WHAT IS...

Stable Commutator Length?

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Kronecker allegedly once said, "God created the natural numbers; all the rest is the work of man." But to a topologist, the natural numbers are just a tool for classifying orientable surfaces, by counting the number of handles (or *genus*).

Commutator length is the algebraic analogue of "number of handles" in group theory. If G is a group, and $a, b \in G$, the *commutator* of a and b is the element $aba^{-1}b^{-1} \in G$. The *commutator subgroup*, denoted $[G, G]$, is the group generated by all commutators, and for $g \in [G, G]$, the *commutator length* of g, denoted $\mathrm{cl}(g)$, is the smallest number of commutators in G whose product is equal to g. The size of $[G, G]$ is one way of measuring the extent to which the group G fails to obey the commutative law $ab = ba$. If G is the fundamental group of a space X, and $g \in G$ is represented by a homologically trivial loop $\gamma \subset X$, the commutator length of g is the smallest genus of a surface that admits a map to X in such a way that the boundary of the surface maps to γ.

Estimating minimal genus is important in many areas of low-dimensional topology. A knot K in the 3-sphere bounds an orientable surface (in its complement) of some genus, called a *Seifert surface*. The least such genus is equal to the commutator length of the *longitude* of the knot, a certain distinguished conjugacy class in the group $\pi_1(S^3 - K)$. As another example, given a 3-manifold M, one can try to find the "simplest" 4-manifold W that bounds it. If M is a certain kind of 3-manifold—for instance, a surface bundle over a circle—one can ask for W to be a surface bundle over a surface, and try to estimate (from below) the genus of the base. This is tantamount to calculating the commutator length of an element in the *mapping class group* of a surface (i.e., the group of self-homeomorphisms of a surface, up to isotopy).

Calculating commutator length (even in finite groups!) is notoriously difficult. A famous conjecture of Ore from 1951, whose proof was announced only very recently, says that in a finite, non-cyclic simple group, $\mathrm{cl} = 1$ for all nonzero elements. So instead, one can stabilize the problem. The *stable commutator length* of g, denoted $\mathrm{scl}(g)$, is the limit

$$\mathrm{scl}(g) = \lim_{n\to\infty} \frac{\mathrm{cl}(g^n)}{n}$$

Commutator length is subadditive in $[G, G]$, so this limit exists.

Computing stable commutator length is still difficult, but feasible in many cases. For instance, there now exist fast algorithms to compute scl in free groups. Since every group is a quotient of a free group, calculating scl on elements in a free group gives universal upper bounds on scl.

Figure 1 plots values of scl by frequency on 64,010 random elements of word length 32 in a free group on two generators. For simplicity, we restrict attention to a subclass of elements for which computation is particularly tractable, namely those represented by *alternating words*.

Some conspicuous features of this plot include the following:

(1) the existence of a *spectral gap* between 0 and 0.5, and another gap immediately above 0.5

(2) the non-discrete nature of the set of values attained

(3) the relative abundance of elements for which $\mathrm{scl} \in \frac{1}{2}\mathbb{Z}$, and (to a lesser extent) $\in \frac{1}{6}\mathbb{Z}$ and so on to other denominators, revealing a "self-similarity" in the histogram, and a power law for the size of the "spikes" of the form $\mathrm{freq}(p/q) \sim q^{-\delta}$, reminiscent of similar power laws that arise in 1-dimensional dynamics (e.g. the phenomenon of *Arnol'd tongues*)

