WHAT IS...

a Kinetic Solution for Degenerate Parabolic-Hyperbolic Equations?

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Nonlinear degenerate parabolic-hyperbolic equations are one of the most important classes of nonlinear partial differential equations. Nonlinearity and degeneracy are two main features of these equations and yield several striking phenomena that require new mathematical ideas, approaches, and theories. On the other hand, because of the importance of these equations in applications, there is a large literature for the design and analysis of various numerical methods to calculate these solutions. In addition, a well-posedness theory (existence, uniqueness, and stability) is in great demand.

A *nonlinear degenerate parabolic-hyperbolic equation* typically takes the form:

$$\partial_t u + \nabla \cdot \mathbf{f}(u) = \nabla \cdot (\mathbf{A}(u)\nabla u), \qquad u \in \mathbb{R}.$$

Here $t \in \mathbb{R}_+ := [0, \infty)$; $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$; $\nabla = (\partial_{x_1}, \ldots, \partial_{x_d})$ is the gradient with respect to $\mathbf{x}$; $\mathbf{f} : \mathbb{R} \to \mathbb{R}^d$ is differentiable and $\mathbf{f}'(u)$ is locally bounded; and the $d \times d$ matrix $\mathbf{A}(u)$ is symmetric, nonnegative, and locally bounded. In physics, such an equation is also called a *nonlinear advection-diffusion equation*. The *diffusion term* $\nabla \cdot (\mathbf{A}(u)\nabla u)$ reflects a transport of molecules, by random molecular motion within the fluid, from a region of higher concentration to one of lower concentration. The *advection term* $\nabla \cdot \mathbf{f}(u)$ describes a transport of the fluid flow.

One of the prototypes is the porous medium equation:

$$\partial_t u = \Delta(u^p), \qquad p > 1,$$

where $\Delta = \sum_{j=1}^d \partial_{x_j x_j}$ is the Laplace operator. The equation is degenerate on the level set $\{u = 0\}$; away from this set, the equation is strictly parabolic. Even though the nonlinear equation is parabolic, the solutions exhibit a certain hyperbolic feature, which results from the degeneracy. One striking family of solutions was found around 1950 in Moscow by Zel'dovich-Kompaneets and Barenblatt. The supports of these solutions propagate at finite speeds (cf. [5]), whereas a solution to a nondegenerate parabolic equation propagates at an infinite speed.

The simplest example for the isotropic case (i.e., $\mathbf{A}(u)$ is diagonal) with both phases is

$$\partial_t u + \partial_x (\frac{u^2}{2}) = \partial_{xx}[u]_+,$$

where $[u]_+ = \max\{u, 0\}$. The equation is hyperbolic when $u < 0$ and parabolic when $u > 0$, and the level set $\{u = 0\}$ is a free boundary separating the hyperbolic phase from the parabolic phase. For any constant states $u^\pm$ with $u^+ < 0 < u^-$ and

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$y_* \in \mathbb{R}$, there exists a unique nonincreasing profile $\phi = \phi(y)$ such that

$$\lim_{y\to-\infty}(\phi(y), \phi'(y)) = (u^-, 0),$$
$$\phi(y) \equiv u^+ \quad \text{for } y > y_*,$$
$$\lim_{y\to y_*-0}(\phi(y), \phi'(y)) = (0, -\infty),$$
$$\lim_{y\to y_*-0}\frac{d[\phi(y)]_+}{dy} = \frac{1}{2}u^+u^-$$

so that $u(t,x) = \phi(x - st)$, $s = \frac{u^+ + u^-}{2}$, is a discontinuous solution connecting u^+ to u^-. Although $u(t,x)$ is discontinuous, $[u(t,x)]_+$ is a continuous function even across the interface $\{u = 0\}$. See Figure 1.

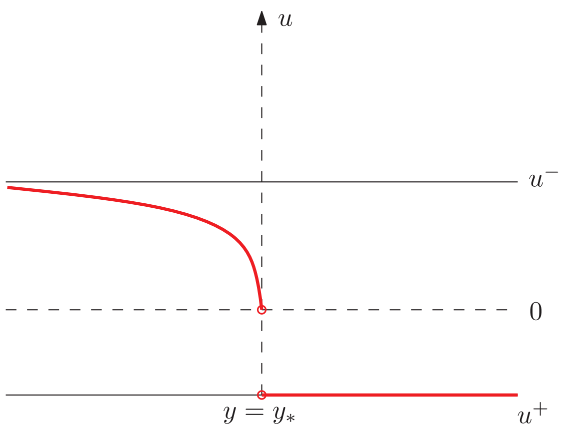

Figure 1. A discontinuous profile connecting the hyperbolic phase to the parabolic phase.

