a Perverse Sheaf?

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Manifolds are obtained by gluing open subsets of Euclidean space. Differential forms, vector fields, etc., are defined locally and then glued to yield a global object. The notion of sheaf embodies the idea of gluing. Sheaves come in many flavors: sheaves of differential forms, of vector fields, of differential operators, constant and locally constant sheaves, etc. A locally constant sheaf (*local system*) on a space X is determined by its monodromy, i.e., by a representation of the fundamental group $\pi_1(X, x)$ into the group of automorphisms of the fiber at $x \in X$: the sheaf of orientations on the Möbius strip assigns $-\mathrm{Id}$ to the generators of the fundamental group $\mathbb{Z}$. A sheaf, or even a map of sheaves, can be glued back together from its local data: exterior derivation can be viewed as a map between sheaves of differential forms; the gluing is possible because exterior derivation is independent of the choice of local coordinates.

The theory of sheaves is made more complete by considering *complexes of sheaves*. A complex of sheaves K is a collection of sheaves $\{K^i\}_{i \in \mathbb{Z}}$ and maps $d^i : K^i \to K^{i+1}$ subject to $d^2 = 0$. The i-th *cohomology sheaf* $\mathcal{H}^i(K)$ is $\ker d^i / \operatorname{im} d^{i-1}$. The (sheafified) de Rham complex $\mathcal{E}$ is the complex with entries the sheaves $\mathcal{E}^i$ of differential i-forms and with differentials $d : \mathcal{E}^i \to \mathcal{E}^{i+1}$ given by the exterior derivation of differential forms. By the Poincaré lemma, the cohomology sheaves are all zero, except for $\mathcal{H}^0 \simeq \mathbb{C}$, the constant sheaf.

The de Rham theorem, stating that the cohomology of the constant sheaf equals closed forms modulo exact ones, points to the fact that $\mathbb{C}$ and $\mathcal{E}$ are cohomologically indistinguishable from each other, even at the local level. The need to identify two complexes containing the same cohomological information via an isomorphism leads to the

notion of *derived category* ([2]): the objects are complexes and the arrows are designed to achieve the desired identifications. The inclusion of complexes $\mathbb{C} \subseteq \mathcal{E}$ is promoted by decree to the rank of isomorphism in the derived category because it induces an isomorphism at the level of cohomology sheaves.

While the derived category brings in a thick layer of abstraction, it extends the reach and flexibility of the theory. One defines the cohomology groups of a complex and extends to complexes of sheaves the ordinary operations of algebraic topology: pull-backs, push-forwards, cup and cap products, etc. There is also a general form of duality for complexes ([2]) generalizing classical Poincaré duality.

Perverse sheaves live on spaces with singularities: analytic spaces, algebraic varieties, PL spaces, pseudo-manifolds, etc. For ease of exposition, we limit ourselves to sheaves of vector spaces on complex algebraic varieties and to perverse sheaves with respect to what is called *middle perversity*. In order to avoid dealing with pathologies such as sheaves supported on the Cantor set, one imposes a technical condition called *constructibility*. Let us just say that the category D_X of *bounded constructible complexes* of sheaves on X sits in the derived category and is stable under the various topological operations mentioned above. If K is in D_X, only finitely many of its cohomology sheaves are nonzero and, for every i, the set supp $\mathcal{H}^i(K)$, the closure of the set of points at which the stalk is nonzero, is an algebraic subvariety.

A *perverse sheaf* on X is a bounded constructible complex $P \in D_X$ such that the following holds for $K = P$ and for its dual $P^\vee$:

$$(1) \qquad \dim_\mathbb{C} \operatorname{supp} \mathcal{H}^{-i}(K) \leq i, \quad \forall i \in \mathbb{Z}.$$

A map of perverse sheaves is an arrow in D_X.

The term "sheaf" stems from the fact that, just as in the case of ordinary sheaves, (maps of) perverse sheaves can be glued; as to "perverse", see below. The theory of perverse sheaves has its roots

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in the two notions of intersection cohomology and of $\mathcal{D}$-module. As we see below, perverse sheaves and $\mathcal{D}$-modules are related by the Riemann-Hilbert correspondence.

