (varieties of) algebras, especially algebras with higher homotopies.

An *algebra A over an operad* $\mathcal{O}$ "is" a map of operads $\mathcal{O} \to \mathcal{E}nd_A$. This is just a compact way of saying that an algebra A has a coherent system of maps $\mathcal{O}(n) \times A^n \to A$.

A major motivation for the development of operads was the desire to have a *homotopy invariant* characterization of based loop spaces and iterated loop spaces. Precisely such coherent systems of higher homotopies provided the answers. For based loop spaces, the operad in question, $\mathcal{K} = \{K_n\}_{n\geq 1}$, consists of the polytopes known as *associahedra*. The usual product of based loops is only homotopy associative. If we fix a specific associating homotopy and consider the five ways of parenthesizing the product of four loops, there results a pentagon whose edges correspond to a path of loops (Figure 3). (Compare

Figure 2 in the article by Devadoss in this issue of the *Notices*.) From the leftmost vertex to the rightmost, consider the two paths of loops across the top or around the bottom. The pentagon can be filled in by a family of such paths from left to right corresponding to further adjustment of parameters.

The associahedron K_n can be described as a convex polytope with one vertex for each way of associating n ordered variables, that is, ways of inserting parentheses in a meaningful way in a word of n letters. The edges correspond to a single application of an associating homotopy. For K_5 see Figure 4 of the article by Devadoss in this issue of the *Notices*; a rotating image is available at `http://igd.univ-lyon1.fr/home/chapoton/stasheff.html`.

The cellular structure of the associahedra is well described by planar rooted trees, the vertices corresponding to binary trees and so forth. The facets are all products of two associahedra of lower dimension, and specific imbeddings can be given to play the role of the $\circ_i$ operations as in an operad. An A_∞*-space* is a space Y which admits a coherent family of maps

$$m_n : K_n \times Y^n \to Y$$

so that they make Y an algebra over the operad (without Σ_n-actions) $\mathcal{K} = \{K_n\}_{n\geq 1}$. The main result is:

A connected space Y (of the homotopy type of a CW-complex) has the homotopy type of a based loop space ΩX *for some X if and only if Y is an* A_∞*-space.*

Homotopy characterization of iterated loop spaces $\Omega^n X_n$ for some space X_n required the full power of the theory of operads with the symmetries.

A connected space Y (of the homotopy type of a CW-complex) has the homotopy type of an n-fold loop space $\Omega^n X_n$ *for some space* X_n *if Y admits a coherent system of maps* $C_n(j) \times Y^j \to Y$.

Although introduced originally in the category of topological spaces, operads were available almost immediately for chain complexes. The analog of an A_∞-space is an A_∞*-algebra,* and there is a Lie analog, an L_∞*-algebra.* These two special cases of "higher homotopy algebras" are particularly important in mathematical physics: A_∞ for open string field theory and L_∞ for closed string field theory and for deformation quantization. The operad for L_∞-algebras is given a very nice and physically relevant geometric interpretation in terms of a real compactification of the moduli space of Riemann spheres with punctures; see the article by Devadoss in this issue of the *Notices*.

One reason for the explosive development of operad theory in the 1990s was the introduction of operadic structures in topological field theories, e.g. CFTs (conformal field theories) and SFTs (string field theories), which in turn was inspired by the importance of moduli spaces of Riemann surfaces with punctures or boundaries (or other decorations) in these physical theories. The little j-disks

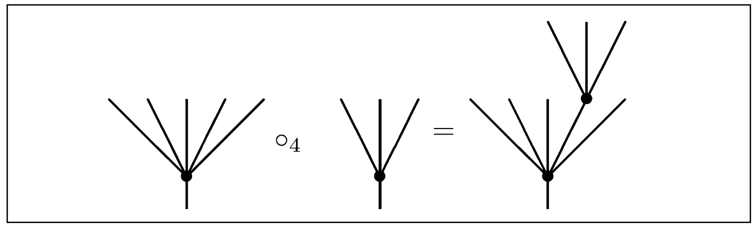

Figure 1. Grafting with the leaves numbered from left to right.

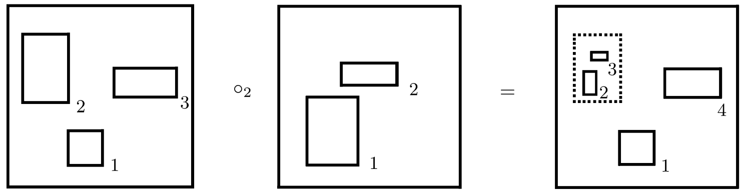

Figure 2. The little 2-cubes operad.

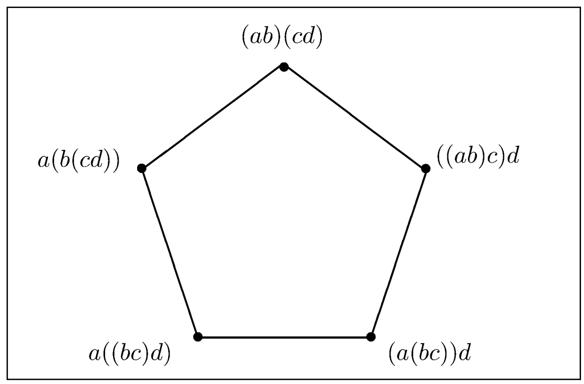

Figure 3. The associahedron K_4.

operad $\mathcal{D}_j$ has a definition quite parallel to that of the little j-cubes operad. Using disks has the advantage of extending nicely to little disks holomorphically embedded in a Riemann surface Σ_g of genus g, leading to a more complicated and subtle notion of a *modular operad*.

Also of importance for applications is the notion of an A_∞*-category*. These have been used by Fukaya for remarkable applications to Morse theory and Floer homology and by Batanin and by May in higher category theory. More recently, they play a role in string and D-brane theory and homological mirror symmetry.

For a reasonably up-to-date introduction and survey, consider [MSS02]. Two particularly important original works are [BV73] and [May72].

Further Reading

[BV73] J. M. Boardman and R. M. Vogt, *Homotopy Invariant Algebraic Structures on Topological Spaces*, Lecture Notes in Math., vol. 347, Springer-Verlag, 1973.

[May72] J. P. May, *The Geometry of Iterated Loop Spaces*, Lecture Notes in Math., vol. 271, Springer-Verlag, 1972.

[MSS02] Martin Markl, Steve Shnider, and Jim Stasheff, *Operads in Algebra, Topology and Physics*, Math. Surveys Monogr., vol. 96, Amer. Math. Soc., Providence, RI, 2002.