WHAT IS... a Gaussian Entire Function?

Fedor Nazarov and Mikhail Sodin

Random analytic functions have been attracting the attention of mathematicians since the 1930s, though the focus of interest has been changing with time. Just as the distribution of eigenvalues is the essence of the random matrix theory, central to the study of random analytic functions are their zero sets. Our random functions are Gaussian and live on the complex plane. The instance when the random zero set is invariant in distribution with regard to (w.r.t., for short) isometries of the plane is the most interesting one. Here we will introduce the reader to a remarkable model of Gaussian entire functions with invariant distribution of zeros.

A Gaussian entire function $f(z)$ is the sum $\sum_k \zeta_k f_k(z)$ of entire functions f_k with independent standard complex Gaussian random coefficients ζ_k (whose density w.r.t. the area measure in $\mathbb{C}$ is $\frac{1}{\pi}e^{-|\zeta|^2}$). We assume that

$$\sum_k |f_k(z)|^2 < \infty \text{ locally uniformly in } \mathbb{C},$$

and also that the functions f_k are linearly independent over ℓ^2, i.e., $\sum_k a_k f_k$ with $\{a_k\} \in \ell^2$ does not vanish identically unless all $a_k = 0$. The first condition implies that almost surely (a.s., for short) the random function f is entire.

Each Gaussian entire function can be uniquely identified with some Hilbert space $\mathcal{H}$ of entire functions (the image of the mapping

$$\ell^2 \ni \{a_k\} \mapsto \sum_k a_k f_k$$

with the scalar product borrowed from ℓ^2) so that the covariance function

$$C_f(z,w) = \mathcal{E}\left\{f(z)\overline{f(w)}\right\} = \sum_k f_k(z)\overline{f_k(w)}$$

is the reproducing kernel in $\mathcal{H}$; i.e.,

$$g(w) = \langle g, C(\cdot, w)\rangle_{\mathcal{H}} \text{ for every } \; g \in \mathcal{H}, w \in \mathbb{C}.$$

The functions f_k form an orthonormal basis in $\mathcal{H}$. Reversing the order, one can start with a Hilbert space $\mathcal{H}$ of entire functions with the reproducing kernel $C_{\mathcal{H}}$, take an orthonormal basis $\{f_k\}$ in $\mathcal{H}$, and build a Gaussian entire function $f_{\mathcal{H}} = \sum_k \zeta_k f_k$ with covariance $C_{\mathcal{H}}$. Since the Gaussian process is determined by its covariance function, this construction does not depend on the choice of the basis in $\mathcal{H}$.

The properties of the (random) zero set $\mathcal{Z}_f = f^{-1}\{0\}$ are encoded in its (random) counting measure n_f defined by $n_f(A) = \#(\mathcal{Z}_f \cap A)$ for any Borel set A. Recall that for every analytic function f, we have

$$n_f = \frac{1}{2\pi}\Delta \log|f|$$

with the Laplacian taken in the sense of distributions. This makes it possible to use complex analysis tools for the study of the distribution of zeroes of Gaussian analytic functions. Using this formula, and taking the expectation of both sides, we get $\mathcal{E}n_f = \frac{1}{2\pi}\Delta\mathcal{E}\log|f|$. Note that $\frac{f(z)}{\sqrt{C_f(z,z)}}$ is the standard complex Gaussian random variable, so

$$\mathcal{E}\log|f| = \frac{1}{2}\log C_f(z,z) + \text{const}\,.$$

This way, we arrive at the elegant Edelman-Kostlan formula

$$\mathcal{E}n_f(z) = \frac{1}{4\pi}\Delta\log C_f(z,z)\,.$$

The surprising Calabi rigidity tells us that the mean $\mathcal{E}n_f$ determines the distribution of $\mathcal{Z}_f$. Alas, this uniqueness gives us no hint as to how to find the distribution of n_f from its mean $\mathcal{E}n_f$.

All the aforementioned results are valid for Gaussian analytic functions in other plane domains. It is the gaussianity that is crucial, not the domain of f.

Fedor Nazarov is professor of mathematics at the University of Wisconsin, Madison. His email address is `nazarov@math.wisc.edu`.

Mikhail Sodin is professor of mathematics at Tel Aviv University. His email address is `sodin@post.tau.ac.il`.