The well-posedness issue for the initial value problem is relatively well understood if—relying on the work of Lax, Oleinik, Volpert, and Kruzhkov, and more recently of Lions-Perthame-Tadmor [4]—one removes the term $\nabla \cdot (\mathbf{A}(u)\nabla u)$. It is equally well understood when the set $\{u : rank(\mathbf{A}(u)) < d\}$ consists only of isolated points with certain orders of degeneracy (cf. [3,5]). For the isotropic case, the well-posedness of entropy solutions was established by various people: solutions with bounded variation by Volpert-Hudjaev (1969), solutions in L^∞ by Carrillo (1999), and unbounded solutions by Chen-DiBenedetto (2001). Here an entropy solution is a discontinuous solution that satisfies both the equation and an additional admissible inequality (called an entropy inequality, motivated by the Second Law of Thermodynamics) in the sense of distributions. However, the general case with solutions in L^1 remained open until 2003, when the kinetic approach was developed in Chen-Perthame [1] in order to deal with both parabolic and hyperbolic phases. This unified approach is motivated by the macroscopic closure procedure of the Boltzmann equation in kinetic theory, the hydrodynamic limit of large particle systems in statistical mechanics, and early works on kinetic schemes to calculate shock waves and on the theoretical kinetic formulation for the pure hyperbolic case (see [1,4]).

More precisely, consider the quasi-Maxwellian kinetic function χ on $\mathbb{R}^2$:

$$\chi(v; u) = \begin{cases} +1 & \text{for } 0 < v < u, \\ -1 & \text{for } u < v < 0, \\ 0 & \text{otherwise.} \end{cases}$$

If $u \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^d))$, then $\chi(v;u) \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^{d+1}))$.

A function $u(t,\mathbf{x}) \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^d))$ is called a *kinetic solution* if $u(t,\mathbf{x})$ satisfies the following:

(i) The kinetic equation:

$$\begin{aligned}&\partial_t\chi(v;u) + \mathbf{f}'(v)\cdot\nabla\chi(v;u)\\ &\quad -\nabla\cdot(\mathbf{A}(v)\nabla\chi(v;u))\\ &= \partial_v(m+n)(t,\mathbf{x};v)\end{aligned}$$

holds in the sense of distributions with the initial data $\chi(v;u)|_{t=0} = \chi(v;u_0)$, for some nonnegative measures $m(t,\mathbf{x};v)$ and $n(t,\mathbf{x};v)$, where $n(t,\mathbf{x};v)$ is defined by

$$\langle n(t,\mathbf{x};\cdot), \psi(\cdot)\rangle := \sum_{k=1}^{d}\left(\nabla\beta_k^{\psi}\right)^2 \in L^1(\mathbb{R}_+\times\mathbb{R}^d)$$

for any $\psi \in C_0^\infty(\mathbb{R})$ with $\psi \geq 0$ and $\beta_k^{\psi}(u) := \int^u \boldsymbol{\sigma}_k(v)\sqrt{\psi(v)}\,dv$, where $\boldsymbol{\sigma}_k(v)$ is the kth column of the matrix $\boldsymbol{\sigma}(v)$ such that $\mathbf{A}(v) = \boldsymbol{\sigma}(v)\boldsymbol{\sigma}(v)^\top$;

(ii) There exists $\mu \in L^\infty(\mathbb{R})$ with $0 \leq \mu(v) \to 0$ as $|v| \to \infty$ such that

$$\int_0^\infty\int_{\mathbb{R}^d}(m+n)(t,\mathbf{x};v)\,dt\,d\mathbf{x} \leq \mu(v);$$

(iii) For any two nonnegative functions $\psi_1, \psi_2 \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned}&\sqrt{\psi_1(u(t,\mathbf{x}))}\sum_{i=1}^{d}\partial_{x_i}\beta_{ik}^{\psi_2}(u(t,\mathbf{x}))\\ &= \sum_{i=1}^{d}\partial_{x_i}\beta_{ik}^{\psi_1\psi_2}(u(t,\mathbf{x}))\end{aligned}$$

for almost every $(t,\mathbf{x})$.

