SEIBERG-WITTEN THEORY ON THREE MANIFOLDS

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Sumário

Capítulo 1

Spin^C Structures

Let V be a finite dimensional $\mathbb{K}$-vector space; in our context $\mathbb{K} = \mathbb{R}$, $K = \mathbb{C}$ or $\mathbb{K} = \mathbb{H}$.

Definition 1.1. A symmetric bilinear form on V is a function $B : V \to \mathbb{K}$ such that,

1. $B(k_1u_1 + k_2u_2, v) = k_1B(u_1, v) + k_2B(u_2, v)$, for all $u_1, u_2 \in V$ and $k_1, k_2 \in \mathbb{K}$.
2. $B(u, v) = B(v, u)$, for all $u, v \in V$.

The quadratic form associated to a symmetric bilinear form $B : V \times V \to \mathbb{K}$ is defined as $q_B(u) = B(u, u)$, $q_B : \mathbb{K} \to \mathbb{R}$.

$$B(u, v) = \frac{1}{2}\left[q_B(u + v) - q_B(u) - q_B(v)\right]. \tag{1.1}$$

Thus, let's focus exclusively on symmetric bilinear forms. In this case, if we fix a basis $\beta = \{e_1, \ldots, e_n\}$ ($dim_{\mathbb{K}}(V) = n$), then the symmetric matrix M representing B is $M_{ij} = (B(e_i, e_j))$ and it is diagonalizable over $\mathbb{K}$. Let

$$V^+ = \oplus_\lambda V_{\lambda>0}, \quad V^- = \oplus_\lambda V_{\lambda<0},$$

where $V_\lambda = \{u \in V \mid M(u) = \lambda u\}$ is the eigenspace associated to the eigenvector λ.

By Sylvester's Theorem, a quadratic form on $\mathbb{R}^n$ is determined, up to similarity, by the pair of natural numbers (rk_q, σ_q), where $rk_q = dim(V^+) + dim(V^-)$ is its rank and $\sigma_q = dim(V^+) - dim(V^-)$ is its signature. The quadratic form is non-degenerated whenever $V_0 = \{0\}$, otherwise it is degenerated. From now on, we will consider that all the quadratic forms are non-degenerated. Thus, Sylvester Theorem claims that if q has $rk_q = r$, $\sigma_q = s$ and $r + s = n$, then it is equivalent to the quadratic form

$$q_{r,s}(x_1, \ldots, x_r, y_1, \ldots, y_s) = \sum_{i=1}^{r} x_i^2 - \sum_{j=1}^{s} y_j^2 \tag{1.2}$$

The classification above justifies the following notation: on $\mathbb{R}^n$, where $n = r+s$, consider $< .,. >_{r,s}$: $V \times V \to \mathbb{R}$ the non-degenerated bilinear form associated to the quadratic form $q_{r,s}$ in 1.2.

1.1 Clifford Algebras

Definition 1.2. Let V be a $\mathbb{K}$-vector space and $q : V \to \mathbb{K}$ be a nondegenerated quadratic form. The Clifford Algebra $Cl(V,q)$ associated to the pair (V,q) is the algebra generated by the relation

$$u.v + v.u = -2B(u,v).1, \quad 1 \in \mathbb{K}. \tag{1.1}$$

where B is the bilinear form defined by the identity 1.1.

Example 1.1. .

1. Let $\mathbb{K} = \mathbb{R}$, $V = \mathbb{R}^n$ and $B(u,v) =< u,v >$ be euclidean inner product on $\mathbb{R}^n$. Consider $\beta = \{e_1, \ldots, e_n\}$ be an orthonormal basis with respect to the inner product. The identity 1.1 induces the following relations in $Cl_n = Cl(\mathbb{R}^n, <,>)$:

$$e_i.e_j + e_j.e_i = -2\delta_{ij}. \tag{1.2}$$

From the relation 1.2, the basis $\beta = \{e_1, \ldots, e_n\}$ generates in Cl_n the elements $e_I = e_{i_1} \ldots e_{i_k}$ of length $\mid I \mid= k$, where $1 \leq k \leq n$ and $I = (i_1, \ldots, i_k)$. For $k \in \{1, \ldots, n\}$, there are $\binom{n}{k}$ linearly independents elements e_I such that $\mid I \mid= k$. Thus,

$$V_k = \{\sum_{i=1}^{a} f_I e_I \mid f_I \in \mathbb{R} \text{ and } , \mid I \mid= k\}$$

is a vector subspace of Cl_n with dimension $\binom{n}{k}$. We note that the vector space structure on $Cl_n = \mathbb{R} \oplus V_1 \oplus V_2 \cdots \oplus V_n$ is isomorphic to the vector space structure on the exterior algebra $\Lambda^*\mathbb{R}^n$, hence its dimension is 2^n.

2. Let $\mathbb{K} = \mathbb{R}$, $V = \mathbb{R}^n$ and $B(u,v) =< .,. >_{r,s}$. If $\beta = \{e_1, \ldots, e_n\}$ is orthonormal with respect to $< .,. >_{r,s}$, then $Cl_{r,s}$ is generated as an algebra by the relations

$$e_i.e_j + e_j.e_i = -2\delta_{ij}, \quad \text{if } i \leq r, \tag{1.3}$$
$$e_i.e_j + e_j.e_i = 2\delta_{ij}, \quad \text{if } i \geq r+1. \tag{1.4}$$

The automorphism $\alpha : V \to V$, $\alpha(u) = -u$, induces the automorphism

$$\alpha : Cl(V,q) \to Cl(V,q), \quad \alpha(\sum_I \phi_I e_I) = \sum_I \phi_I \alpha(e_I),$$

where if $e_I = e_{i_1} \dots e_{i_k}$, then $\alpha(e_I) = \alpha(e_{i_1}) \dots \alpha(e_{i_k})$. Furthermore, the relation $\alpha^2 = I$ induces the decomposition

$$Cl(V,B) = Cl^0 \oplus Cl^1,$$

where $Cl^0 = \{\phi \in Cl(V,q); \alpha(\phi) = \phi\}$ and $Cl^1 = \{\phi \in Cl(V,q); \alpha(\phi) = -\phi\}$. The subspace Cl^0 is an subalgebra generated by $< 1, e_I : \mid I \mid \text{ even} >$ and Cl^1 is a vector space generated by $< e_I : \mid I \mid \text{ odd} >$.

Proposition 1.1. *For all r, s, there is an algebra isomorphism $Cl_{r,s} \simeq Cl^0_{r+1,s}$. In particular, $Cl_n \simeq Cl^0_{n+1}$ for all n.*

Demonstração. Choose a $q_{r,s}$-orthonormal basis $\beta = \{e_1, \dots, e_{r+s+1}\}$ of $\mathbb{R}^{r+s+1}$ so that $q_{r,s}(e_i) = 1$ for $1 \leq i \leq r+1$ and $q_{r,s}(e_i) = -1$ for $r+1 < i \leq r+s+1$ and consider the map $f : \mathbb{R}^{r+s} \to Cl^0_{r+1,s}$ defined by setting $f(e_i) = e_i e_{r+1}$ on the basis $\beta_{r,s} = \{e_1, \dots, e_r, e_{r+2}, \dots, e_{r+s+1}\}$ and extend it linearly to $\mathbb{R}^{r+s}$. For $u = \sum_{i \neq r+1} u_i e_i$, we have that

$$f(u)^2 = \sum_{i,j} u^i u^j e_i e_{r+1} e_j e_{r+1} = \sum_{i \neq r+1} u_i u_j e_i e_j = u.u = -q(u).1$$

It follows from the universal property that f extends to an algebra momomorphism, which restriction to the subalgebra $Cl^0_{r+1,s}$ is surjective.

□

Definition 1.3. Fixed a orthonormal basis $\beta = \{e_1, \dots, e_n\}$ of $(\mathbb{R}^n, < .,. >_{r,s})$, the volume form of $Cl_{r,s}$ is the element $w = e_1 \dots e_n$.

Proposition 1.2. *Let $w = e_1 \dots e_n$ be the volume form of $Cl_{r,s}$, then;*

1. *w is well defined (it independs on the choosen basis).*
2. *$w^2 = (-1)^{\frac{n(n+1)}{2}+s}$.*
3. *For all $u \in V$, $uw = (-1)^{n-1} wu$. In particular, if n is odd, then w is central in $Cl_{r,s}$. If n is even, then $\phi w = w\alpha(\phi)$, for all $\phi \in Cl_{r,s}$.*

Demonstração. .