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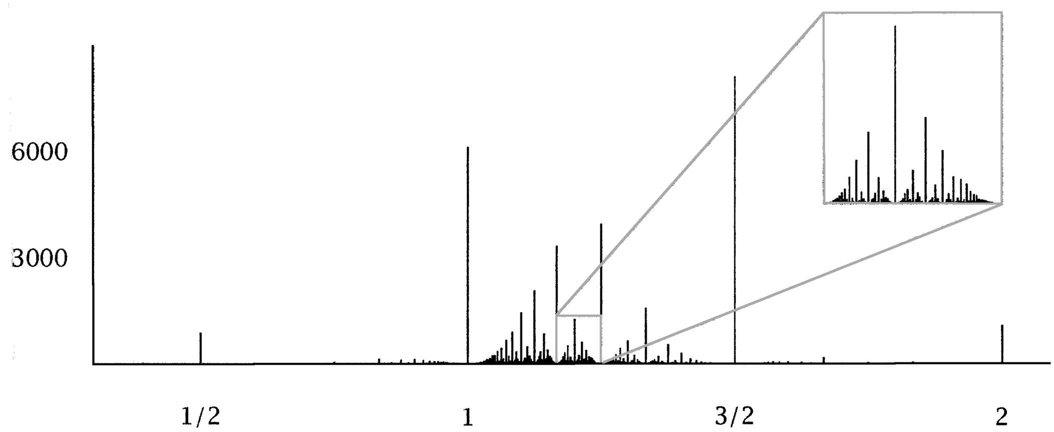

Figure 1. Values of scl on 64,010 alternating words of length 32. The horizontal axis is scl and the vertical axis is frequency.

See [2] for a theoretical explanation of some of these features; also see the references of [2] and [1] for further reading.

A fact hinted at in this figure is that the values of scl attained in a free group are all *rational*. This is not a universal phenomenon: there are examples of finitely presented groups with irrational scl, but interestingly enough, no known examples where scl is irrational and algebraic. This rationality (or otherwise) has consequences in dynamics. For certain groups G of homeomorphisms of the circle, there is a natural central extension $\widehat{G}$ of G with the property that rationality of stable commutator length in $\widehat{G}$ is directly related to the existence of periodic orbits in S^1 for elements $g \in G$. A similar relationship between rationality and dynamics exists for certain groups of symplectic matrices.

One can learn a lot about an invariant by studying when it vanishes. There are many important classes of groups G for which scl is identically zero on $[G, G]$, including

(1) torsion groups

(2) solvable groups, and more generally, amenable groups

(3) $\mathrm{SL}(n, \mathbb{Z})$ for $n \geq 3$, and many other lattices (uniform and nonuniform) in higher rank Lie groups

(4) groups of piecewise-linear homeomorphisms of $[0, 1]$; Thompson's group of piecewise dyadic rational linear homeomorphisms of the circle

On the other hand, there are many other classes of groups for which scl is nonzero on typical elements, including

(1) free groups, hyperbolic groups

(2) mapping class groups

(3) groups of area-preserving diffeomorphisms of surfaces

The problem of computing scl can be recast in homological terms, by counting triangles (or, formally, 2-chains) instead of genus. The (real) singular chain groups of a space, and the terms in the bar resolution of a group, are vector spaces with canonical bases, and one can use these bases to make these vector spaces into *normed* spaces. *Bounded (co)-homology*, introduced by Gromov [3], arises when one studies the natural L^1 and L^∞ norms on these vector spaces using the tools of homological algebra. One can interpret stable commutator length as the infimum of the L^1 norm (suitably normalized) on chains representing a certain (relative) class in group homology.

The unit balls in the L^1 and L^∞ norms on finite-dimensional vector spaces are rational polyhedra. Computing the L^1 norm of a homology class is a kind of linear programming problem. In certain groups, computing scl reduces to a *finite-dimensional* linear programming problem, which explains the rationality of scl in some cases. The polyhedral nature of L^1 norms is manifest in several closely related contexts. Most well-known is the *Thurston norm* on the homology of a 3-manifold, which turns up again and again throughout low-dimensional topology, in the theories of taut foliations, symplectic 4-manifolds, quasigeodesic flows, Heegaard Floer homology, and so on.

Thus stable commutator length gives insight into bounded (co-)homology of groups and spaces, and conversely.

Further Reading

[1] C. Bavard, Longeur stable des commutateurs, *L'Enseign. Math.* **37** (1991), 109–150.

[2] D. Calegari, *scl*, to be published in the MSJ monograph series; available from the author's website.

[3] M. Gromov, Volume and bounded cohomology, *IHES Publ. Math.* **56** (1982), 5–99.