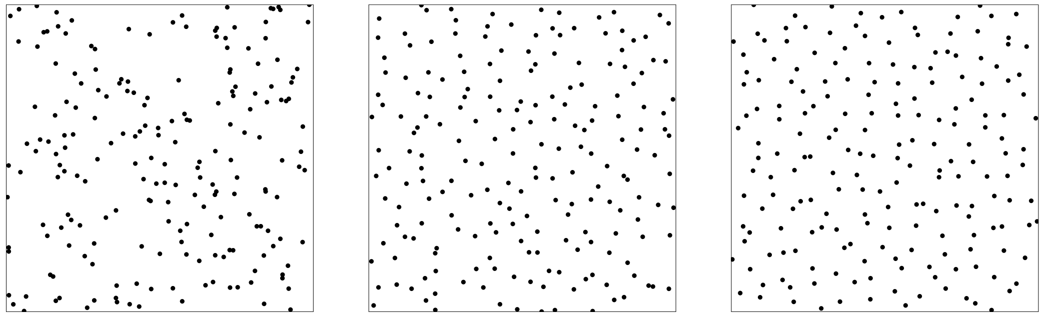

Figure 1. Samples of the Poisson process (figure by B. Virág), limiting Ginibre process, and zeroes of a GEF (figure by M. Krishnapur). The last two processes are quite different, though the eye does not easily distinguish them.

It is not at all obvious that there exist Gaussian entire functions with zeros having a translation-invariant distribution. It is not difficult to see that Gaussian entire functions cannot be translation invariant themselves.[1] Fortunately, a weaker property called projective invariance is sufficient for the translation invariance of zeros. Namely, if there is a family of nonrandom functions ϕ_λ ($\lambda \in \mathbb{C}$) without zeros such that the random functions $\phi_\lambda(z)f(z+\lambda)$ and $f(z)$ have the same distribution, then the distribution of $\mathcal{Z}_f$ is translation invariant.

Letting $f_k(z) = z^k/\sqrt{k!}$, we get $C_f(z,w) = e^{z\overline{w}}$, which is the kernel for the classical Fock-Bargmann space of entire functions, that is, the closure of polynomials in $L^2(\mathbb{C}, \frac{1}{\pi}e^{-|z|^2})$. The Gaussian entire function associated with this Hilbert space is projective invariant w.r.t. isometries of $\mathbb{C}$. The rotation and reflection invariance are obvious. To show the translation invariance, note that the Gaussian entire function

$$f(z+\lambda)e^{-z\overline{\lambda}-\frac{1}{2}|\lambda|^2}, \quad \lambda \in \mathbb{C},$$

has the same covariance function as f.

By the Edelman-Kostlan formula,

$$\mathcal{E}n_f = \frac{1}{4\pi}\Delta|z|^2 = \frac{1}{\pi}m,$$

where m is the area measure (we treat the average $\mathcal{E}n_f$ as a measure). Replacing f by $f_L(z) = f(\sqrt{\frac{L}{\pi}}\,z)$, this average can be changed to Lm with any $L > 0$. On the other hand, if zeros of a Gaussian entire function F have a translation-invariant distribution, then the mean $\mathcal{E}n_F$ is a translation-invariant measure on $\mathbb{C}$. Hence, it is proportional to the area measure m; i.e., $\mathcal{E}n_F = Lm$ with a constant $L > 0$. Then by the Calabi rigidity, the zero sets $\mathcal{Z}_F$ and $\mathcal{Z}_{f_L}$ have the same distribution. In other words, the only freedom in this construction is the scaling $z \mapsto tz$ with $t > 0$, and the Gaussian Entire Function (GEF, for short) with translation-invariant zeros is essentially unique. Geometers know this in a different wording: $z \mapsto \{z^k/\sqrt{k!}\}_{k\ge 0}$ is an isometric embedding of the Euclidean plane into the projective Hilbert space $P(\ell_2)$ equipped with the Fubini-Study metric, and this embedding is essentially unique.

The construction leading to projective invariance has been known since the 1930s, though the corresponding Gaussian functions were introduced only in the 1990s by Kostlan, Bogomolny-Bohigas-Lebouef, Shub-Smale, and Hannay. It is worth mentioning that there are similar constructions for other domains with transitive groups of isometries (hyperbolic plane, Riemann sphere, cylinder, and torus).

Few natural translation-invariant random point processes on the plane are known. The most widely studied one is the Poisson process, where for any collection of disjoint subsets of the plane, the numbers of points in these subsets are independent, and the mean number of points in a set is proportional to its area. This process is invariant w.r.t. all measure-preserving transformations of the plane, which is far more than we asked for. Another example is a one-component plasma of charged particles of one sign confined by a uniform background of the opposite sign. It contains as a special case the large N limit of Ginibre ensemble of eigenvalues of $N \times N$ matrices with independent standard complex Gaussian entries.[2] One more example is the random zero set $\mathcal{Z}_f$ of GEF f.