1. Let $\beta' = \{v_1, \dots, v_n\}$ be another basis and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the orientation preserving orthogonal automorphism of $\mathbb{R}^n$ taking the representation of a vector in β to the representation in β',

$$v_i = T(e_i) = \sum_{l=1}^{n} t_{li} e_l, \quad (T)_{kl} = t_{lk}.$$

Then

$$w^{\beta'} = v_1 \dots v_n = T(e_1) \dots T(e_n) = det(T).e_1 \dots e_n = w.$$

2. Let $w = e_1 \dots e_n$ and $u = \sum_i v_i e_i$, so

$$u.w = \sum v_i e_i e_1 \dots e_i \dots e_n = \sum_i v_i (-1)^{i-1} e_1 \dots e_i e_i \dots e_n =$$
$$= (-1)^{[(i-1)+(n-i)]} v_i e_1 \dots e_i \dots e_n . e_i = (-1)^{n-1} w.u.$$

□

Proposition 1.3. *Let either $n = 4k$ or $n = 4k + 3$. In this cases, $w^2 = 1$ in Cl_n generates the decomposition $Cl_n = Cl^+ \oplus Cl^-$, where $Cl^{\pm}$ is the eigenspace associated to the eigenvalue ± 1 of w.*

Demonstração. Let $\pi^{\pm} = \frac{1}{2}(1 \pm w)$, so $\pi^+ + \pi^- = 1$ and $(\pi^+)^2 = \pi^+$, $(\pi^-)^2 = \pi^-$ and $\pi^+\pi^- = \pi^-\pi^+ = 0$. Now, define $Cl^{\pm} = \pi^{\pm}(Cl_n)$. □

Corollary 1.1. *Consider $n = 4k$ and let V be a Cl_n-module. Then there is a decomposition*

$$V = V^+ \oplus V^-$$

into the $+1$ and -1 eigenspaces for the multiplication by w ($V^{\pm} = \pi^{\pm}(V)$). Also, for any $v \in V$ with $q(v) \neq 0$, the module multiplication by v gives the isomorphisms

$$v : V^+ \to V^-, \quad v : V^- \to V^+.$$

Demonstração. The last claim concerning the vector $v \in V$ is obtained from the fact that n being even then $vw = -wv$, and so

$$v.\pi^{\pm} = \frac{1}{2} v.(1 \pm w) = \frac{1}{2}(1 \mp w).v = \pi^{\mp}.v.$$

□

Definition 1.4. Let V be a real vector space endowed with a non-degenerated quadratic form $q : V \to \mathbb{R}$. The complexification of $Cl(V, q)$ is

$$\mathbb{C}l(V, q) = Cl(V, q) \otimes \mathbb{C}.$$

notation: $\mathbb{C}l_{r,s} = \mathbb{C}l_n = Cl_n \otimes \mathbb{C}$.

1.1.1 Classification

Next it is described the main steps to classify the Cliford Algebra an to show that they are all algebras of matrixes.

Proposition 1.4.

$$Cl_{1,0} = \mathbb{C}, \quad Cl_{0,1} = \mathbb{R} \oplus \mathbb{R}, \tag{1.5}$$
$$Cl_{2,0} = \mathbb{H}, \quad Cl_{0,2} = M(2, \mathbb{R}), \tag{1.6}$$
$$Cl_{1,1} = M(2, \mathbb{R}). \tag{1.7}$$

Proposition 1.5. *For all n, r e s, there are isomorphisms*

$$Cl_{0,n+2} \simeq Cl_{n,0} \otimes Cl_{0,2} \tag{1.8}$$
$$Cl_{n+2,0} \simeq Cl_{0,n} \otimes Cl_{2,0} \tag{1.9}$$
$$Cl_{r+1,s+1} \simeq Cl_{r,s} \otimes Cl_{1,1}. \tag{1.10}$$

Theorem 1.1. *For all n, there are periodicity isomorphisms*

$$Cl_{n+8,0} \simeq Cl_{n,0} \otimes Cl_{8,0} \tag{1.11}$$
$$Cl_{0,n+8} \simeq Cl_{0,n} \otimes Cl_{0,8} \tag{1.12}$$
$$\mathbb{C}l_{n+2} \simeq \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2, \tag{1.13}$$

where $Cl_{8,0} = Cl_{0,8} = M(16, \mathbb{R})$ e $\mathbb{C}l_2 = M(2, \mathbb{C})$. Therefore,

$$\mathbb{C}l_{2k-1} = M(2^{k-1}, \mathbb{C}) \oplus M(2^{k-1}, \mathbb{C}), \tag{1.14}$$
$$\mathbb{C}l_{2k} = M(2^k, \mathbb{C}), \tag{1.15}$$
$$\mathbb{C}l_{2k+1} = M(2^k, \mathbb{C}) \oplus M(2^k, \mathbb{C}). \tag{1.16}$$

Example 1.2. Let $V = \mathbb{R}^n$ and $\beta = \{e_1, \dots, e_n\}$ be a orthonormal basis. Using β, we can describe the inclusion $\iota : \mathbb{R}^n \hookrightarrow Cl_n$ and the generators set of Cl_n.

1. $Cl_1 \simeq \mathbb{C}$
   In this case $V = \mathbb{R}$, let $\beta = \{e_1 = 1\}$ be the basis. Since $e_1^2 = -1$, we take $\iota : V \to \mathbb{C}$ by setting $\iota(e_1) = i$.

2. $Cl_2 \simeq \mathbb{H}$ In this case $V = \mathbb{R}^2$, let $\beta = \{e_1 = (1,0), e_2 = (0,1)\}$ be canonical orthonormal basis. The quaternions $\mathbb{H}$ can be identified with $\mathbb{C}^2$ using the fact that $i = jk$, as shown next;

$$q_0 + q_1 i + q_2 j + q_3 k = q_0 + q_1 jk + q_2 j + q_3 k = (q_0 + q_3 k) + j(q_2 + kq_3).$$

As a algebra, consider $\mathbb{H}$ generated by $\{j, k\}$ and let $\iota(e_1) = j$ and $\iota(e_2) = k$. If we consider the representation $\sigma : \mathbb{H} \to M(2, \mathbb{C})$, given by

$$\sigma(a + bj) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

we assign

$$\sigma\eta(e_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma\eta(e_2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{1.17}$$

3. $Cl_3 \simeq \mathbb{H} \oplus \mathbb{H}$
$V = \mathbb{R}^3$ and $\beta = \{e_1, e_2, e_3\}$ the canonical basis

4. Cl_4

1.1.2 Representations of Clifford Algebras

Let W be a Cl_n-module and $\rho : CL_n \to Hom(W, W)$ be a linear representation. The volume form plays important role in the classification of irredutible Cl_n-modules because $\rho(w)^2 = I$ and w is central whenever n is odd.

Proposition 1.6. *If $n = 4m + 3$, then the eigenspaces $Cl_n^{\pm}$ of the volume form w are inequivalent and irredutible representations of Cl_n.*

Demonstração. In this case $w^2 = 1$ and w is central. Of course, $Cl_n^{\pm}$ are Cl_n-modules. The decomposition $Cl_n = Cl_n^{+} \oplus Cl_n^{-}$ together with the fact that each component $Cl_n^{\pm}$ is Cl_n-invariant means that they are irredutible representations of Cl_n, say $\rho_{\pm}$. To see that they are inequivalent, we observe that $\rho_{\pm}(w) = \pm I$ and that there is no isomorphism $F : Cl_n^{+} \to Cl_n^{-}$ such that $\rho_{+}(w) = F^{-1} \circ \rho_{-}(w) \circ F$. □

Proposition 1.7. *Let $n = 4m$ and $\rho : Cl_n \to Hom(W, W)$ be a irredutible representation. Then, there is the decomposition $W = W^{+} \oplus W^{-}$, where each space $W^{\pm}$ is Cl_n^0 invariant, and each one corresponds to a irredutible representation of $Cl_{n-1} \simeq Cl_n^0$.*

Demonstração. It is enough to observe that $\phi.w = w.\phi$, for all $\phi \in Cl_{4m}$ and $w \in Cl_n^0$. However, $u.w = -w.u$ for all $u \in V$, and so, $u.\pi^{\pm} = \pi^{\mp} u$. □

Corollary 1.2. *Consider the complex Clifford Algebras $\mathbb{C}l_n$ and $w_{\mathbb{C}} = i^{[\frac{n+1}{2}]} w$ the complex volume form;*

1. *if $n = 2k+1$, then there are two irredutible and inequivalent representations of $\mathbb{C}l_{2k+1}$.*
2. *if $n = 2k$, then there is only one irredutible representation $W \simeq \mathbb{C}^4$ admitting a decomposition $W = W^+ \oplus W^-$, where $W^\pm$ are the irredutible representations of $\mathbb{C}l_{2k-1} \simeq \mathbb{C}l^0_{2k}$.*

Demonstração. First we observe that $w_{\mathbb{C}} = 1$, since

$$w_{\mathbb{C}}^2 = i^{2.[\frac{n+1}{2}]}.e_1 \dots e_n.e_1 \dots e_n = (-1)^{\frac{n(n-1)}{2}}.(-1)^{\frac{n(n-1)}{2}} = 1.$$

Thus, $w_{\mathbb{C}}$ has eigenvalues ± 1 and the corresponding eigenspaces $\mathbb{C}l_n^\pm$ induce the decomposition $\mathbb{C}l_n = \mathbb{C}l_n^+ \oplus \mathbb{C}l_n^-$.

1. If n is odd, $w_{\mathbb{C}}$ is central, then $\mathbb{C}l_n^\pm$ are invariant as $\mathbb{C}l_n$-modules and inequivalent as representations.
2. if n is even, then any $v \in V$ induces an isomorphism $v : \mathbb{C}l_n^\pm \to \mathbb{C}l_n^\mp$. Besides, each $\mathbb{C}l_n^\pm$ is $\mathbb{C}l_n^0 \simeq \mathbb{C}l_{n-1}$ invariant.

□

<u>remark:</u> In the Corollary above, the dimension of the irredutible representation spaces W can be computed as follows: (hint: $dim(M(n,\mathbb{R})) = n^2$)

1. $n = 2k-1$;
   Since $dim(\mathbb{C}l_{2k-1}) = 2^{2k-1}$, it follows that $dim(\mathbb{C}l^\pm_{2k-1}) = 2^{2k-2} = (2^{k-1})^2$. Hence, $dim(W) = 2^{k-1}$.
2. $n = 2k$;
   Since in this case $\mathbb{C}l_{2k}$ is irredutible as a $\mathbb{C}l_{2k}$-module and $dim(\mathbb{C}l_{2k}) = (2^k)^2$, it follows that $dim(W) = 2^k$.

