WHAT IS...

a Cluster Algebra?

Andrei Zelevinsky

Cluster algebras, first introduced in [2], are constructively defined commutative rings equipped with a distinguished set of generators *(cluster variables)* grouped into overlapping subsets *(clusters)* of the same finite cardinality (the *rank* of an algebra in question). Among these algebras one finds coordinate rings of many algebraic varieties that play a prominent role in representation theory, invariant theory, the study of total positivity, etc. For instance, homogeneous coordinate rings of Grassmannians, Schubert varieties, and other related varieties carry a cluster algebra structure (after a minor adjustment). Potential applications of this structure include explicit constructions of the (dual) canonical basis and toric degenerations for these varieties.

Since its inception, the theory of cluster algebras has found a number of exciting connections and applications: quiver representations, preprojective algebras, Calabi-Yau algebras and categories, Teichmüller theory, discrete integrable systems, Poisson geometry... The current state of these developments, including links to papers, working seminars, conferences, etc., is represented at the online Cluster Algebras Portal created and maintained by S. Fomin [1].

Although some of the above connections are rather technical, cluster algebras themselves are defined in an elementary manner not requiring any tools beyond high-school algebra. On the other hand, they have an unusual feature that both generators and algebraic relations among them are not given from the outset but are produced by an iterative process of *seed mutations*.

Andrei Zelevinsky is professor of mathematics at Northeastern University. His email address is `andrei@neu.edu`.
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Before discussing the general definition, let us look at cluster algebras of rank two. One associates such an algebra $\mathcal{A}(b,c)$ with any pair (b,c) of positive integers. The cluster variables in $\mathcal{A}(b,c)$ are the elements x_m, for $m \in \mathbb{Z}$, defined recursively by the *exchange relations*

$$x_{m-1}x_{m+1} = \begin{cases} x_m^b + 1 & \text{if } m \text{ is odd;} \\ x_m^c + 1 & \text{if } m \text{ is even.} \end{cases}$$

Iterating these relations, we can express each x_m as a rational function of x_1 and x_2. Thus, $\mathcal{A}(b,c)$ is the subring generated by all the x_m inside the field of rational functions $\mathbb{Q}(x_1,x_2)$. The clusters are the pairs $\{x_m, x_{m+1}\}$ for $m \in \mathbb{Z}$. Starting with the initial cluster $\{x_1,x_2\}$, we can reach any other cluster by a series of exchanges

$$\cdots \leftrightarrow \{x_0,x_1\} \leftrightarrow \{x_1,x_2\} \leftrightarrow \{x_2,x_3\} \leftrightarrow \ldots .$$

For an arbitrary rank n, the construction is similar. Each cluster $\mathbf{x} = \{x_1,\ldots,x_n\}$ is a collection of algebraically independent elements of some ambient field, and each cluster variable x_k can be exchanged from $\mathbf{x}$ by forming a new cluster $\mathbf{x}' = \mathbf{x} - \{x_k\} \cup \{x'_k\}$. Here x_k and x'_k are related by an exchange relation of the following form: the product $x_k x'_k$ is equal to the sum of two disjoint monomials in the variables from $\mathbf{x} \cap \mathbf{x}' = \mathbf{x} - \{x_k\}$. (For simplicity, we restrict ourselves to the *coefficient-free* case, where both monomials appear with the coefficient 1.) The exponents in these two monomials are encoded by an $n \times n$ integer matrix $B = (b_{ij})$ called the *exchange matrix*; it is usually assumed to be skew-symmetrizable, that is, $d_i b_{ij} = -d_j b_{ji}$ for some positive integers $d_1,\ldots,d_n$. The corresponding exchange relations take the form

$$x_k x'_k = \prod_i x_i^{[b_{ik}]_+} + \prod_i x_i^{[-b_{ik}]_+},$$

where we use the notation $[b]_+ = \max(b,0)$.

A pair $(\mathbf{x}, B)$ as above is called a *seed*. To begin the iterative process, we extend, for each index k, the transformation $\mathbf{x} \mapsto \mathbf{x}'$ of clusters to the transformation $(\mathbf{x}, B) \mapsto (\mathbf{x}', B')$ of seeds called the *seed mutation* in direction k. Its key ingredient is the *matrix mutation* $\mu_k : B \mapsto B' = (b'_{ij})$ given by the rule

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + [b_{ik}]_+[b_{kj}]_+ & \\ \quad -[-b_{ik}]_+[-b_{kj}]_+ & \text{otherwise.} \end{cases}$$

The corresponding cluster algebra is then defined as the subring of $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all cluster variables, that is, by the union of all clusters obtained from the initial cluster $\mathbf{x}$ by iterating seed mutations in all directions.