It is time for examples. If X is nonsingular, then $\mathbb{C}_X[\dim X]$, i.e., the constant sheaf in degree $-\dim_\mathbb{C} X$, is self-dual and perverse. If $Y \subseteq X$ is a nonsingular closed subvariety, then $\mathbb{C}_Y[\dim Y]$, viewed as a complex on X, is a perverse sheaf on X. If X is singular, then $\mathbb{C}_X[\dim X]$ is usually not a perverse sheaf. On the other hand, the intersection cohomology complex (see below) is a perverse sheaf, regardless of the singularities of X. The extension of two perverse sheaves is a perverse sheaf. The following example can serve as a test case for the first definitions in the theory of $\mathcal{D}$-modules. Let $X = \mathbb{C}$ be the complex line with origin $o \in X$, let z be the standard holomorphic coordinate, let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X, let a be a complex number, and let D be the differential operator $D : f \mapsto zf' - af$. The complex P_a

$$(2) \qquad 0 \longrightarrow P_a^{-1} := \mathcal{O}_X \xrightarrow{D} P_a^0 := \mathcal{O}_X \longrightarrow 0$$

is perverse. If $a \in \mathbb{Z}^{\geq 0}$, then $\mathcal{H}^{-1}(P_a) = \mathbb{C}_X$ and $\mathcal{H}^0(P_0) = \mathbb{C}_o$. If $a \in \mathbb{Z}^{<0}$, then $\mathcal{H}^{-1}(P_a)$ is the extension by zero at o of the sheaf $\mathbb{C}_{X \setminus o}$ and $\mathcal{H}^0(P_a) = 0$. If $a \notin \mathbb{Z}$, then $\mathcal{H}^{-1}(P_a)$ is the extension by zero at o of the local system on $X \setminus o$ associated with the branches of the multivalued function z^a and $\mathcal{H}^0(P_a) = 0$. In each case, the associated monodromy sends the positive generator of $\pi_1(X \setminus o, 1)$ to $e^{2\pi i a}$. The dual of P_a is P_{-a} (this fits well with the notions of adjoint differential equation and of duality for $\mathcal{D}$-modules). Every P_a is the extension of the perverse sheaf $\mathcal{H}^0(P_a)[0]$ by the perverse sheaf $\mathcal{H}^{-1}(P_a)[1]$. The extension is trivial (direct sum) if and only if $a \notin \mathbb{Z}^{>0}$.

A local system on a nonsingular variety can be turned into a perverse sheaf by viewing it as a complex with a single entry in the appropriate degree. On the other hand, a perverse sheaf restricts to a local system on some dense open subvariety. We want to make sense of the following slogan: *perverse sheaves are the singular version of local systems*. In order to do so, we discuss the two widely different ideas that led to the birth of perverse sheaves about thirty years ago: *the generalized Riemann-Hilbert correspondence* (RH) and *intersection cohomology* (IH) ([3]).

(RH) Hilbert's 21st problem is concerned with Fuchs-type differential equations on a punctured Riemann surface Σ. As one circuits the punctures, the solutions are transformed: the sheaf of solutions is a local system on Σ (see (2)).

The 21st problem asked whether any local system arises in this way (it essentially does). The sheafification of linear partial differential equations on a manifold gives rise to the notion of $\mathcal{D}$-module. A *regular holonomic* $\mathcal{D}$-module on a

complex manifold M is the generalization of the Fuchs-type equations on Σ. The sheaf of solutions is now replaced by a *complex of solutions*, which, remarkably, belongs to D_M. In (2), the complex of solutions is P_a, the sheaf of solutions to $D(f) = 0$ is $\mathcal{H}^{-1}(P_a)$, and $\mathcal{H}^0(P_a)$ is related to the (non)solvability of $D(f) = g$. Let $\mathcal{D}_{r,h}^b(M)$ be the bounded derived category of $\mathcal{D}$-modules on M with regular holonomic cohomology. RH states that the assignment of the (dual to the) complex of solutions yields an equivalence of categories $\mathcal{D}_{r,h}^b(M) \simeq D_M$. Perverse sheaves enter the center of the stage: they correspond via RH to regular holonomic $\mathcal{D}$-modules (viewed as complexes concentrated in degree zero).