Let's compute the volume form $w_{\mathbb{C}}$ in $\mathbb{C}l_{2k-1} = M(2^{k-1},\mathbb{C}) \oplus M(2^{k-1},\mathbb{C})$ and $\mathbb{C}l_{2k} \simeq M(2k,\mathbb{C})$ are performed;

Proposition 1.8. *Let $w_{\mathbb{C}} \in \mathbb{C}l_n$ be the volume form; so,*

$$w_{\mathbb{C}} \rightsquigarrow \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad or \quad w_{\mathbb{C}} \rightsquigarrow \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \tag{1.18}$$

Demonstração. From the decompositions $\mathbb{C}l_n = \mathbb{C}^+ \oplus \mathbb{C}^-$, we have $w_{\mathbb{C}} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, hence $A^2 = B^2 = I$, and $A = \pm I$ and $B = \pm I$. Since $1 \to I$ and $-1 \to -I$, the only possibilities are 1.18. In the first case, we compute

$$\pi^+ = \frac{1 + w_{\mathbb{C}}}{2} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \pi^- = \frac{1 - w_{\mathbb{C}}}{2} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

□

1.1.3 $(\mathbb{C}l_4^0)^+$

As vector spaces, we have the isomorphism $\mathbb{C}l_4 \simeq \Lambda^*(\mathbb{R}^4) \otimes \mathbb{C}$. One of the Seiberg-Witten equations requires a relationship among the vector spaces $(\mathbb{C}l_4^0)^+$ and $\Lambda_+^2(\mathbb{R}^4)$;

Proposition 1.9. *Let* $(\mathbb{C}l_4^0)^+ = \mathbb{C}l_4^0 \cap \mathbb{C}l_4^+$*, so*

$$(\mathbb{C}l_4^0)^+ \simeq \{\left\langle \frac{1+w_{\mathbb{C}}}{2} \right\rangle \oplus \Lambda_+^2(\mathbb{R}^4)\} \otimes \mathbb{C} \tag{1.19}$$

Demonstração. Its is straight forward that

$$\mathbb{C}l^0 \simeq (\Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4) \otimes \mathbb{C}.$$

The subspaces $\Lambda^2\mathbb{C}$ and $(\Lambda^0 \oplus \Lambda^4) \otimes \mathbb{C}$ are invariant by $w_{\mathbb{C}}$. Let $\beta = \{e_1, e_2, e_3, e_4\}$ be a orthonormal basis of $\mathbb{R}^4$, so $w_{\mathbb{C}} = -e_1e_2e_3e_4$ and

$$w(e_1e_2) = e_3e_4, \quad w(e_1e_3) = -e_2e_4, \quad w(e_1e_4) = e_2e_3.$$

The multiplication by $w_{\mathbb{C}}$ on $\mathbb{C}l_n$ is similar to the action of the Hodge $*$-operator on $\Lambda^*(\mathbb{R}^n)$, in the sense that the diagram below commutes;

$$\begin{array}{ccc} \mathbb{C}l_n & \xrightarrow{w_{\mathbb{C}}} & \mathbb{C}l_n \\ \simeq\downarrow & & \simeq\downarrow \\ \Lambda^*(\mathbb{R}^n) & \xrightarrow{*} & \Lambda^*(\mathbb{R}^n) \end{array}$$

So, the elements

$$\frac{e_1e_2 + e_3e_4}{2}, \frac{e_1e_3 - e_2e_4}{2}, \frac{e_1e_4 + e_2e_3}{2}$$

form a basis of Λ_+^2. So, a generator set of $(\mathbb{C}l_4^0)^+$ is given by

$$\left\langle \frac{1+w_{\mathbb{C}}}{2} \right\rangle \oplus \left\langle \frac{dx^1 \wedge dx^2 + dx^3 \wedge dx^4}{2}, \frac{dx^1 \wedge dx^3 - dx^2 \wedge dx^4}{2}, \frac{dx^1 \wedge dx^4 + dx^2 \wedge dx^3}{2} \right\rangle$$

□

1.2 Spin Group

The multiplicative group of units in $Cl(V,q)$ is the set

$$Cl^{\times}(V,q) = \{\phi \in Cl(V,q) \mid \phi \quad \text{invertible}\}$$

This group contains all elements $v \in V$ with $q(v) \neq 0$, since $v^{-1} = -\frac{v}{q(v)}$. It is a Lie group of dimension $2^{dim(V)}$. The adjoint representation $Ad : Cl^{\times}(V,q) \to Aut(Cl(V,q))$ is given by $Ad_{\phi}(x) = \phi x \phi^{-1}$. For $v \in V$,

$$-Ad_v(w) = w - 2\frac{q(v,w)}{q(v)}v. \tag{1.1}$$

The right hand side of expression 1.1 is a q-reflection over the q-orthogonal plane to v, hence it preserves q. In order to obtain a q-reflection, or to get rid of the minus sign on the left hand side, we introduce the twisted adjoint representation $\overline{Ad} : Cl^{\times}(V,q) \to Aut(Cl(V,q))$,

$$\overline{Ad}_{\phi}(y) = \alpha(\phi)y\phi^{-1}, \tag{1.2}$$

Note the following: (1) $\overline{Ad}_{\phi} = Ad_{\phi}$ iff ϕ is even and (2) $\overline{Ad}_{\phi} = -Ad_{\phi}$ iff ϕ is odd. It follows that for all $v \in V$

$$\overline{Ad}_v(w) = w - 2\frac{q(v,w)}{q(v)}v. \tag{1.3}$$

Thus, $\overline{Ad}_v$ is a q-reflection. Define $P(V,q)$ to be the subgroup of $Cl^{\times}(V,q)$ generated by $v \in V$ with $q(v) \neq 0$. In this way, we have got a representation $P(V,q) \to O(V,q)$.

Definition 1.5. The Pin group $Pin(V,q)$ of (V,q) is the subgroup of $P(V,q)$ generated by the elements $v \in V$ with $q(v) = \pm 1$. The associated Spin group of (V,q) is defined by

$$Spin(V,q) = Pin(V,q) \cap Cl^0(V,q).$$

Now, letting $\overline{P}(V,q) = \{\phi \in Cl^{\times}(V,q) \mid \overline{Ad}_{\phi}(V) = V\}$ $(P(V,q) \subset \overline{P}(V,q))$, the twisted adjoint representation induces the representation $\overline{P}(V,q) \to O(V,q)$.

Proposition 1.10. *Suppose that V is finite dimensional and that q is non-degenerated. Then the kernel of the homomorphism $Pin(V,q) \to O(V,q)$ is ± 1.*

remark: The groups Pin and Spin are generated by the generalized unit sphere
$S = \{v \in V \mid q(v) = \pm 1\}$; that is,

$$\begin{aligned} Pin(V,q) &= \{v_1 \ldots v_r \mid q(v_i) = \pm 1\} \\ Spin(V,q) &= \{v_1 \ldots v_r \in Pin(V,q) \mid r \quad \text{even}\} \end{aligned}$$

By setting $O_{r,s} = O(V, q_{r,s})$ and $SO_{r,s} = SO(V, q_{r,s})$, we have the following important results;

Theorem 1.2. *Let V be a finite dimensional $\mathbb{R}$-vector space and suppose q is a non-degenerate quadratic form on V. Then there are short exact sequences*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin_{r,s} \xrightarrow{\overline{Ad}} SO_{r,s} \longrightarrow 1,$$

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow Pin_{r,s} \xrightarrow{\overline{Ad}} O_{r,s} \longrightarrow 1.$$

Furthermore, if $(r,s) \neq (1,1)$, these two-sheeted coverings are non-trivial over each componente of $O_{r,s}$. In particular, the exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin_n \xrightarrow{\overline{Ad}} SO_n \longrightarrow 1 \tag{1.4}$$

represents the universal covering homomorphism of SO_n, for all $n \geq 3$.

Demonstração. The exact sequence are a direct consequence of proposition 1.10. The non-triviality of the coverings is achieved by exibiting in $Spin_{r,s}$ a connected path from 1 to -1: choose orthogonal vectors $e_1, e_2 \in \mathbb{R}^n$ with $q(e_1) = q(e_2) = \pm 1$ (this is possible since $(r,s) \neq (1,1)$). Then the curve $\gamma : [0,1] \to Spin_{r,s}$,

$$\gamma(t) = [cos(t)e_1 + sen(t)e_2].[-cos(t)e_1 + sen(t)e_2] = \pm cos(2t) + sen(2t)e_1e_2$$

satisfies $\gamma(0) = \pm 1$ and $\gamma(\pi/2) = \mp 1$. □

remark: .

1. $Spin_3 = S^3$ and $Spin_4 = S^3 \times S^3$.
2. The fundamental groups of SO_n and $SO^0_{r,s}$ are
   (a) $\pi_1(SO_n) = \mathbb{Z}_2$, $n \geq 3$,
   (b) $\pi_1(SO^0_{r,s}) = \pi(SO_r) \times \pi_1(SO_s)$ for all r, s.
3. Fixed an orthogonal basis $\beta = \{e_1, \ldots, e_n\}$ in $\mathbb{R}^n$, the curves $\gamma_{ij} : [0,1] \to Spin_n$, $\gamma_{ij}(t) = cos(2t) + sen(2t)e_ie_j$ satisfy $\gamma(0) = 1$ and $\gamma'(0) = e_ie_j$. So, the Lie Algebra of $Spin_n$ is generated by the pairs e_ie_j and its dimension is $\frac{n(n-1)}{2}$.