The definition of matrix mutations may look strange at first. (Remarkably, the same rule came up in recent work on Seiberg dualities in string theory.) One of its main consequences—and one of the main reasons for introducing it—is the *Laurent phenomenon*: every cluster variable, which a priori is just a rational function in the elements of a given cluster, is in fact a Laurent polynomial with integer coefficients. For instance, in each rank 2 algebra $\mathcal{A}(b, c)$, every cluster variable x_m is a Laurent polynomial in x_1 and x_2. As a corollary, if we specialize all elements of some cluster to 1 then all cluster variables become integers. This is rather unexpected since, in the process of seed mutations, every cluster variable eventually appears as the denominator of the expression used for producing a new one. The cluster algebra machinery provides a unified explanation of several previously known phenomena of this kind. One example is the *Somos-5* sequence discovered some years ago by M. Somos: its first five terms are equal to 1, and the rest are given by the recurrence relation $x_m x_{m-5} = x_{m-1}x_{m-4} + x_{m-2}x_{m-3}$. The fact that all terms of this sequence are integers can be deduced from the Laurent phenomenon for cluster algebras.

The Laurent polynomial expressions for cluster variables are not yet well understood. S. Fomin and the author conjectured that all coefficients in these Laurent polynomials are *positive*; this is still open in general. The strongest known result in this direction (by P. Caldero and M. Reineke), which establishes the conjecture in many special cases, uses some heavy machinery: it is based on a beautiful geometric interpretation (due to P. Caldero and F. Chapoton) of the coefficients in question as Euler-Poincaré characteristics of certain *quiver Grassmannians*.

A cluster algebra is of *finite type* if it has finitely many seeds. As shown in [3], these algebras are classified by the same Cartan-Killing types (or Dynkin diagrams) as semisimple Lie algebras, finite root systems, and many other important structures. In particular, the rank-two cluster algebra $\mathcal{A}(b, c)$ is of finite type if and only if $bc \le 3$; the reader is invited to check that if $bc = 1$ (respectively 2; 3) then the sequence of cluster variables is periodic with period 5 (respectively 6; 8). These three cases are naturally associated with the root systems A_2, B_2, and G_2.

The study of cluster algebras of finite type brings to light new combinatorial and geometric structures associated to root systems. For example, by a result of F. Chapoton, S. Fomin, and the author, the *cluster complex* (the simplicial complex whose vertices are cluster variables and whose maximal simplices are clusters) can be identified with the dual face complex of a simple convex polytope, the *generalized associahedron*. These polytopes include as special cases the Stasheff associahedron (in type A_n), and the Bott-Taubes *cyclohedron* (in type B_n).

We conclude this brief tour of cluster algebras with the following informal question: how can one detect a cluster algebra structure in a commutative ring of interest? A possible strategy is to look for three-term relations satisfied by some naturally arising elements of the ring and try to interpret them as exchange relations. The mutation mechanism will take care of the rest. One recent example (due to S-W. Yang and the author): consider the variety X_n of tridiagonal $(n+1) \times (n+1)$ unimodular complex matrices $U = (u_{ij})$ with $u_{i,i+1} = u_{i+1,i} = 1$ for all i. The top-left-corner principal minors $D_1, \ldots, D_n$ of U satisfy the recurrence relations $D_{k+1} = u_{k+1,k+1}D_k - D_{k-1}$, which feature prominently in the classical theory of orthogonal polynomials. Rewritten in the form $D_k u_{k+1,k+1} = D_{k+1} + D_{k-1}$, these relations acquire distinct "cluster flavor". Indeed, the coordinate ring of X_n can be made into a cluster algebra (of finite type A_n) with the initial cluster $\mathbf{x} = \{D_1, \ldots, D_n\}$, so that the above relations are exactly the exchange relations from $\mathbf{x}$. Other examples of three-term relations leading to cluster algebras include short Plücker relations between Plücker coordinates on Grassmannians, and Ptolemy relations between Penner coordinates on the decorated Teichmüller space of a bordered Riemann surface.

Further Reading

1. S. Fomin, `http://www.math.lsa.umich.edu/~fomin/cluster.html`.
2. S. Fomin and A. Zelevinsky, Cluster algebras I: Foundations, *J. Amer. Math. Soc.* **15** (2002), 497–529.
3. ———, Cluster algebras II: Finite type classification, *Invent. Math.* **154** (2003), 63–121.