# WHAT IS...
# a Lefschetz Pencil?

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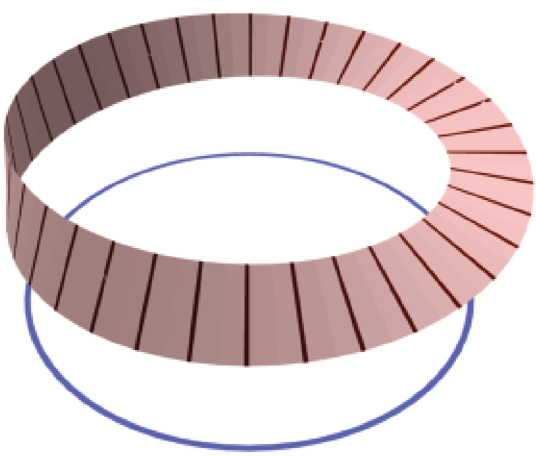
One of the most challenging subjects in topology is the study of smooth 4-manifolds. For a simple approach to this, we list some examples. After the 4-sphere, the best known compact 4-manifolds are Cartesian products of surfaces. For more variety, we can "twist" the product structure to obtain a *fiber bundle*. Consider the Möbius band, with its projection $\pi : M \to S^1$ to the circle. Each *fiber* (point preimage) is an interval $I$, and any sufficiently small neighborhood $U$ in $S^1$ has preimage given by $U \times I$, with $\pi$ corresponding to projection to the first factor. Thus, $M$ is locally indistinguishable from the product $S^1 \times I$ with its projection to $S^1$.



**The Möbius band fibered by intervals.**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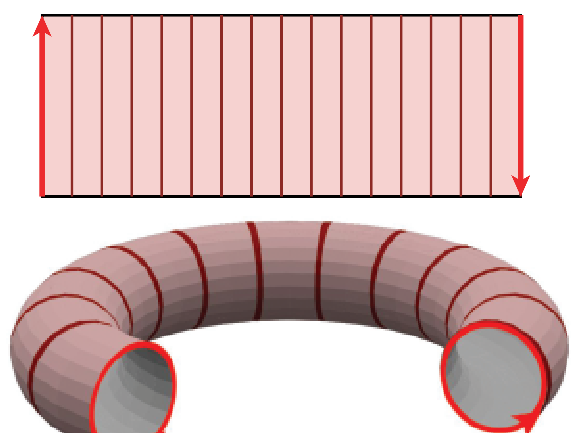
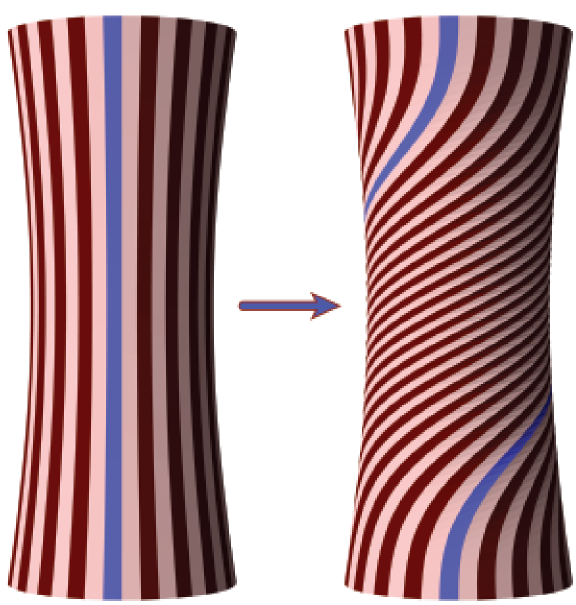
Of course, $M$ is not globally a product, since its boundary is connected, unlike the fiber $I$. There are also two bundles over $S^1$ with $S^1$-fibers: the torus $T = S^1 \times S^1$ and the Klein bottle $\pi : K \to S^1$. (Try to visualize each of these surfaces filled by a family of disjoint circles.) For each pair of compact surfaces $\Sigma$ and $F$, we can now consider bundles $\pi : X \to \Sigma$ with fiber $F$. Most choices of $\Sigma$ and $F$ will yield infinitely many 4-manifolds $X$ in this manner. When $\Sigma = F = T$, for example, we can obtain each of the 4-manifolds $T \times T$, $T \times K$, and $K \times K$ as a product of two $S^1$-bundles over $S^1$ (with the product of the two projection maps). Alternatively, we can obtain infinitely many examples by thinking of



**Just as the Möbius band is obtained from a rectangle by identifying opposite ends, the Klein bottle is obtained by an identification of opposite ends of a tube.**

$\Sigma = T$ as being obtained from the cylinder $S^1 \times I$ by gluing the two boundary components together. Then every self-diffeomorphism of $F$ gives a way to glue the boundary components of $(S^1 \times I) \times F$ to obtain a bundle over $T$. (Compare with the Klein bottle $K \to S^1 = I/\partial I$ as pictured above.) For every bundle $\pi : X \to \Sigma$, the preimage of each circle $C \subset \Sigma$ is itself a bundle over $C$, determined by a self-diffeomorphism of $F$ called the *monodromy* around $C$. (What is the monodromy around each factor of $S^1 \times S^1$ in each of the above examples?)