Definition 1.6. A $Spin_n$ representation $\triangle : Spin_n \to GL(W)$ is a representation induced by a Clifford Representation $Spin_n \subset Cl_n \overset{\triangle}{\to} GL(W)$. If the representation space W is irredutible, we say that $\triangle$ is a fundamenatl representation

<u>remark:</u> Since $Spin_n \subset Cl^0_n$, a $Spin_n$ representation are induced by a Cl_{n-1}-representation. Therefore, for each n there is only one fundamental spin representation.

1.3 Spin Structure

In the last section it was shown how to construct the Clifford Algebra $Cl(V,q)$ associated to a $\mathbb{K}$-vector space V endowed with a quadratic form $q : V \to \mathbb{K}$. In this section, the aim is to extent the construction to riemannian vector bundles $E \overset{\pi}{\to} X$ over a smooth manifold X. Let $E_x = \pi^{-1}(x)$ be the fiber over $x \in X$ and consider that E is a orientable vector bundle of rank n endowed with a quadratic form q, where $q_x : E_x \to \mathbb{K}$. This constructions starts by considering the Clifford Algebras $Cl_n(E_x, q)$ and then the bundle $Cl_n(E,q) = \cup_{x\in X} Cl_n(E_x, q_x)$. In order to get grips on the transition functions of the bundle $Cl_n(E,q)$ it is better to construct it as an associated bundle. For this purpose, let $SO(E) = \{T \in SO(E_x) \mid \forall x \in X\}$ and $P_{SO(E)}$ be the orthogonal frame bundle associated to E (its fibers are diffeomorphic to SO_n). We recall that the transition function of $P_{SO(E)}$ are the same as E. In fact, by considering the representation $\rho : SO_n \to GL(V)$ induced by the inclusion, we have that

$$E = P_{SO(E)} \times_\rho V.$$

Now, letÂ´s consider the representation $\tilde{\rho} : SO_n \to Aut(Cl_n)$

$$\tilde{\rho}(g)(e_{i_1}, \dots, e_{i_n}) = (\rho(g)(e_{i_1})) \dots (\rho(g)(e_{i_n})).$$

In this way, $Cl(E) = P_{SO(E)} \times_{\tilde{\rho}} Cl_n$.

Now, we would like to construct a principal bundle P_{Spin_n} over X associated to the exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin_n \xrightarrow{\overline{Ad}} SO_n \longrightarrow 1. \tag{1.1}$$

However, this is not always possible due to an obstruction named 2^{nd} Stiefel-Whitney class $w_2(E) \in H^2(X, \mathbb{Z}_2)$.

Definition 1.7. Let X be a smooth manifold and $dim(X) \geq 3$. A spin structure over a vector bundle $\pi : E \to X$ is a principal bundle $P_{Spin_n}(E)$ and a map $\zeta : P_{Spin_n} \to P_{SO(E)}$, such that $\zeta(p.g) = \zeta(p).\overline{Ad}(g)$, for all $p \in P_{Spin_n}(E)$ and $g \in Spin_n$, and such that the diagram below is commutative ($\pi \circ \zeta = \pi'$)

$$\begin{tikzcd} & P_{Spin} \arrow[dl, "\zeta"'] \arrow[d, "\pi'"] \\ P_{SO(E)} \arrow[r, "\pi"] & X \end{tikzcd} \tag{1.2}$$

The exact sequence in 1.1 induces an exact sequence at the Cech Cohomology level;

$$H^1(X, \mathbb{Z}_2) \longrightarrow H^1(X, Spin_n) \xrightarrow{\overline{Ad}} H^1(X, SO_n) \xrightarrow{w_2(E)} H^2(X, \mathbb{Z}_2), \tag{1.3}$$

where each class in $H^1(X, G)$ represents the set of transition functions of a G-principal bundle. Therefore, a class in $H^1(X, Spin_n)$ represents a $Spin_n$-bundle lifted from a SO_n-bundle $P_{SO(E)}$ (as in diagram 1.2) if, and only if, $w_2(E) = 0$.

Definition 1.8. A vector bundle $\pi : E \to X$ is spin if $w_2(E) = 0$.

1.3.1 Classification of Spin Bundles

From its initial concept, the map $\zeta : P_{Spin_n} \to P_{SO_E}$ is a double cover when restricted to the fibers. Let's investigate the possible maps doing this. A basic fact about 2-covers of a manifold M is that they are classified by $H^1(X, \mathbb{Z}_2)$, since a 2-cover is determined by the kernel of a homomorphism $\pi_1(M) \to \mathbb{Z}_2$, which descends to a homomorphism $H_1(M, \mathbb{Z}_2) \to \mathbb{Z}_2$ determining a class in $H^1(M, \mathbb{Z}_2)$. Therefore, the space of spin structure over $P_{SO(E)}$ is $1 \leftrightarrow 1$ with the classes $\phi \in H^1(P_{SO(E)}, \mathbb{Z}_2)$ that are non-trivial when restricted to the fibers of $P_{SO(E)}$.

example: Let X and $f : T^4 \to T^4$ is a double. The pull-back bundle $f^* P_{SO(E)}$ doesn't corresponds to a $Spin_4$ structure on a vector bundle $\pi : E \to T^4$.

The bundle sequence $SO_n \xrightarrow{i} P_{SO(E)} \xrightarrow{\pi} X$ induces the exact sequence

$$0 \longrightarrow H^1(X, \mathbb{Z}_2) \xrightarrow{\pi^*} H^1(P_{SO(E)}, \mathbb{Z}_2) \xrightarrow{i^*} H^1(SO_n, \mathbb{Z}_2) \xrightarrow{w_E} H^2(X, \mathbb{Z}_2), \tag{1.4}$$

where $w_2(E) = w_E(g_1)$ and g_1 is the generator of $H^1(SO_n, \mathbb{Z}_2) \simeq \mathbb{Z}_2$.

Theorem 1.3. *Let E be a orientable vector bundle over X. There exists a spin structure over E if, and only if, $w_2(E) = 0$. Moreover, $H^1(X, \mathbb{Z}_2)$ can be identified as the space $Spin(E)$ of spin structures on E.*

Demonstração. If there is a spin structure, then we have seen that $w_2(E) = 0$. Let's prove the converse. Suppose $w_2(E) = 0$, by the exact sequence **??** an element $\phi \in H^1(P_{SO(E)}, \mathbb{Z}_2)$, non-trivial along the fibers, can be written as $\phi = \pi^*(\alpha) + \beta$, where $\alpha \in H^1(X, \mathbb{Z}_2)$ and $i^*(\beta) = g_1$. Since π^* is a monomorphism, it follows that whenever $\alpha \neq \alpha'$ in $H^1(X, \mathbb{Z}_2)$ we have

$$\phi' = \pi^*(\alpha') + \beta \neq \pi^*(\alpha) + \beta = \phi.$$

Therefore, after fixing $\beta \notin Ker(i^*)$, for each $\alpha \in H^1(X, \mathbb{Z}_2)$ corresponds only one spin structure. □

Example 1.3. LetÂ´s compute the spin structures on some explicit examples;

1. Consider $\pi : E \to S^1$ as a $rank = 2$ riemannian vector bundle with structural group $SO(2)$. In this case, the frame bundle must be trivial, hence $P_{SO(E)} = S^1 \times S^1 = T^2$ and $w_2(E) = 0$. According with the theorem 1.3, there are only four spin structures on E, since $H^1(T^2, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The sequence $1 \to SO_2 \to T^2 \to S^1$ induces the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(S^1, \mathbb{Z}_2) & \xrightarrow{\pi^*} & H^1(T^2, \mathbb{Z}_2) & \xrightarrow{i^*} & H^1(SO_2, \mathbb{Z}_2) & \xrightarrow{w_E} & 0 \\ \downarrow & & \simeq\downarrow & & \simeq\downarrow & & =\downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow{w_E} & 0 \end{array} \tag{1.5}$$

Each spin structure is equivalent to an element of form $(\alpha, 1) \in H^1(T^2, \mathbb{Z}_2)$, $\alpha \in H^1(S^1, \mathbb{Z}_2)$. So, they are $(0,1)$ and $(1,1)$; each one corresponding to the following 2-covers of T^2:

(a) $p_1 : T^2 \to T^2$, $p_1(e^{i\theta}, e^{i\zeta}) = (e^{i\theta}, e^{i2\zeta})$; $T^2 = S^1 \times (S^1 / \mathbb{Z}_2)$.

(b) $p_2 : T^2 \to T^2$, $p_2(e^{i\theta}, e^{i\zeta}) = (e^{i2\theta}, e^{i2\zeta})$; $T^2 = S^1 \times_{\mathbb{Z}_2} S^1$.
In this example we can see that although the principal bundles are diffeomorphic to T^2, they carry different spin structures.

