WHAT IS...
a Compacton?

Philip Rosenau

A soliton is a special solitary traveling wave that after a collision with another soliton eventually emerges unscathed. Solitons are solutions of partial differential equations that model phenomena like water waves or waves along a weakly anharmonic mass-spring chain. The existence of solitons critically depends on special mathematical properties of the model equations. Typically such equations have solitary wave solutions whose interaction is almost, but not exactly, clean: the reemerging waves appear perturbed, and *the mathematical miracle whereby, among other things, one can explicitly describe the interaction is lost.* Since all solitary waves have infinite tails, *it is natural to seek model equations that generate solitary waves with a finite span.* Two such waves would interact only for a finite time and then, unlike solitons, would be completely oblivious to each other. This is somewhat analogous to a search for wavelets with compact support. We define a *compact wave* as a robust solitary wave with *compact support* beyond which it vanishes identically [1]. We then define a *compacton* as a compact wave that preserves its shape after interacting with another compacton.

How Compact Waves Emerge. Consider

$$u_t + (u^m)_x + \left[u^a(u^b)_{xx}\right]_x = 0, \tag{1}$$

with $a + b \equiv n \geq 1$, and $m \geq a - 1$. For $n = b = 1$ and $m = 2, 3$, (1) reduces to the celebrated *Kdv* and *m-KdV* equations, respectively, which are the home base of solitons. But it is when $n > 1$ that the second nonlinearity enables formation of compact patterns.

Consider first the special case of (1) wherein $m = 2$ and $a = b = 1$. We seek a traveling solution, $U(s = x - \lambda t)$, with velocity λ. Integrating once gives $U[-\lambda + U + U_{ss}] = C_0$. To avoid singular solutions, we set $C_0 = 0$ and obtain $U = \lambda(1 + A_0 \cos s)$. If $A_0 = 1$, then the trough of the periodic wave touches $U = 0$, *where the last term in equation (1) degenerates (and the solution's uniqueness is lost), turning any period between two troughs into an isolated entity.* We may now remove any such period and connect its edges with the trivial $U = 0$ solution

$$U(x - \lambda t) = 2\lambda \cos^2\left[\frac{x - \lambda t}{2}\right], \tag{2}$$

where $|x - \lambda t| \leq \pi$ *and* U *vanishes elsewhere.* This is the sought-after *compact wave.* Its second derivative has a jump at $U = 0$, but since $UU_{ss} \sim s^2 H(s) \downarrow 0$, where $H(s)$ is the Heaviside function, it satisfies our equation.

Unlike the usual solitonic case, in which there is an instant runaway of its initial support, in our case, because u vanishes at the edge of the support, the degeneration of the dispersive mechanism $(uu_{xx})_x$ blocks an instantaneous spread of the front. Instead of infinite tails we obtain a wave of finite span that propagates with constant velocity.

We return to (1) and look for solitary traveling waves. Two easy integrations give

$$U^{2(b-1)}\left[\frac{1}{2}U_s^2 - \lambda A_1 U^{3-n} + A_m U^{2+m-n}\right] = 0, \tag{3}$$

with $A_k^{-1} = b(k + b - a)$. The expression in the brackets describes a periodic wave with a peak at $U = [\lambda A_1/A_m]^{\frac{1}{m-1}}$ and a trough at $U = 0$, where $U \sim s^{\frac{2}{n-1}}$. Again, since for $\omega \equiv b + 1 - a > 0$, $U^{b-1}U_s \sim s^{\frac{\omega}{n-1}} H(s) \downarrow 0$, $U(s)$ satisfies (1) and each period between the troughs is an isolated entity. Removed and connected with the trivial states, the 1-period wave solution turns into a *compact wave.* For $m = n$, we have

$$U = \left[\frac{\lambda b}{\omega} \cos^2\left(\frac{(m-1)}{2b} s\right)\right]^{\frac{1}{m-1}} \tag{4}$$

when $|s| \leq \frac{b}{m-1}$ and zero elsewhere. Figure 1 shows a clean interaction of three typical $K(2,2)$ compact waves ($b = n = m = 2$ in (1)). This and many other experiments tempt us to declare them as compactons despite the formation of small ripples. The infinite number of conservation laws in a conventional solitonic case would imply that collisions are slightly inelastic. However, $K(2,2)$ has only four local conservation laws (u, u^3, $ucosx$, and $usinx$), so clean interactions and a ripple can, in principle,

Philip Rosenau is professor of applied mathematics at Tel Aviv University. His email address is rosenau@post.tau.ac.il.

coexist [1]. Though analysis is needed to settle the question, extensive numerical studies of (1) indicate that the mechanism underlying the interaction of compact waves is very different from that of solitary waves or solitons.

Stationary Compactons: When $\omega < 0$, (1) supports *stationary compact waves*. Some of them may be seen as *solitons in "mass units"*, and thus they inherit the integrability of their antecedents. Let u be a density of some quantity. Then the map $x \to z = \int_{-\infty}^{x} u(x,t)dx$ defines the "mass" of u in $(-\infty, x)$. For a typical soliton the "total mass" is finite, and in these coordinates u is compact. Moreover, since shifts in time do not change the mass distribution of the soliton, it is stationary in mass units. Thus in the *m-KdV* case $u = \sqrt{2\lambda}\cosh^{-1}[\sqrt{\lambda}(x - \lambda t)] = \sqrt{2\lambda}\sin(z/\sqrt{2}),\ 0 \le z \le \sqrt{2}\pi$. In mass units interaction of N-solitons turns ultimately into *N-stationary compactons,* and *KdV* and *m-KdV equations* are mapped into (1) with $m \to m+1,\ a = 3,\ \ b = 1,$ and thus $\omega = -1$. The singularity now confines the dynamics to the initial support. Application to a motion of curves in a plane is given in [2].

