WHAT IS...

Measure Rigidity?

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Measure rigidity is a commonly used shorthand term for rigidity of invariant measures. Here the term rigidity does not have a formal mathematical definition. Rather, it is an informal description of the frequently appearing phenomenon that, for certain mathematical objects, the only examples have much more algebraic structure than was originally demanded. A simple example would be the statement that we have rigidity of continuous homomorphisms of the real line. This is a shorthand way of saying that all continuous homomorphisms of the additive group $\mathbb{R}$ to itself are in fact linear. (A more subtle result is that one even has rigidity of measurable homomorphisms of the real line.) As the requirement to be a homomorphism is much weaker than the requirement to be linear, this is indeed an example where additional algebraic structure is forced. One may say that a linear map cannot be perturbed to a nonlinear continuous homomorphism or that linear maps are rigid.

To explain what an invariant measure is, we need to introduce a transformation on a space—a dynamical system. Let X be a metric space. For example, we could set $X = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, which is often called the torus or the circle group and consists of the cosets $x = r + \mathbb{Z}$ for $r \in \mathbb{R}$. Let us agree that a measure μ on a metric space X is simply a way of assigning to every continuous function $f \in C(X)$ a number—its integral $\int f(x)d\mu(x)$ or $\int f d\mu$ with respect to μ—so that the usual properties of an integral hold: we require that $f \to \int f d\mu$ is linear and that $f \geq 0$ implies $\int f d\mu \geq 0$. In fact, we will be studying probability measures on X, and so the integral of the constant function 1_X is one. The Riemann integral $\int_0^1 f(r+\mathbb{Z})dr$ for $f \in C(\mathbb{R}/\mathbb{Z})$ would in this sense define a probability measure, which is called the Lebesgue measure $m_{\mathbb{T}}$.

Now let $T : X \to X$ be a continuous map. A probability measure μ on X is invariant if $\int f d\mu = \int f \circ T d\mu$. One should think of T as the time evolution of the dynamical system, of f as the outcome of a physical experiment, and of the integral as the expected value for the outcome of f. Then the invariance of μ is simply the requirement that the expected value of the outcome is the same now and one time unit later. The set $\mathcal{M}(T)$ of invariant probability measures depends crucially on the transformation T. For many maps T this set $\mathcal{M}(T)$ is rather large, and it is impossible to give a reasonable description. However, sometimes we also have rigidity of invariant measures: the set of invariant measures shows a surprising amount of structure. We will give examples of both scenarios.

Let us start by studying the case of the circle rotation defined by $R_\alpha(x) = x+\alpha$, where $x \in \mathbb{T}$, $\alpha \in \mathbb{R}$ is a given number, and addition is understood as modulo $\mathbb{Z}$. Let us assume $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is irrational, as this makes the dynamical system more interesting. The standard substitution rule of the Riemann integral shows in fact that the Lebesgue measure $m_{\mathbb{T}}$ is invariant under R_α. We claim that $m_{\mathbb{T}}$ is the only R_α-invariant measure. One way to see this uses the characters $e_n(x) = e^{2\pi i n x} \in C(\mathbb{T})$ with $n \in \mathbb{Z}$. Suppose μ is an unknown R_α-invariant probability measure. Then by definition $\int e_n d\mu = \int e_n \circ R_\alpha d\mu$. For $n = 0$ this contains no new information, as $e_0 = 1_{\mathbb{T}}$ is constant. So assume $n \neq 0$; then we have $e_n(R_\alpha(x)) = e^{2\pi i n\alpha} e_n(x)$, which shows that $\int e_n \circ R_\alpha d\mu = e^{2\pi i n \alpha} \int e_n d\mu$. As $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we have $e^{2\pi i \alpha n} \neq 1$, and so $\int e_n d\mu = 0$. This agrees with the value $\int e_n dm_{\mathbb{T}}$ that the Lebesgue measure $m_{\mathbb{T}}$ assigns to e_n. Hence we have $\int f d\mu = \int f dm_{\mathbb{T}}$ for any finite linear combination of characters e_n. As the latter can be used to approximate any other continuous function uniformly (by the Stone-Weierstrass theorem), one sees that $\mu = m_{\mathbb{T}}$. This is the most basic example of rigidity of invariant measures and also the strongest form of it: If $\mathcal{M}(T)$ contains only one measure, then T is called uniquely ergodic.