2. Let SO_n, $n \geq 3$, and $\pi : TSO_n \to SO_n$ be its tangent bundle. It is well known that the tangent bundle of a Lie Group is trivial, so $TSO_n = SO_n \times SO_n$ and $w_2(TSO_n) = 0$. Therefore, it admits a spin structure and there are only two of them because $H^1(SO_n, \mathbb{Z}_2) = \mathbb{Z}_2$, for $n \geq 3$. The spin structures correspond to the following 2-covers;

$$SO_n \times Spin_n \stackrel{p_1}{\to} SO_n \times SO_n, \quad Spin_n \times Spin_n \stackrel{p_2}{\to} Spin_n \times_{\mathbb{Z}_2} Spin_n.$$

3. Whenever X is an almost complex manifold, then

$$w_2(X) = c_1(X)\ mod\ 2 \tag{1.6}$$

By considering $X = F_g$ a closed surface of genus g, then $c_1(F_g) = \chi(F_g) = 2(1-g)$, and $w_2(F_g) = 0$. Thus, TF_g admits a spin structure and, since $H^1(F_g, \mathbb{Z}_2) \simeq \mathbb{Z}_2^{2g}$, there are 2^{2g} spin structures on TF_g.

Definition 1.9. An orientable riemannian manifold X is spin if $w_2(TX) = 0$. In this case, X carries a spin structure by fixing a spin structure $s_X \in H^2(X, \mathbb{Z}_2)$ on TX (notation: $w_2(X) = w_2(TX)$)

remarks:

1. the spin structure on TX independs on the riemannian metric defined on X, since the inclusion $P_{SO(E)} \to P_{GL(E)}$ is a homotopy equivalence.

2. A diffeomorphism $f : X \to X$ may change the spin structure defined on X. The induced isomorphism $f^* : H^1(X, \mathbb{Z}_2) \to H^1(X, \mathbb{Z}_2)$ may not be the trivial one.

1.3.2 Geometric Meaning of a Spin Structure

Let $w = w_1 + w_2 + \cdots + w_n \in H^*(X, \mathbb{Z}_2)$ be the Total Stiefel-Whitney class, $n = dim(X)$. Using the Steenrod squaring operations $Sq^k : H^j(X) \to H^{j+k}$, there exists only one element $v_k \in H^k(X, \mathbb{Z})$ such that

$$(v_k \cup u) = Sq^k(u) \in H^n(X, \mathbb{Z}), \quad \forall u \in H^{n-k}(X, \mathbb{Z}_2).$$

Let $Sq = \sum_{i=0}^{k} Sq^k$ be the total Steenrod squaring and $v = 1 + v_1 + v_2 + \dots$ be the total Wu class. Clearly, v satisfies the identity

$$v \cup u = Sq(u), \quad \forall u \in H^*(X, \mathbb{Z}_2).$$

Proposition 1.11. *(Wu's formula [3]) Let $w \in H^*(X, \mathbb{Z}_2)$ be the total Stiefel-Whitney class of X:*

$$w = Sq(v).$$

(since $Sq^i(u) = 0$ if $i > deg(u)$, then $v_k = 0$ for $k > [\frac{n}{2}]$).

Example 1.4. Let X be an orientable manifold with dimension ≤ 4. By Wu's formula,

$$\sum_{i=0}^{4} w_i = \sum_{i=0}^{2} Sq^i(v) = 1.(1 + v_1 + v_2) + v_1.(1 + v_1 + v_2) + v_2(1 + v_1 + v_2) =$$

$$= 1 + v_1 + v_1^2 + v_2 + v_1 v_2 + v_2^2 \quad \Rightarrow \quad \begin{cases} w_1 = v_1 = 0, \\ w_2 = v_1^2 + v_2 = v_2, \\ w_3 = v_1 v_2 = 0, \\ w_4 = v_2^2. \end{cases}$$

The tangent bundle of closed 3-manifolds is always trivial, so $w_2(X) = 0$ and $w = 1$. If X is an orientable 4-manifold, then $w = 1 + w_2$, where $w_2 = v_2$ and

$$w_2 \cup u = u \cup u, \quad \forall u \in H^2(X, \mathbb{Z}_2).$$

Proposition 1.12. *If X is a spin 4-manifold, then $Q(u, u) = 0$ mod 2 for all $u \in H^2(X, \mathbb{Z})$.*

Theorem 1.4. *Let X be a n-dimensional manifold.*

1. *If $n \geq 5$, then X is spin if, and only if, all embeded orientable surface in X has trivial normal bundle.*

2. *If $n = 4$, then X is spin if, and only if, the euler class of the normal bundle of an embeded orientable surface in X is even.*

Demonstração. The group $H_2(X, \mathbb{Z})$ is generated by compact orientable surfaces and so is the group $H_2(X, \mathbb{Z}_2) = H_2(X, \mathbb{Z}) \otimes \mathbb{Z}_2$. By Whitney's Embedding Theorem, if $n \geq 5$ these surfaces can be embedded in X, and if $n = 4$ they may have transversal self-intersections which can be removed by the price of increasing the genus of the surface. In both cases, the generating classes are smooth. Let $\iota : \Sigma \to X$ be such embedding with normal bundle $\nu(\Sigma)$. Then, since $w_2(\Sigma) = 2(1 - g)$ *mod* $2 \equiv 0$ *mod* 2,

$$\iota^* w_2(X) = \iota^* w_2(TX) = w_2(\iota^* TX) = w_2(T\Sigma \oplus \nu(\Sigma)) = \tag{1.7}$$
$$w_2(T\Sigma) + w_2(\nu(\Sigma)) = w_2(\nu(\Sigma)). \tag{1.8}$$

Hence,

$$\iota^* w_2(X)[\Sigma] = w_2(X)[\iota_*(\Sigma)] = w_2(\nu(\Sigma))[\Sigma] = \tag{1.9}$$
$$= \chi(\nu(\Sigma))\ mod\ 2 = Q(\Sigma,\Sigma)\ mod\ 2. \tag{1.10}$$

Therefore, $w_2(X) = 0$ if, and only if, $w_2(\nu(\Sigma))[\Sigma] = 0$ for all embedded surface in X. Furthermore, if $dim(\nu(\Sigma)) \geq 3$ (or $n \geq 5$), then $\nu(\Sigma)$ is trivial and so $w_2(\nu(\Sigma)) = 0$. Whenever $dim(\nu(\Sigma)) = 2$ (or $n = 4$), $w_2(X) = 0$ iff for any surface its self-intersection number is even. □

Corollary 1.3. *Let X be simply connected.*

1. *If $dim(X) \geq 5$, then X is spin iff every embedded 2-sphere in X has trivial normal bundle.*
2. *If $dim(X) = 4$, then X is spin iff $Q(u,u) \equiv 0\ mod\ 2$, for all $u \in H^2(X,\mathbb{Z})$.*

Demonstração. It is enough to remark that X being simply connected imply, by Hurewicz's theorem, that $H_2(X,\mathbb{Z}) \simeq \pi_2(X)$ is generated by spheres. If $dim(X) = 4$, the generating spheres may not be smoothly embedded. □

Today, the main question in 4-dimensional smooth topology is about the classification of smooth, closed, simply connected 4-manifolds. However, the question concerning the realization of quadratic forms as intersection forms of smooth manifolds is still unsolved. There are two very deep theorems about the last question;

Theorem 1.5. *(Rohlin) Let X be a smooth, closed 4-dimensional manifold with signature σ_X. If X is spin, then*

$$\sigma_X \equiv 0\ mod\ 16.$$

remark: For all 4-manifold X, $\sigma_X \equiv 0\ mod\ 8$.

Theorem 1.6. *(Donaldson) Let X be a smooth, closed 4-dimensional manifold. If Q_X is positive (negative) definite, then*

$$Q_X \simeq 1 \oplus 1 \oplus \cdots \oplus 1, \quad (-1 \oplus -1 \oplus \cdots \oplus -1).$$

Corollary 1.4. *X spin and positive definite, then $rk(Q_X) \equiv 0\ mod\ 16$.*

1.3.3 Interpretation of w_2 as an Obstruction

A a smooth manifold X admits a CW-complex structure $K = \cup_{i=0}^{n} K_i$, where $K^{(i)}$ is the i-skeleton and the underlying polyhedron is $\mid K \mid = X$. Besides, the CW-structure can be induced by a handle decomposition, as decribed in the next section.

For a vector bundle $p : E \to X$, the 2^{nd} Stiefel-Whitney class measures the extendability of a trivialization τ over the 1-skeleton $K^{(1)}$ to the 2-skeleton $K^{(2)}$. Let $C_i(X)$ be the i^{th}-chain $\mathbb{Z}$-module, $Z_i(X) = Ker(\partial)$ be i^{th}-cycles submodule and $B_i(X) = Im(\partial)$ be the i^{th}-boundaries submodule. Also, there is the dual submodules $C^i(X)$, $Z^i(X)$ and $B^i(X)$.

Let's start by discussing the orientability of a bundle $p : E \to X$. Consider $c \in C_1(X)$, $c : [0, 1] \to X$, with a frame β_0 fixed at $c(0)$ and another one β_1 fixed at $c(1)$. It is natural to ask if it is possible to continuously extend the frames β_0, β_1 over c. Clearly, if β_0 and β_1 belong to the same connected component of O_n then the extension can be continously performed, otherwise it can not be performed since they lie in distinct connected component. This is better interpreted as a map $w_1(E, \tau) : S^0 \to \pi_0(O_n)$ associating to a trivialization τ on $\partial c = S^0$ a class in $\pi_0(O_n) \simeq \mathbb{Z}_2$, where the value is 0 if the frames are in the same component and 1 otherwise. This procedure when applied to 1-cycles in $Z_1(X, \mathbb{Z})$ induces a homomorphism $w_1(E, \tau) : Z_1(X, \mathbb{Z}) \to \mathbb{Z}_2$.