In agreement with the slogan mentioned above, the category of perverse sheaves shares the following formal properties with the category of local systems: it is Abelian (kernels, cokernels, images, and coimages exist, and the coimage is isomorphic to the image), stable under duality, Noetherian (the ascending chain condition holds), and Artinian (the descending chain condition holds), i.e., every perverse sheaf is a finite iterated extension of *simple* (no subobjects) perverse sheaves. In our example, the perverse sheaves (2) are simple if and only if $a \in \mathbb{C} \setminus \mathbb{Z}$.

What are the simple perverse sheaves? Intersection cohomology provides the answer.

(IH) The *intersection cohomology groups* of a singular variety X with coefficients in a local system are a topological invariant of the variety. They coincide with ordinary cohomology when X is nonsingular and the coefficients are constant. These groups were originally defined and studied using the theory of geometric chains in order to study the failure, due to the presence of singularities, of Poincaré duality for ordinary homology, and to put a remedy to it by considering the homology theory arising by considering only chains that intersect the singular set in a controlled way. In this context, certain sequences of integers, called *perversities*, were introduced to give a measure of how a chain intersects the singular set, whence the origin of the term "perverse". The intersection cohomology groups thus defined satisfy the conclusions of Poincaré duality and of the Lefschetz hyperplane theorem.

On the other hand, the intersection cohomology groups can also be exhibited as the cohomology groups of certain complexes in D_X: the *intersection complexes* of X with coefficients in the local system. It is a remarkable twist in the plot of this story that the simple perverse sheaves are precisely the intersection complexes of the irreducible subvarieties of X with coefficients given by simple local systems!

We are now in a position to clarify the earlier slogan. A local system L on a nonsingular subvariety $Z \subseteq M$ gives rise to a regular holonomic

$\mathcal{D}$-module supported over the closure $\overline{Z}$. The same L gives rise to the intersection complex of $\overline{Z}$ with coefficients in L. Both objects extend L from Z to $\overline{Z}$ across the singularities $\overline{Z} \setminus Z$. By RH, the intersection complex is precisely the complex of solutions of the $\mathcal{D}$-module.

A pivotal role in the applications of the theory of perverse sheaves is played by the *decomposition theorem*: let $f : X \to Y$ be a proper map of varieties; then the intersection cohomology groups of X with coefficients in a simple local system are isomorphic to the direct sum of a collection of intersection cohomology groups of irreducible subvarieties of Y, with coefficients in simple local systems. For example, if $f : X \to Y$ is a resolution of the singularities of Y, then the intersection cohomology groups of Y are a direct summand of the ordinary cohomology groups of X. This "as-simple-as-possible" splitting behavior is the deepest known fact concerning the homology of complex algebraic varieties and maps. It fails in complex analytic and in real algebraic geometry. The decomposition of the intersection cohomology groups of X is a reflection in cohomology of a finer decomposition of complexes in D_Y. The original proof of the decomposition theorem uses algebraic geometry over finite fields (perverse sheaves make perfect sense in this context). For a discussion of some of the proofs see [1].

One striking application of this circle of ideas is the fact that the intersection cohomology groups of projective varieties enjoy the same classical properties of the cohomology groups of projective manifolds: the Hodge (p, q)-decomposition theorem, the hard Lefschetz theorem, and the Hodge-Riemann bilinear relations. This, of course, in addition to Poincaré duality and to the Lefschetz hyperplane theorem mentioned above.

The applications of the theory of perverse sheaves range from geometry to combinatorics to algebraic analysis. The most dramatic ones are in the realm of representation theory, where their introduction has led to a truly spectacular revolution: proofs of the Kazhdan-Lusztig conjecture, of the geometrization of the Satake isomorphism, and, recently, of the fundamental lemma in the Langlands program (see the survey [1]).

References

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