One may ask why one should care about rigidity of invariant probability measures. The answer lies in a simple construction. Let $T : X \to X$ be an arbitrary continuous map from a compact metric space X to itself. Then notice that for any $f \in C(X)$, $x \in X$, and any large N, the ergodic (time) average $\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x)$ is bounded by $\|f\|_\infty = \max\{|f(x)| : x \in X\}$ and is close to being

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invariant under T in the sense that the difference between the average for f and for $f \circ T$ is bounded by $\frac{2\|f\|_\infty}{N}$. Fixing a function f, we can choose a subsequence N_k along which the ergodic average for f converges—we may think of the limit as $\int f d\mu$ for some $\mu \in \mathcal{M}(T)$. To completely define μ on all functions $f \in C(X)$, one has to continue picking subsequences until one finally arrives at a subsequence for which the ergodic average converges for all functions. In a sense, the measure μ describes the statistical behavior of the orbit $x, T(x), T^2(x), \ldots$, at least for certain very long stretches of time. Combined with measure rigidity this can have very interesting consequences. If T is uniquely ergodic and $\mu \in \mathcal{M}(T)$, then no matter which subsequence (of a subsequence, etc.) one may have picked, the ergodic average for that subsequence must converge to $\int f d\mu$ because μ is the only T-invariant measure. However, this independence of the subsequence means that $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int f d\mu$, both for every $f \in C(X)$ and for every $x \in X$. One may summarize this by saying that unique ergodicity implies equidistribution (with respect to μ) of the orbit $x, T(x), T^2(x), \ldots$ for any $x \in X$. Not only does this help to describe the closure of the orbit (optimally as being equal to the support of μ), but it also asymptotically describes (in terms of μ) the amount of time the orbit will spend in various parts of the space.

Another example of a uniquely ergodic transformation is $S : (x_1, x_2) \to (x_1 + 2\alpha, x_2 + x_1)$ on $\mathbb{T}^2$ where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Furstenberg proved the unique ergodicity of S and used the orbit of $(\alpha, 0)$ to derive the equidistribution of the sequence $\alpha, 4\alpha, 9\alpha, \ldots, n^2\alpha, \ldots$ in $\mathbb{T}$, which re-proved a theorem of Weyl. The examples above are still quite simple, but a similar understanding of invariant probability measures for more general classes of transformations can be a very powerful tool. In fact, Marina Ratner proved a very general measure classification theorem (concerning unipotent group actions of quotients of Lie groups) and derived from it equidistribution and orbit closure theorems; the orbit closure theorem is known as Raghunathan's conjecture. The measure classification is indeed a good example of rigidity: by assumption the measure is known to be invariant under a small subgroup, and in the end it is known to be a highly structured measure (called algebraic or Haar) for which the support is the orbit of a bigger group. However, unlike the above cases, in general there will be many different invariant probability measures. These theorems and their extensions by Dani, Eskin, Margulis, Mozes, Ratner, Shah, Tomanov, and others have found many applications, in particular in number theory.

Another transformation on $\mathbb{T}$ is the times-two map $T_2(x) = 2x$ for $x \in \mathbb{T}$. Here the Lebesgue measure $m_{\mathbb{T}}$ is again invariant under T_2, and so is the measure defined by $\int f d\delta_0 = f(0)$. Moreover, we can in fact apply our above construction to produce many invariant probability measures. Represent $x \in [0, 1)$ by its binary expansion, and notice that application of T_2 corresponds to shifting the binary expansion of x. Choosing this infinite sequence in the digits 0 and 1, we may ensure, for example, that we never see the finite sequence 000 or 111. Then the above construction will lead to T_2-invariant probability measures that have support disjoint from $(-\frac{1}{8}, \frac{1}{8}) + \mathbb{Z} \subset \mathbb{T}$. For this transformation the abundance of T_2-orbits of different types (any sequence in 0s and 1s defines an orbit) is reflected in the abundance of T_2-invariant measures.

A highly interesting question was raised by Furstenberg around 1967. He showed that the orbit set $\{2^k 3^\ell (x) : k, \ell \geq 1\}$ is dense in $\mathbb{T}$ whenever the starting point $x \in \mathbb{T} \setminus \mathbb{Q}$ is irrational; here the orbit is taken with respect to the semigroup generated by T_2 and the times-three map T_3. As we have discussed, there is often a correspondence between orbits and invariant measures. Hence it is natural to ask the following: What are the probability measures on $\mathbb{T}$ that are at the same time invariant under T_2 and under T_3? Certain rational numbers $r \in \mathbb{Q}$ are periodic for both T_2 and T_3, and with these one can easily define invariant probability measures. Also, we know that the Lebesgue measure is invariant. Are these (and their convex combinations) the only ones? The best-known result towards this conjecture is due to Dan Rudolph, and several generalizations have been obtained by Kalinin, A. Katok, Lindenstrauss, Spatzier, and me. Similar to Raghunathan's conjecture, these generalized conjectures are phrased for dynamical systems defined on quotients of Lie groups. However, in this case the dynamical system is defined by (several commuting) diagonal matrices instead of unipotent matrices. Even though Furstenberg's question and its generalizations are still open, the partial results have already found several applications. The most striking of these may be Lindenstrauss's proof of the equidistribution of the arithmetic Laplace-eigenfunctions (Hecke-Maaß cusp forms) on certain quotients of the hyperbolic plane.

References

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