Proposition 1.13. *Let E be a rank n real vector bundle over a smooth manifold X. So,*

1. $\delta w_1(E, \tau) = 0$.
2. *Let τ' and τ be distincts trivializations over the 1-skeleton $K^{(1)}$. Then there exists a class $\eta_0 \in C_0(X)$ such that*

$$w_1(E, \tau') - w_1(E, \tau) = \delta\eta_0.$$

Hence, $w_1(E, \tau) \in H^1(X, \mathbb{Z}_2)$ independs on τ.

Demonstração. .

1. For $s \in C_2(X)$, $\delta w_1(E, \tau)(s) = w_1(E, \tau)(\partial s)$. From the classification of compact surfaces, it is known that $\mid s \mid$ has the homotopy type of a bouquet $\bigvee_{i=1}^{2g} S^1$, where g is the genus of $\mid s \mid$. Besides, ∂s is homotopic to the bouquet, so $w_1(E, \tau)(\partial s) = \prod_{i=1}^{2g} w_1(E, \tau) = 0 \ mod\ 2$. Therefore, $w_1(E, \tau)(\partial s) = 0$.
2. If τ' and τ are distincts trivialization, then $w_1(E, \tau')$ and $w_1(E, \tau)$ take different values in $\pi_0(O_n)$. So, for any 1-chain $c \in C_1(X)$, $(w_1(E, \tau') - w_1(E, \tau))(c)$ measures the difference on a 0-chain $q = \partial c \in C_0(X)$, which is nothing else than just a coboundary $\delta\eta_0$.

□

Definition 1.10. The 1^{st} Stiefel-Whitney class of a vector bundle E is $w_1(E) \in H^1(X, \mathbb{Z}_2)$. E is an orientable vector bundle if $w_1(E) = 0$.

The same sort of question can be asked by analysing the case of extending a trivialization (frame) of an oriented vector bundle E over the 1-complex $K^{(1)}$ to a trivialization over the 2-chain complex $K^{(2)}$. First of all, let's fix an trivialization τ on $K^{(1)}$. For each 1-cycle $\gamma \in Z_1(X)$, τ is a map $\tau : \gamma^* E \to S^1 \times \mathbb{R}^n$ given by

$$\tau(p^{-1}(t)) = (t; e_1(t), \ldots, e_n(t)).$$

Thus, for each $\gamma \in C_1(X)$, there is a map $\tau_\gamma : S^1 \to SO_n$, $\tau_\gamma(t) = (e_1(t), \ldots, e_n(t))$. Therefore, a trivialization τ over a closed curve $\gamma : S^1 \to X$ induces a class $[\tau_\gamma] \in \pi_1(SO_n)$, where $\pi_1(SO_n) \simeq \mathbb{Z}_2$ whenever $n > 2$, and $\pi_1(SO_2) = \mathbb{Z}$. Assuming that E is an oriented vector bundle, it admits a trivialization τ over the 1-skeleton $K^{(1)}$. The restriction of τ over the boundary $\partial\triangle$ of a 2-simplex $\triangle$ extends over $\triangle$ iff $[\tau_{\partial\triangle}] = 0$. By defining $w_2(E, \tau)(\triangle) = [\tau_{\partial\triangle}]$, we have a homomorphism $w_2(E, \tau) : C_2(X) \to \pi_1(SO_n)$. Now, let $S \subset K^{(2)}$ be a submodule of $Z_2(X)$. From the classification theorem of compact surfaces, $\mid S \mid$ is homotopic to $(\bigvee_{i=1}^{2g} S^1) \sqcup D^2$, where g is the genus of $\mid S \mid$. Let $\hat{S}$ be a submodule of S such that $\mid S \mid = \mid \hat{S} \mid \sqcup D^2$. Therefore, the bundle E is trivial over $\hat{S}$ and it extents over S iff $[\tau_{\partial\hat{S}}] = 0$. Thus,

$$w_2(E, \tau)(\mid S \mid) = [\tau_{\partial\hat{S}}]$$

and E extends over S iff $w_2(E, \tau)(\mid S \mid) = 0$.

Proposition 1.14. *Let E be a rank n oriented real vector bundle over a smooth manifold X. So,*

1. $\delta w_2(E, \tau) = 0$.
2. *Let τ' and τ be distincts trivialization over the 1-skeleton $K^{(1)}$. Then there exists a class $\eta_1 \in C^1(X)$ such that*

$$w_2(E, \tau') - w_2(E, \tau) = \delta\eta_1.$$

Hence, $w_2(E, \tau) \in H^2(X, \mathbb{Z}_2)$ independs on τ.

Demonstração. .

1. For $Q \in C_3(X)$, $\delta w_2(E, \tau)(Q) = w_2(E, \tau)(\partial Q)$. First of all, let's decompose Q as $Q = \sum_{i=1}^n q_i \triangle_i^3$, where $\triangle_i^3$, $i = 1, \ldots, n$, are 3-simplex. For each $i \in \{1, \ldots, n\}$, we have $w_2(E, \tau)(\partial\triangle_i^3) = 0$ because every trivialization over $\partial\triangle_i^3$ extends over $\triangle_i^3$, since $\pi_2(SO_n) = 0$ for $n > 2$. Therefore, E is trivial over $K^{(3)}$. Hence, E is trivial over $\partial Q \in Z_2(X)$ and $w_2(E, \tau)(\partial Q) = 0$.

2. By considering τ' and τ distincts trivialization, $w_2(E, \tau')$ and $w_2(E, \tau)$ assume different values in $\pi_1(SO_n)$ when computed on $s \in C_2(X)$. In this way, for any $s \in C_2(X)$,

$$(w_2(E, \tau') - w_2(E, \tau))(s) = [\tau'_{\partial s}] - [\tau_{\partial s}]$$

measures the difference on a 1-chain $c = \partial s \in C_1(X)$, $c = \partial s$, which is nothing else than just a coboundary $\delta\eta_1 : C_1(X) \to \mathbb{Z}_2$.

$\square$

Definition 1.11. The 2^{nd} Stiefel-Whitney class of an oriented, real vector bundle E is $w_2(E) \in H^2(X, \mathbb{Z}_2)$. E is a spin bundle if $w_2(E) = 0$.

Example 1.5. Every closed, oriented, compact surface Σ_g of genus g is spin. As described above, by considering $\Sigma_g = \hat{\Sigma}_g \sqcup D^2$, the class $w_2(X) \in H^2(X, \mathbb{Z}_2)$ is measured by fixing a trivilization τ of $T\Sigma_g$ over $\hat{\Sigma}_g$ and computing the class $[\tau_{\partial\hat{\Sigma}}]$. In this way, τ defines a frame $\beta = \{e_1, e_2\}$ over $\hat{\Sigma}$. The Hopf theorem states that the index of each vector field e_1 and e_2 must be equal to the Euler characteristic $\chi(\Sigma_g) = 2(1-g)$, what means that the maps $e_i : \partial D^2 \to SO_n$ induces the class $[\tau_i] = 0 \in \mathbb{Z}_2$. Therefore, the frame β extends over D^2, hence over Σ_g.

It follows from the former discussion that, for all oriented real vector bundle E, the space of orientations on E is parametrized by $H^0(X, \mathbb{Z}_2) = \mathbb{Z}_2$ and the space of spin structure is parametrized by $H^1(X, \mathbb{Z}_2)$.

1.3.4 Spin Structure on a Handlebody

A k-handle h_k of dimension n is, by definition, the space $h_k = D^k \times D^{n-k} \overset{homeo}{\simeq} D^n$. A k-handle has the following subsets;

1. the core of h_k is $D^k \times \{0\}$ and its cocore is $\{0\} \times D^{n-k}$.
2. the attaching a-sphere of h_k is $A = S^{k-1} \times \{0\}$ and the belt b-sphere is $B = \{0\} \times S^{n-k-1}$. (convention: $S^{-1} = \{1\}$.)

A handle decomposition of a smooth manifold X is a decomposition

$$X = \mathcal{H}_0 \cup \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_k \cup \cdots \cup \mathcal{H}_n,$$

where $\mathcal{H}_k = \cup_{i=1}^{n_k} h_k^i$ is the union of n_k k-handles, each one corresponding to a critical value of a Morse function $f : X \to \mathbb{R}$ with Morse index k. Fortunetely, it can be assumed that the critical values of f are in crescent order according with their indexes. Thus, let

$$X^0 = \mathcal{H}^0,\ X^1 = X^0 \cup \mathcal{H}_1, \ldots, X^k = X^{k-1} \cup \mathcal{H}_k, \ldots,\ X^n = X^{n-1} \cup \mathcal{H}_n.$$

Each piece X^k is a n-manifold with boundary. In order to attach a k-handle over ∂X^{k-1}, we need to perscribe two pieces of data:

1. the isotopy class of an embedding $\phi : S^{k-1} \times \mathbb{R}^{n-k}$ with trivial normal bundle. (ϕ is the attaching map);

2. a normal framing τ of $\phi(S^{k-1})$ corresponding to the identification of the normal bundle $\nu(\phi(S^{k-1}))$ with $S^{k-1}\times\mathbb{R}^{n-k}$.