The Poisson process can be easily recognized since its points can clump together while, in contrast, the Ginibre eigenvalue process and the GEF zero process have local repulsion between points: it is unlikely that one would see two points very close to each other. The latter two look rather alike, although some of their characteristics are quite different. For instance, as Forrester and Honner observed, if h is a smooth function with

[1] *B. Weiss showed that, unexpectedly, there are translation-invariant random entire functions, not Gaussian, of course.*

[2] *Though one-component plasma has been studied by physicists for a long time, it seems that almost all rigorous mathematical results still pertain only to the special case of Ginibre ensemble.*

compact support, then the variance of the linear statistics of zeros $n_f(r;h) = \sum_{z_f} h(\frac{a}{r})$ decays as $\|\Delta h\|^2_{L^2} r^{-2}$ for $r \to \infty$, while in the Ginibre case the corresponding variance tends to the limit proportional to $\|\nabla h\|^2_{L^2}$ (for the Poisson process the variance grows with r as $\|h\|^2_{L^2} r^2$).

The decay of the variance of smooth linear statistics for zeros of GEF yields another surprising rigidity. We fix a bounded plane domain G and suppose that we know the configuration of zeros of f outside of G. Then taking any smooth compactly supported test-function h that equals 1 in some neighborhood of the origin, we recover the number of zeros of f inside G:

$$n_f(G) = \lim_{r\to\infty}\left\{\frac{r^2}{\pi}\iint_{\mathbb{C}} h\,\mathrm{d}m - \sum_{a\in Z_f\setminus G} h\left(\frac{a}{r}\right)\right\} \quad \text{a.s.}$$

At the end of this introductory tour, we will take a brief look at the random potential

$$U_f(z) = \log|f(z)| - \tfrac{1}{2}|z|^2$$

and at its gradient field ∇U_f. Their distributions are invariant w.r.t. isometries of $\mathbb{C}$, and

$$\tfrac{1}{2\pi}\Delta U_f = \mathrm{div}(\nabla U_f) = n_f - \tfrac{1}{\pi}m.$$

The potential U_f equals $-\infty$ on $\mathcal{Z}_f$ and has no other local minima since its Laplacian is negative on $\mathbb{C}\setminus\mathcal{Z}_f$. The gradient curves oriented in the direction of decay of U_f and terminating at $a \in \mathcal{Z}_f$ form a basin B_a. Different basins are separated by the gradient curves joining local maxima with saddle points. Remarkably, all bounded basins have the same area π:

$$1 - \frac{1}{\pi}m(B_a) = \frac{1}{2\pi}\iint_{B_a}\Delta U_f = \frac{1}{2\pi}\int_{\partial B_a}\frac{\partial U_f}{\partial n} = 0\,.$$

One can prove that the probability of a long gradient curve decays exponentially with its diameter, so, a.s., all basins are bounded. Thus, one obtains a random partition of $\mathbb{C}$ into nice bounded domains of equal area with many intriguing properties.

We hope that we have aroused the reader's curiosity by now. Note that we have presented only a tiny portion of results and questions concerning Gaussian analytic functions and their zeros.

Further Reading

For those new to this subject, we recommend the book *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*, J. B. Hough, M. Krishnapur, Y. Peres, B. Virág, Amer. Math. Soc., 2009. The electronic version is available at `stat-www.berkeley.edu/~peres/GAF_book.pdf`.

The lecture by M. Sodin at the 4th ECM, Stockholm, 2004 (`arXiv:math/0410343`), surveys results obtained by that time. Further developments can be found in recent papers written by the authors with A. Volberg, by B. Tsirelson, and by A. Nishry, and posted in the `arXiv`.

Complex-geometry-oriented readers might be interested in reading the papers by P. Bleher, M. Douglas, B. Shiffman, and S. Zelditch, which are also posted in the `arXiv`.

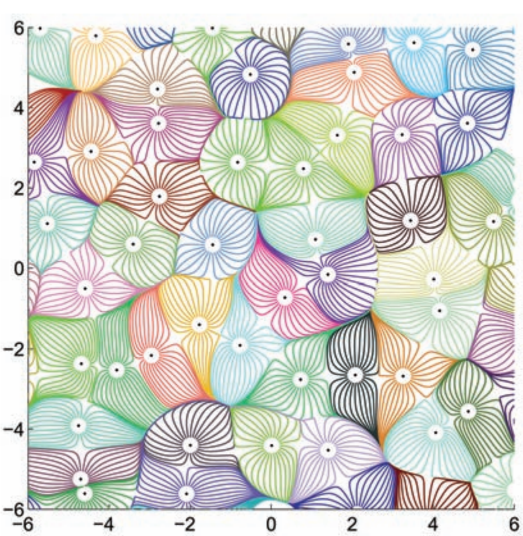

Figure 2. Random partition of the plane into domains of equal area generated by the gradient flow of the random potential U_f (figure by M. Krishnapur). The lines are gradient curves of U_f, the black dots are random zeros. Many basins meet at the same local maximum, so that two of them meet tangentially, while the others approach it cuspidally, forming long, thin tentacles.