WHAT IS...

a Flip?

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A *flip* is a special codimension-2 surgery in algebraic geometry.

Flips turn up, for example, in the study of compactifications of moduli spaces. Constructions in algebraic geometry depend on parameters (moduli). For instance, Riemann surfaces of genus g are in natural 1-to-1 correspondence with the points of an algebraic variety M_g. Usually, the moduli space is noncompact, making it unsuitable for the study of enumerative and topological questions. Several meaningful compactifications are possible, and they are related by flips. Understanding the different compactifications in terms of flips can lead to beautiful and difficult combinatorial questions. Among the first examples of this point of view is the work of Thaddeus on stable pairs on curves; a recent variation on this theme is the treatment of 3-fold flops by Bridgeland, which I describe below (a flop is a kind of flip).

Flips are steps in Mori's *minimal model program*. Starting with a nonsingular projective variety X, the minimal model program is an analog of the geometrization program in topology; its aim is to perform surgery on X until the *canonical line bundle* $K_X = \wedge^{\mathrm{top}} T^*X$ has global positivity properties.

The simplest example of surgery in algebraic geometry is the *blow-up* $f\colon Y \to X$ of a nonsingular point P of a surface X. The surface Y is formed by removing $P \in X$ and sticking the projectivized tangent space $E = \mathbb{P}T_P X$ in its place. The morphism $f\colon Y \to X$ identifies $Y \setminus E$ with $X \setminus \{P\}$ and contracts the exceptional set E to P. If $X = \mathbb{C}^2$ with coordinates (x, y), and $P = (0, 0)$, then

$$Y = \{xm_1 - ym_0 = 0\} \subset \mathbb{C}^2 \times \mathbb{P}^1$$

where m_0, m_1 are homogeneous coordinates on $\mathbb{P}^1$. The function $m = m_1/m_0$ is well defined on the chart $\{m_0 \neq 0\}$, which is identified with the set $\{y = mx\} \subset \mathbb{C}^3$. The point at infinity corresponds to the vertical line $\{x = 0\}$.

Doing this construction inside real algebraic geometry produces the picture of a helix; topologically, one cuts out a small disk and replaces it with a Möbius strip.

In the language of algebraic geometry, a surgery is called a *birational map*. By definition, a birational map $\varphi\colon Y \dashrightarrow X$ is an isomorphism $\varphi\colon Y \setminus E \to X \setminus F$, where the *exceptional sets* $E \subset Y$ and $F \subset X$ are algebraic subvarieties.

In the case of a flip, the exceptional sets $E \subset Y$ and $F \subset X$ are *small*, that is, they have codimension ≥ 2. By contrast, in the blow-up $f\colon Y \to X$ of a nonsingular point, the exceptional set $E \subset Y$ is of codimension 1.

Topological surgery arises in nature when we cross a critical value of a C^∞ Morse function $h\colon M \to \mathbb{R}$. As we cross a critical value t_0, the level set $h^{-1}(t_0 - \varepsilon)$ is surgically modified into $h^{-1}(t_0 + \varepsilon)$. By the Morse lemma, a local model of this situation is $M = \mathbb{R}^n \times \mathbb{R}^m$ with coordinates (x, y) such that

$$(x, y) \to h(x, y) = -\|x\|^2 + \|y\|^2.$$

We see that, as t crosses t_0, the level set undergoes a surgery in which $S^{n-1} \times D^m$ is replaced by

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$D^n \times S^{m-1}$ (where S^n and D^n denote the n-dimensional sphere and ball). Integrating along the gradient of h, we get an action of the additive group $\mathbb{R}$ on M; the level sets $h^{-1}(t)$ for $t = t_0 \pm \varepsilon$ are diffeomorphic to the quotients $M^{\pm}/\mathbb{R}$, where $M^- = M \setminus \{x = 0\}$ and $M^+ = M \setminus \{y = 0\}$.