The normal framing τ corresponds to a map $\tau : S^{k-1}\to Gl_{n-k}$, and so, each normal framing is defined, up to homotopy, as an element in $\pi_{k-1}(Gl_{n-k})$. In dimension 4,

$$\pi_{k-1}(Gl_{4-k}) = \begin{cases} \mathbb{Z}_2, & k=1,\\ \mathbb{Z}, & k=2,\\ 1, & k=3,4. \end{cases} \tag{1.11}$$

Therefore, in dimension 4 the framing of a 2-handle is specified by an integer number. A trivialization τ over the attaching sphere $A = S^1\times\{0\}$ extends to a trivialization over the core of a 2-handle iff the framing $\tau_A \in 2\mathbb{Z}$.

Definition 1.12. A n-dimensional handlebody is a n-manifold X admiting a handle decomposition

$$X = D^n \cup \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_k \cup \cdots \cup \mathcal{H}_{n-1},$$

where $\mathcal{H}_k = \cup_{i=1}^{n_k} h_k^i$ is the union of n_k k-handles.

Example 1.6. Let $X = D^4\cup h_1\cup h_2$ be a 4-manifold obtained by attaching two 2-handles to the ball $X = D^4$. The attaching spheres $A_1, A_2 \subset S^3 = \partial D^4$ are knots in S^3. The Seifert surfaces $S_1,\ S_2$ associated to each knot, respectively, when capped off by the core of the 2-handles define surfaces $\Sigma_1,\ \Sigma_2$. Thus, $H_2(X,\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}$ is generated by $[\Sigma_1]$ and $[\Sigma_2]$. The quadratic form $Q_X : H_2(X,\mathbb{Z})\times H_2(X,\mathbb{Z})\to\mathbb{Z}$ is defined as the linear extension of

$$Q_X([\Sigma_i],[\Sigma_j]) = lk(A_i,A_j),\ i,j = 1,2,$$

where $lk(A_i,A_j)$ is the linking number among A_i and A_j. In the case $i=j$ the linking number is exactly the framing of A_i. Therefore, X is spin iff the framings of $[\tau_{A_1}]$ and $[\tau_{A_2}]$ are in $2\mathbb{Z}$.

In the example above it is shown how a 2-handle h attached to D^4 determines a surface Σ_h in $X = D^4\cup h$. By sliding a 2-handle h_1 over a 2-handle h_2 we obtain a new 2-handle h because its attaching sphere is not A_1 anymore. The surface Σ_h obtained by attaching h represents the homology class $[\Sigma_h] = [\Sigma_{h_1}] + [\Sigma_{h_2}]$, so

$$Q_X([\Sigma_h],[\Sigma_h]) = \{Q_X([\Sigma_{h_1}],[\Sigma_{h_1}]) + Q_X([\Sigma_{h_2},[\Sigma_{h_2}])\}\ mod\ 2. \tag{1.12}$$

Proposition 1.15. *Let Y^3 be a closed 3-manifold. Thus, Y bounds a handlebody X^4 constructed by using only one 0-handle and several 2-handles.*

Demonstração. Its is known that every closed 3-manifold is the boundary of a 4-manifold W. A handle decomposition of W can be obtained by using only one 0-handle and no 4-handles. The 1-handles can be cancelled by attaching 2-handles along the cores of the 1-handles killing the $\pi_1(W)$ generators. By turning the handlebody upside down the 3-handles became 1-handles and the same process can be appllied to end up with only 2-handles and one 0-handle. □

Proposition 1.16. *Let Y be a closed 3-manifold endowed with a spin structre s_Y. Thus, there exists a closed surface Σ_g of genus g and a spin structure s_M on $M = S^1 \times \Sigma_g$ such that (Y, s_Y) is spin cobordant to (M, s_M).*

Demonstração. Consider $X = (Y \times [0,1]) \cup_{i=1}^n h_i$, where h_i are 2-handles attached to $Y \times \{1\}$ with framing n_i. If the framings n_i are even numbers, then the spin structure s_Y extends to a spin structure s_X on X and we define $M = \varnothing$. In case there are 2-handles with odd framings, suppose they are h_1 and h_2, we can slide h_2 over h_1 in order to replace h_2 by the new 2-handle h_2' with frame $n \in 2\mathbb{Z}$ (compute it using equation 1.12). Therefore, we are left with just one 2-handle h with odd framing. Whenever the number of odd framed 2-handles is greater than two, this procedure can be carried out to end up with only one odd framed 2-handle denoted by h.

By erasing the cocore of h, the effect of attaching h is canceled out. However, we can consider the Seifert surface $S_{bs} \subset Y \times \{1\}$ of the belt sphere of h and capp it off with the cocore of h to construct the genus g closed surface $F_g = (\{1\} \times D^2) \cup S_{bs}$. Let $\nu(F_g)$ be F_g normal bundle. In this way, it has been constructed a cobordism $\widehat{X}$ among Y and $\partial\nu(F_g)$. It may not be true that $\nu(F_g) = D^2 \times F_g$. By modifying $\widehat{X}$ we can obtain a cobordism $\widetilde{X}$ among Y and $S^1 \times F_g$. The bundle $\nu(F_g)$ is trivial iff the U_1-bundle $\partial(\nu(F_g))$ is trivial. In order to turn the U_1-bundle $\partial\nu(F_g)$ into a trivial bundle it is necessary to became null its 1^{st}-Chern class c_1. If $c_1(\partial\nu(F_g)) > 0$, then by connecting sum $\widetilde{X}$ with $\overline{\mathbb{C}P}^2$ and tubing F_g with $\mathbb{C}P^1 \subset \overline{\mathbb{C}P}^2$, the 1^{st} Chern class of the U_1-bundle $\partial(\nu(F_g))$ is decreased by 1, and the tubing process can go on until $c_1(\partial\nu(F_g)) = 0$. If $c_1(\partial\nu(F_g)) < 0$, by connecting sum with $\mathbb{C}P^2$ and tubing with $\mathbb{C}P^1 \subset \mathbb{C}P^2$ then $c_1(\partial\nu(F_g))$ is increased by 1. Now, the spin structure s_Y can be extended over X, hence defines a spin structure s_M on $M = S^1 \times F_g$. □

Next, let's see that $S^1 \times F_g$ is spin cobordant to a finite union of 3-toris T^3.

Lemma 1.1. *Consider $S^1 \times F_g$ endowed with a spin structure s. Thus, there are a finite number of spin 3-toris (T_i^3, s_i) endowed with a spin structure s_i, $i \in \{1, \dots, n\}$ and a spin 4-manifold (W, s_W) such that:*

1. $\partial W = (S^1 \times F_g) \sqcup (\cup T_i^3)$,
2. $s_W |_{S^1 \times F_g} = s$ *and* $s_W |_{T_i^3} = s_i$

Demonstração. The kernel of the proof relies on the fact that F_g is spin cobordant to $\sqcup_{i=1}^g T_i^2$. In order to verify this fact it is enough to consider a set of curves $\{\gamma_1, \dots, \gamma_{g-1}\}$

splitting F_g into surfaces of genus 1. Consider the 3-manifold $F_g \times [0,1]$. By attaching 2-handles $h_i = D^2 \times D^1$, wich attaching spheres are $A_i = \gamma_i \times \{1\}$, we obtain a coboundary among F_g and $\sqcup_{i=1}^g T_i^2$. Any spin structure on F_g, when restricted to γ_i, $i = 1, \ldots, g-1$, can be extended to the core of h_i. So, it extends to T_i^2. Therefore, if F_g is spin cobordant to a union of g 2-toris, then $S^1 \times F_g$ is spin cobordant to a union of g 3-toris T^3.

□

Lemma 1.2. *Let $s \in Spin(T^3)$. Thus, there is a spin manifold (X, s_X) such that (T^3, s) bounds $X, s_X)$.*

Theorem 1.7. *All closed spin 3-manifold (Y, s_Y) bounds a compact spin 4-manifold (X, s_X) such that $s_X \mid_Y = s_Y$.*

1.4 Almost Complex Structures

An almost complex structure on a smooth manifold X is an automorphism $J : TX \to TX$ such that $J^2 = -1$. In this case, we say that (X, J) is an almost complex manifold.
remark: If (X, J) is an amost complex manifold, then X is even dimensional.

The vector bundle (TX, J) being a complex bundle allows one to consider the Chern classes $c_i(X, J) = c_i(TX, J)$, $i = 0, \ldots, dim_{\mathbb{C}}(X)$. In the case X is a 4-manifold, the almost complex surface (X, J) has two Chern classes: $c_1(X, J)$ and $c_2(X, J)$. Besides, $c_2(X, J) = \chi(X)$ (the euler class of TX) and $c_1(X, J) = c_1(K_J^*)$, where $K_J^* = det_{\mathbb{C}}(TX, J)$ is the anti-canonical bundle[1] of (X, J).

In this section, let X be an oriented $2n$-dimensional manifold.