Well-posedness in L^1

The advantage of the notion of kinetic solutions is that the kinetic equation is well defined even when $\mathbf{f}(u(t,\mathbf{x}))$ and $\mathbf{A}(u(t,\mathbf{x}))$ are not locally integrable so that L^1 is a natural space in which the kinetic solutions are posed. This notion also covers the so-called renormalized solutions used in the context of scalar hyperbolic conservation laws by Bénilan-Carrillo-Wittbold (2000). Based on this notion, a new approach has been developed in [1] to establish a well-posedness theory for the initial value problem for kinetic solutions in L^1. In particular, the L^1-contraction of kinetic solutions is established, which implies that the kinetic solution forms a semigroup with respect to $t > 0$.

Consistency

The uniqueness result implies that any kinetic solution in L^∞ must be an entropy solution. On the other hand, any entropy solution is actually a kinetic solution. Therefore, the two notions are equivalent for solutions in L^∞, although the notion of kinetic solutions is more general.

Connection with the Classical Entropy Method

By the very construction of the kinetic approach, any results using the classical entropy method can easily be translated in terms of the old Kruzkov entropies by integrating in v. In the case of uniqueness for the general case, this was performed by Bendahmane-Karlsen (2004).

Condition (iii), which is a fundamental and natural property similar to a chain rule, automatically holds in the isotropic case. It is also the cornerstone for the uniqueness in the general case. Moreover, condition (ii) implies that $m + n$ has no support at $u = \infty$.

Furthermore, let $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ be the unique entropy solution with periodic initial data $u_0 \in L^\infty$ for period $\mathbb{T}_P = \Pi_{i=1}^d [0, P_i]$. Assume that $\mathbf{f}(u)$ is in C^1, $\mathbf{A}(u)$ is continuous, and both satisfy the *nonlinearity-diffusivity condition*: For any $\tau \in \mathbb{R}$ and $\mathbf{y} \in \mathbb{R}^d$ with $|\mathbf{y}| = 1$, the set

$$\{v : \tau + \mathbf{f}'(v) \cdot \mathbf{y} = 0,\ \mathbf{y}\mathbf{A}(v)\mathbf{y}^\top = 0\} \subset \mathbb{R}$$

has Lebesgue measure zero. Then

$$\|u(t,\cdot) - \frac{1}{|\mathbb{T}_P|}\int_{\mathbb{T}_P} u_0(\mathbf{x})\,d\mathbf{x}\|_{L^1(\mathbb{T}_P)} \to 0 \qquad \text{as } t \to \infty,$$

where $|\mathbb{T}_P|$ is the volume of the period $\mathbb{T}_P$.

The nonlinearity-diffusivity condition implies that there is no interval of v in which $\mathbf{f}(v)$ is affine and $\mathbf{A}(v)$ is degenerate. Unlike the pure hyperbolic case, the equation is no longer self-similar invariant. The argument for achieving this decay result is based on the kinetic approach developed in [1], involves a time-scaling and a monotonicity-in-time property of the entropy solution, and employs the advantages of the kinetic equation, in order to recognize the role of the nonlinearity-diffusivity of the equation (see [2]).

Follow-up results based on this approach include L^1-error estimates and continuous dependence of solutions both on $\mathbf{f}$ and $\mathbf{A}$, and more general degenerate parabolic-hyperbolic equations. Further regularity results of solutions have been established by Tadmor-Tao (2007).

Further Reading

[1] G.-Q. Chen and B. Perthame, Well-posedness for nonisotropic degenerate parabolic-hyperbolic equations, *Annales de l'Institut Henri Poincaré: Analyse non linéaire* **20** (2003), 645–668.

[2] ———, Large time behavior of periodic solutions to anisotropic degenerate parabolic-hyperbolic equations, *Proc. Amer. Math. Soc.* **137** (2009), 3003–3011.

[3] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer-Verlag, New York, 1993.

[4] P.-L. Lions, B. Perthame, and E. Tadmor, A kinetic formulation of multidimensional scalar conservation laws and related equations, *J. Amer. Math. Soc.* **7** (1994), 169–191.

[5] J. L. Vázquez, *The Porous Medium Equation: Mathematical Theory*, Oxford University Press, Oxford, 2007.