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DIRICHLET AND NEUMAN PROBLEMS FOR THE SEIBERG-WITTEN EQUATIONS

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Abstract

It is shown that the non-homogeneous Dirichlet and Neuman problems for the 2^{nd}-order Seiberg-Witten equation on a compact 4-manifold X admit a regular solution once the $\mathcal{H}$-condition, which is a non-homogeneous Palais-Smale Condition, is satisfied. The approach consist in applying the elliptic techniques to the variational setting of the Seiberg-Witten equation. The gauge invariance of the functional allows to restrict the problem to the Couloumb subspace $\mathcal{C}_\alpha^{\mathfrak{c}}$ of configuration space. The coercivity of the $\mathcal{SW}_\alpha$-functional, when restricted into the Coulomb subspace, imply existence of a weak solution, which turn out to be a strong solution. Its is shown that whenever (A,ϕ) is a solution ($\phi \neq 0$) the spinor field ϕ is L^∞ bounded. The regularity then follows from the boundness of L^∞-norms of a spinor solution and the Gauge Fixing Lemma.

The Dirichlet ($\mathcal{D}$) and Neumann ($\mathcal{N}$) boundary value problems associated to the $\mathcal{SW}_\alpha$-equations are the following: Let's consider $(\Theta,\sigma) \in \Omega^1(ad(\mathfrak{u}_1)) \oplus \Gamma(\mathcal{S}_\alpha^+)$ and (A_0,ϕ_0) defined on the manifold ∂X (A_0 is a connection on $\mathcal{L}_\alpha\mid_{\partial X}$, ϕ_0 is a section of $\Gamma(\mathcal{S}_\alpha^+\mid_{\partial X})$). In this way, find $(A,\phi)\in\mathcal{C}_\alpha^{\mathcal{D}}$ satisfying $\mathcal{D}$ and $(A,\phi)\in\mathcal{C}_\alpha^{\mathcal{N}}$ satisfying $\mathcal{N}$, where

$$\mathcal{D}=\begin{cases} d^*F_A+4\Phi^*(\nabla^A\phi)=\Theta,\\ \Delta_A\phi+\frac{(|\phi|^2+k_g)}{4}\phi=\sigma,\\ (A,\phi)\mid_{\partial X}\overset{\text{gauge}}{\sim}(A_0,\phi_0),\end{cases}\qquad \mathcal{N}=\begin{cases} d^*F_A+4\Phi^*(\nabla^A\phi)=\Theta,\\ \Delta_A\phi+\frac{(|\phi|^2+k_g)}{4}\phi=\sigma,\\ i^*(*F_A)=0,\ \nabla_\nu^A\phi=0,\end{cases}\tag{0.1}$$

and

1. the operator $\Phi^*:\Omega^1({\mathcal{S}_\alpha}^+)\to\Omega^1(\mathfrak{u}_1)$ is locally given by

$$\Phi^*(\nabla^A\phi)=\frac{1}{2}\nabla^A(\mid\phi\mid^2)=\sum_i<\nabla_i^A\phi,\phi>\eta_i,\tag{0.2}$$

   and $\eta=\{\eta_i\}$ is an orthonormal frame in $\Omega^1(ad(\mathfrak{u}_1))$.

2. $i^*(*F_A)=F_4$, where
   $F_4=(F_{14},F_{24},F_{34},0)$ is the local representation of the 4^{th}-component (normal to ∂X) of the 2-form of curvature in the local chart (x,U) of X;
   $x(U)=\{x=(x_1,x_2,x_3,x_4)\in\mathbb{R}^4;\ \|x\|<\epsilon, x_4\geq 0\}$, and
   $x(U\cap\partial X)\subset\{x\in x(U)\mid x_4=0\}$. Let $\{e_1,e_2,e_3,e_4\}$ be the canonical base of $\mathbb{R}^4$, so $\nu=-e_4$ is the normal vector field along ∂X.

A global formulation for problems $\mathcal{D}$ and $\mathcal{N}$ is made using the Seiberg-Witten functional, which for each $\alpha\in Spin^c(X)$, is defined as

$$\mathcal{SW}_\alpha(A,\phi)=\int_X\{\frac{1}{4}\mid F_A\mid^2+\mid\nabla^A\phi\mid^2+\frac{1}{8}\mid\phi\mid^4+\frac{k_g}{4}\mid\phi\mid^2\}dv_g+\pi^2\alpha^2.\tag{0.3}$$

where k_g= scalar curvature of (X,g).

The $\mathcal{G}_\alpha$-action on $\mathcal{C}_\alpha$ has the following properties;