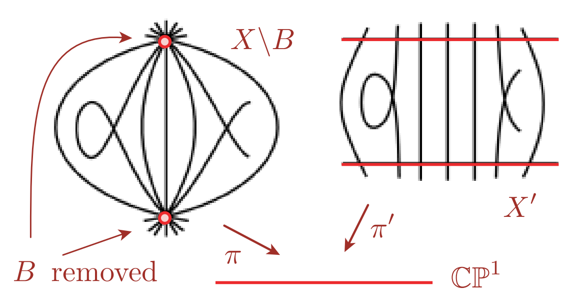
Unfortunately, fiber bundles do not form a very representative class of 4-manifolds, especially in the simply connected case, where the two $S^2$-bundles over $S^2$ are the only examples. To obtain more generality, we relax the requirement that $\pi$ be locally a product by allowing critical points of the simplest type, locally modeled by the complex quadratic map $q : \mathbb{C}^2 \to \mathbb{C}$, $q(u, v) = u^2 + v^2$. The resulting maps $\pi : X \to \Sigma$ (for $X, \Sigma$ oriented) are called *Lefschetz fibrations* (e.g., [3], Chapter 8). These have only finitely many critical points, and each singular fiber (preimage of a critical value) looks like a surface with a transverse self-intersection. (In the local model, $q^{-1}(0)$ is the union of the two planes $v = \pm iu$.) The complement in $X$ of the singular fibers is then a fiber bundle, and the monodromy around a curve in $\Sigma$ encircling a single critical value is given by a right-handed *Dehn twist* $\varphi$. That is, a certain subset of a nonsingular fiber $F$ is identified with the oriented cylinder $S^1 \times [0, 2\pi]$, and $\varphi$ is given there by $\varphi(\theta, t) = (\theta + t, t)$, adjusted near $S^1 \times \{0, 2\pi\}$ to fit smoothly together with the identity map elsewhere on $F$.



**The bundle over a circle around a critical value of a Lefschetz fibration is obtained from the product $[0, 1] \times F$ by identifying the boundary surfaces through a Dehn twist, shown here on the subset of F outside which it is fixed.**

For any word $w$ (i.e., finite sequence) in right-handed Dehn twists on $F$, we can construct a Lefschetz fibration $X \to D^2$ over the disk, whose monodromies around consecutive critical values realize $w$, by suitably gluing copies of the model critical point onto the trivial fibration $D^2 \times F$. If the composite of all Dehn twists in $w$ is the identity on $F$ (up to homotopy through diffeomorphisms), then the boundary of $X$ is $S^1 \times F$, so we can glue on another copy of $D^2 \times F$ to obtain a Lefschetz fibration over $S^2$. In fact, Lefschetz fibrations over $S^2$ are essentially classified by such words with trivial composite, up to a suitable equivalence relation corresponding to rearranging the critical values in $S^2$. The resulting classification problem for words in the self-diffeomorphism group of $F$ is still unsolved when $F$ has genus $\geq 2$ and is the subject of ongoing research. Lefschetz fibrations over surfaces $\Sigma$ of higher genus can be studied similarly, but the resulting 4-manifolds will never be simply connected. Fortunately, the case $\Sigma = S^2$ already includes an extensive collection of simply connected 4-manifolds.

To construct a typical example, we begin with a generic pair $p_0, p_1$ of homogeneous degree-$d$ polynomials on $\mathbb{C}^3$. That is, each $p_j$ satisfies $p_j(\lambda z) = \lambda^d p_j(z)$, so its zero-locus is a well-defined subset of $\mathbb{CP}^2 = \mathbb{C}^3 \setminus \{0\}$ modulo complex scalar multiplication. For each $(t_0, t_1) \in \mathbb{C}^2 \setminus \{0\}$, the homogeneous polynomial $t_0 p_0 + t_1 p_1$ also has a well-defined zero locus $C_t$ in $\mathbb{CP}^2$, and this depends only on $t = \frac{t_0}{t_1} \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} = S^2$. We would like to identify each $C_t$ as $\pi^{-1}(t)$, for some map $\pi$ to $S^2$. However, the subset $B$ given by $\{z \in \mathbb{CP}^2 \mid p_0(z) = p_1(z) = 0\}$ consists of $d^2$ points, and it is easy to see that for distinct $t, t' \in \mathbb{CP}^1$ we have $C_t \cap C_{t'} = B$. The resulting map $\pi : \mathbb{CP}^2 \setminus B \to \mathbb{CP}^1$ is an example of a *Lefschetz pencil* [3]. At each point of $B$ it is locally modeled by $p : \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$, $p(u, v) = \frac{u}{v}$, and the critical points are quadratic as before. A Lefschetz pencil $\pi : X \setminus B \to \mathbb{CP}^1$ can always be extended to a Lefschetz fibration $\pi' : X' \to \mathbb{CP}^1 = S^2$ by *blowing up*