I use an analog of this construction over the complex numbers to give an example of a flip. Consider the action of the multiplicative group $\mathbb{C}^\times$ on $B = \mathbb{C}^4$ with weights $(-2, -1, 1, 1)$:

$$(x_1, x_2, y_1, y_2) \mapsto (\lambda^{-2}x_1, \lambda^{-1}x_2, \lambda y_1, \lambda y_2).$$

The quotient topology by this action is not Hausdorff. Indeed, a general orbit is asymptotic to an orbit in $\{y = 0\}$ as $\lambda \to 0$ and to an orbit in $\{x = 0\}$ as $\lambda \to \infty$. There are two meaningful ways to get a Hausdorff topological space, and they are related by a flip. Indeed, consider open subsets $B^- = B \setminus \{x = 0\}$ and $B^+ = B \setminus \{y = 0\}$. Then $X^- = B^-/\mathbb{C}^\times$ and $X^+ = B^+/\mathbb{C}^\times$ are Hausdorff topological spaces with a natural structure of algebraic varieties, and the obvious birational map $\varphi\colon X^- \dashrightarrow X^+$ is a flip. Note that X^- is covered by charts $\{x_1 \neq 0\} \cong 1/2(1, 1, 1)$ (the quotient of $\mathbb{C}^3$ by a reflection of all coordinates) and $\{x_2 \neq 0\} \cong \mathbb{C}^3$.

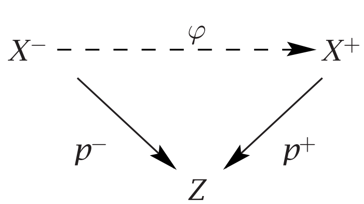
This example also illustrates Włodarczyk's view of a flip as a birational cobordism, which leads to the proof by him, with Abramovich, Karu, and Matsuki, of the factorization theorem, stating that a birational map between nonsingular varieties is a composition of blow-ups and blow-downs along nonsingular centres.

The few nineteenth-century birational geometers who ventured into higher dimensions were on some level aware of codimension-2 surgery. However, flips were only discovered recently as steps of the minimal model program. If X is a projective variety, the minimal model program performs surgery on X until the canonical line bundle K_X is *nef*, that is,

$$\deg K_{X|C} = \int_C c_1(K_X) \geq 0$$

for all algebraic curves $C \subset X$ (c_1 denotes the first Chern class). To achieve this, it is necessary to allow X to have mild (to be precise, terminal) singularities, for example certain orbifold singularities, for which K_X still makes sense.

The formal definition of flip requires a diagram



where $p^{\pm}$ are small birational morphisms with compact fibres, such that $-K_{X^-}$ is nef over Z, that is, $-\int_C c_1(K_{X^-}) \geq 0$ on curves C contracted by p^-, and K_{X^+} is nef over Z.

Usually, $p^-\colon X^- \to Z$ is given, and the problem is to show that $p^+\colon X^+ \to Z$ exists. The flip can exist only in very restricted conditions, and the few known existence results are very difficult to establish. Mori proved that if X^- is a 3-fold with terminal singularities, then the flip exists. The example above is one of these flips considered by Mori.

In the case of *flops*, where $K_{X^-} = K_{X^+} = 0$, Bridgeland shows that X^+ is the moduli space of certain sheaves on X^- (more precisely, complexes of coherent sheaves in the derived category), thus providing a construction of X^+. This idea has since been applied, with mixed success, to the problem of existence of flips.

Flips are fundamental in algebraic geometry in dimension ≥ 3; the best place to start learning about them is [1].

For many applications, it is desirable to have explicit equations of all 3-fold flips. A paper of Mori [2], which is a great place to look for examples of flips, classifies the important special case of semistable flips.

In his work [3], Shokurov proves the existence of flips in dimension 4. His work is based on an important extension of the Mori category, the category of *log terminal pairs* (X, B) of a variety X and a *boundary* $\mathbb{Q}$-divisor $B = \sum b_i B_i$. In this notation, the $B_i \subset X$ are irreducible subvarieties of codimension 1 and the coefficients $0 < b_i \leq 1$ are rational numbers. Initially at least, B is psychologically a boundary, in the sense that one's emotional investment is in the complement $X \setminus B$. The proof uses the graded ring $R = \oplus_{n \geq 0} \Gamma(X, n(K_X + B))$ of global holomorphic differentials satisfying growth conditions at the boundary. For an introduction, see A. Corti, *3-fold flips after Shokurov*, `http://www.dpmms.cam.ac.uk/~corti/flips.html`.

References

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[2] S. Mori, *On Semistable Extremal Neighborhoods*, Higher Dimensional Birational Geometry (Kyoto, 1997), 157–184, Adv. Stud. Pure Math., 35, Math. Soc. Japan, Tokyo, 2002.

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