Compact Breathers. Consider the vibrations of a chain of particles interconnected by springs:

(5)
$$y_n'' + \Phi'(y_n) = \frac{1}{h}\left[T(\frac{y_{n+1} - y_n}{h}) - T(\frac{y_n - y_{n-1}}{h})\right].$$

Here h is the interparticle distance, T is the attraction between two adjacent mass points, and $\Phi'(y_n)$ is the force exerted by the ambience on the nth node. The continuum limit, $y(x,t) \equiv y_n(t)$, yields

(6) $$y_{tt} + \Phi'(y) = T(y_x)_x,$$

which describes the motion of a string. When the stretch $u = y_x$ is small, $T(u) \sim u + u^\alpha,\ \alpha > 1$. Balanced with a weak force due to the discreteness, it begets a *KdV*-like equation. Being interested in the *opposite limit of a very strong anharmonicity,* we assume that $T(u) \sim u^3$. (6) is now singular whenever the stretch and thus the wave speed $C^2(u) \sim u^2$ vanish. If $\Phi'(y) = y - y^3$, we find a space-time separable solution for both the string and the chain. It yields a periodic motion in both time and space. For the string the degeneracy at $y_x = 0$ enables us to connect a 1-period of the solution with the trivial state, yielding a stationary compact solution, a *breather*, which oscillates in time.

How does the discrete breather behave? Though in principle the discrete lattice does not support compact solutions, a careful analysis reveals that the spread beyond the compact domain is confined to *a very narrow boundary layer where the decay is super-exponential* [3]. Figure 2 reveals that neither the shape of the breather nor its support seems to depend on the number of mass points.

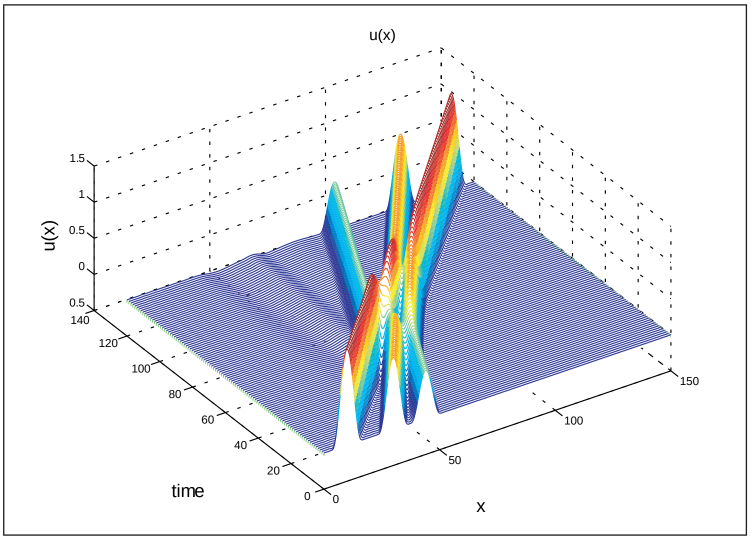

Figure 1. Interaction of three $K(m = 2,\ n = 2)$ compact waves (courtesy of M. Staley). Numerically they seem to emerge from the interaction intact, yet the interaction site is marked by a very small ripple which decomposes into compacton-anticompacton pairs [1].

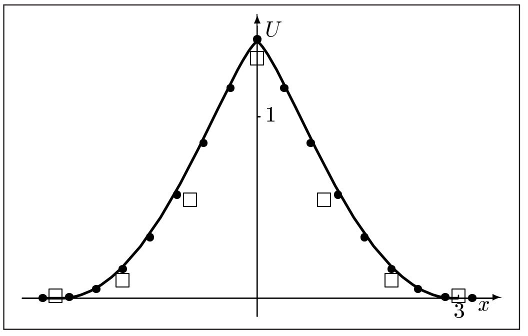

Figure 2. Continuous (line) and two discrete breather profiles for $h = 0.4, 1$, respectively.

As h increases, breathers become stable at higher and higher amplitudes, and their basin of stability increases dramatically as well [3].

Clearly the singularities presented are not esoteric mathematical entities, but a natural limit of a very localized boundary layer. This appears to be a generic property of many discrete genuinely nonlinear dynamical systems. For instance, a Lotka-Volterra-like problem, $2h\dot{u}_j = u_{j+1}^2 - u_{j-1}^2$, may be seen as a discrete antecedent of the $K(2,2)$ equation with the solitary waves having a super-exponentially decaying front, which in the quasi-continuum limit becomes strictly compact. A similar conclusion emerges from a recently published work with A. Pikovsky on phase compactons in chains of dispersively coupled oscillators.

References

[1] P. Rosenau and J. M. Hyman, *Phys. Rev. Lett.* **70** (1993), 564.
[2] P. Rosenau, *Phys. Lett.* A. **275** (2000), 193.
[3] P. Rosenau and S. Schochet, *Phys. Rev. Lett.* **94** (2005), 045503.