**A Lefschetz pencil and the corresponding Lefschetz fibration.**

$B$, or one-point compactifying each fiber separately at each $b \in B$. This changes $X$ by connected summing with a copy of $\mathbb{CP}^2$, with orientation opposite the complex orientation, at each $b \in B$. Thus, our example results in a Lefschetz fibration on each $\mathbb{CP}^2 \# d^2\overline{\mathbb{CP}^2}$, $d \in \mathbb{Z}^+$, each obtained from a Lefschetz pencil on $\mathbb{CP}^2$.

Which 4-manifolds admit Lefschetz pencils? In the early twentieth century, Lefschetz constructed such a structure on every *algebraic surface*, i.e., 4-manifold arising as the zero-locus in $\mathbb{CP}^n$ of a collection of homogeneous polynomials. This allowed him to intensively study the topology of algebraic surfaces, a large class of 4-manifolds including many simply connected examples. A decade ago Donaldson showed that the much larger class of *symplectic* 4-manifolds admits Lefschetz pencils. These admit *symplectic forms*, closed differential 2-forms that are nondegenerate as bilinear forms [2]. Symplectic manifolds have themselves been extensively studied for several decades. Unlike algebraic surfaces, symplectic 4-manifolds realize all finitely presented groups as their fundamental groups. A typical simply connected 4-manifold is homeomorphic to infinitely many diffeomorphism types of symplectic manifolds, only finitely many of which are algebraic, and to infinitely many other manifolds that do not admit symplectic structures [3]. It can be shown [2], [3] that a Lefschetz pencil on a 4-manifold determines a symplectic form on it. Thus, the class of 4-manifolds admitting Lefschetz pencils is identical to the class admitting symplectic structures. This class is large and well studied but still somewhat mysterious. It is hoped that the interplay between the two structures will shed new light on both of them.

There are various ways to generalize our discussion of Lefschetz fibrations and pencils. First, we can consider 4-manifolds with boundary. If $\Sigma = D^2$, and $F$ also has boundary, then the boundary of $X$ will be a 3-manifold with an *open book* decomposition [1] whose monodromy is a composite of right-handed Dehn twists. It is not fully understood which 3-manifolds admit such right-handed open books, but such structures correspond to holomorphically fillable contact structures. In fact, the corresponding 4-manifolds are precisely those admitting *Stein structures* (with finite topology), a classical notion from complex analysis. (A Stein manifold is a complex manifold that properly and biholomorphically embeds in some $\mathbb{C}^n$.) Alternatively, we can move to higher dimensions [2]. Donaldson's work still produces Lefschetz pencils $X \setminus B \to \mathbb{CP}^1$, where the critical points are locally $q(u_1, \ldots, u_n) = \sum u_j^2$ and $B$ has codimension 4 with local model $u_1/u_2$ as before. The corresponding classification theory via diffeomorphisms of the fiber is analogous to the 4-dimensional case but harder. One would like to extend Donaldson's theory to *linear k-systems* $X \setminus B \to \mathbb{CP}^k$ for all $k$, as Auroux has done when $k = 2$. If this can be done for $2k = \dim X - 2$ (so the fibers are surfaces), then the corresponding linear systems (*hyperpencils*) exist precisely on manifolds admitting symplectic structures, characterizing the latter as Lefschetz pencils do on 4-manifolds. While these generalizations are receiving well-deserved study, much remains to be done in the basic setting of closed, simply connected 4-manifolds.

## Further Reading


[1] E. Giroux, What is an open book? *Notices Amer. Math. Soc.* **52** (2005), 42–43.

[2] R. Gompf, The topology of symplectic manifolds, Proceedings of 7th Gökova Geometry-Topology Conference, *Turkish J. Math.* **25** (2001), 43–59; http://www.ma.utexas.edu/~gompf.

[3] R. Gompf and A. Stipsicz, *4-Manifolds and Kirby Calculus*, Grad. Studies Math., vol. 20, Amer. Math. Soc., Providence, RI, 1999.