1.4.1 Complex Structures on $\mathbb{R}^{2n}$

In $\mathbb{R}^{2n}$, the canonical complex structure is

$$J_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Let $GL_n(J_0, \mathbb{C}) = \{A \in GL_{2n}(\mathbb{R}) \mid AJ_0 = J_0 A\}$, so

$$M \in GL_n(J_0, \mathbb{C}) \quad \Leftrightarrow \quad \exists A, B \in GL_n(\mathbb{R}) \quad \text{such that} \quad \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = A + iB.$$

In this way, the space of complex structures on $\mathbb{R}^{2n}$ is $\mathcal{C} = GL_{2n}(\mathbb{R})/GL_n(J_0, \mathbb{C})$. An amost complex structure on X is equivalent to a map $X \to \mathcal{C}$, $x \mapsto J_x$ where $J_x : T_xX \to T_xX$ and $J_x^2 = -I$. So, the space of complex structures on T_xX is $\mathcal{C}_x = GL_{2n}(\mathbb{R})/GL_n(J_x, \mathbb{C})$. By considering the bundle

$$\mathfrak{C}_X = \cup_{x \in X} GL_{2n}(\mathbb{R})/GL_n(J_x, \mathbb{C}),$$

[1] the canonical bundle is $K_J = det_{\mathbb{C}}(T^*X, J^*)$, $c_1(K_J) = -c_1(X, J)$

with fiber $\mathcal{C}$, an almost complex structure on X is a section in $\mathfrak{C}_X$.

The following theorem sets a necessary and sufficient condition to the existence of a almost complex stucture on a 4-manifold X;

Theorem 1.8. *Let X be a closed smooth 4-manifold.*

1. *If X admits an almost complex structure J, then*

$$c_1^2(X, J) = 3\sigma_X + 2\chi(X), \tag{1.1}$$

and $c_1(X, J)$ must be an integral lift of $w_2(X)$. Furthermore, $b_2^+(X) + b_1(X)$ must be odd.

2. *If there exists an class $\vartheta \in H^2(X, \mathbb{Z})$ being a integral lift of $w_2(X)$ and satififying*

$$\vartheta^2 = 3\sigma_X + 2\chi(X),$$

then X admits an almost complex structure J with $c_1(X, J) = \vartheta$.

Moreover, assuming either that X is simply connected or has indefinite intersection form, such class ϑ exist whenever $b^+(X) + b_1(X)$ is odd.

Demonstração. 1. ($\Rightarrow$) For any compex vector bundle we have $w_2(E) = c_1(E) \mod 2$ and also its 1^{st} Pontrjagin class is

$$p_1(E) = -c_2(E \otimes_{\mathbb{R}} \mathbb{C}) = -c_2(E \oplus E^*) = c_1(E).c_1(E) - 2c_2(E).$$

By restricting to our case $E = (TX, J)$, where $p_1(TX) = 3\sigma_X$ and $c_2(TX) = \chi(X)$, it follows that $c_1^2(TX, J) = 3\sigma_X + 2\chi(X)$.

2. ($\Leftarrow$) Let L be the complex line bundle over X with $c_1(L) = \vartheta$, and consider $E_\vartheta = L \oplus \mathbb{C}$, so $c_1(E_\vartheta) = \vartheta$. Now, we cut off a ball $D^4 \subset X$ and glue it back using a SU_2-twist in order to obtain a bundle $E_{\vartheta,\chi(X)}$ with $c_2 = \chi(X)$; the class $c_1 = \vartheta$ is preserved through this sort of surgery. The bundle $E_{\vartheta,\chi}$ is complex and its characteristic numbers are

$$\begin{aligned} &w_2(E_{\vartheta,\chi(X)}) = w_2(TX), \quad e(E_{\vartheta,\chi(X)}) = c_2(E_{\vartheta,\chi(X)}) = \chi(X) = c_2(TX), \\ &p_1(E_{\vartheta,\chi(X)}) = c_1^2(E_{\vartheta,\chi(X)}) - 2c_2(E_{\vartheta,\chi(X)}) = \vartheta^2 - 2\chi(X) = p_1(TX) \end{aligned}$$

By Dold-Whitney theorem [1], the isomorphism classes of bundles over a 4-complex are classified by their numbers w_2, p_1 and the euler class e. Consequently, the bundles $E_{\vartheta,\chi(X)}$ and TX are isomorphic and the complex structure on the fibers of E can be transported to an almost complex structure on TX.

$\square$

Corollary 1.5. *If X^4 admits an almost complex structure, then it must be that $b_2^+(X) + b_1(X)$ is odd.*

Demonstração. Let $c_1(J)$ be the 1^{st} Chern class of X, so $c_1^2(J) = 3\sigma_X + 2\chi(X)$. By van der Blij's lemma, we have

$$c_1^2(J) = \sigma_X \; mod \; 8,$$

and thus $\sigma_X + \chi(X) = 0 \; mod \; 4$. Further, $\sigma_X = b_2^+ - b_2^-$ and $\chi(X) = 2 - 2b_1 + b_2^+ + b_2^-$, and hence $2b_2^+ - 2b_1(X) + 2 = 0 \; mod \; 4$. Therefore,

$$b_2^+ + b_1 = 1 \; mod \; 2.$$

□

Corollary 1.6. *For every characteristic element $\underline{w} \in H^2(X, \mathbb{Z}_2)$, there is a partial almost-complex structure $J \mid_3$ over the 3-skeleton of X, with $c_1(J \mid_3) = \underline{w}$.*

Demonstração. It may exists a characteristic element $\underline{w}$ which integral lifts do not satisfy the identity 1.1. In this case, in the proof of theorem 1.8, the bundles $E_{\vartheta,\chi(X)}$ and TX are isomorphic over the 3-skeleton of X, or equivalently, over $X - \{point\}$[2]. □

Definition 1.13. Let X be a manifold endowed with an almost complex structure J. A surface $S \subset X$ is called a J-holomorphic curve (or pseudo-holomorphic curve) if its tangent bundle is J-invariant ($J(TS) = TS$).

Theorem 1.9. *(Adjunction Inequality) Let (X, J) be a almost complex 4-manifold and S is a pseudo-holomorphic curve in X, then we have*

$$\chi(S) + S.S = K^*.S \tag{1.2}$$

Demonstração. Observing that $c_1(K^*) = c_1(TX)$ (notation $K^* = c_1(K^*)$), we have

$$\begin{aligned} K^*.S = c_1(TX)(S) = c_1(TX \mid_S) = c_1(TS \oplus \nu S) = \\ = c_1(TS) + c_1(\nu S) = \chi(S) + S.S \end{aligned}$$

□

<u>remarks:</u> Let's fix a riemannian metric on X^{2n}, so the following facts are relevant in the presence of an almost complex structure J defined on X;

1. the adjunction inequality 1.2 is the main ingredient to estimate the lower genus of a surface $\Sigma \subset X$ representing the class S.
2. J induces a reduction to U_n of the structural group SO_{2n} of TX.
3. If $n = 2$, J is equivalent to the existence of a foliation of codimension 2 of X^4.

[2] this partial almost complex structure does offer enough data to be lifted and extended to a unique *spin*$^{\mathbb{C}}$ structure across all X.

1.5 Spin^c Structures

The monomorphism $\phi : \mathbb{C} \to M_(2, \mathbb{R})$, $\phi(a+ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ induces the standard inclusion $\iota : U_n \to SO_{2n}$ and the canonical homomorphism $\iota \times det : U_n \to SO_{2n} \times U_1$, given by $(\iota \times det)(A) = (\iota(A), det(A))$. The covering map $p : Spin_n \to SO_n$, under an almost complex structure J, is given by

$$p\left[\prod_{k=1}^{n} (cos(\theta_k) + sin(\theta_k) e_k J(e_k))\right] = \begin{pmatrix} R_{2\theta_1} & 0 & 0 & \dots & 0 \\ 0 & R_{2\theta_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & R_{2\theta_n} \end{pmatrix},$$

where $R_{\theta_k} = \begin{pmatrix} cos(\theta_k) & -sen(\theta_k) \\ sen(\theta_k) & cos(\theta_k) \end{pmatrix}$. The kernel of p is $Ker(p) = \{\pm 1\} \simeq \mathbb{Z}_2$.

The scenario induces one to lift the homomorphism $\iota \times det : U_n \to SO_{2n} \times U_1$ to $\hat{p} : Spin_{2n} \times U_1 \to SO_{2n} \times U_1$

$$\begin{tikzcd} & Spin_{2n} \times U_1 \arrow[d, "p"] \\ U_n \arrow[ur, "\hat{p}"] \arrow[r, "\iota \times det"] & SO_{2n} \times U_1 \end{tikzcd}$$

Neverthless, the lif above does not reveal any interesting new structure. Once $H^1(SO_{2n} \times U_1, \mathbb{Z}_2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$, there are three non trivial 2-covers for $SO_{2n} \times U_1$, and the interesting one is $Spin_{2n} \times_{\mathbb{Z}_2} U_1$;

Definition 1.14. The $Spin^c_n$ group is

$$Spin^c_n = Spin_n \times_{\mathbb{Z}_2} U_1.$$

In this way, it is natural to consider the lift as shown in the diagramm below:

$$\begin{tikzcd} & Spin_{2n} \times_{\mathbb{Z}_2} U_1 \arrow[d, "\xi"] \\ U_n \arrow[ur, "j"] \arrow[r, "\iota \times det"] & SO_{2n} \times U_1 \end{tikzcd}$$

Now, there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin^c_n \xrightarrow{\xi} SO_n \times U_1 \longrightarrow 1, \tag{1.1}$$

where $\mathbb{Z}_2 \subset Spin^c_n$ is generated by the classes $[(-1,1)] = \{(-1,1),(1,-1)\}$ and $[(1,1)] = \{(-1,-1),(1,1)\}$. Besides, $Spin^c_n \subset \mathbb{C}l_n = Cl_n \otimes \mathbb{C}$ as a multiplicative subgroup of the group of units.

A $Spin^{\mathbb{C}}$-structure on a complex bundle E is defined as folows;

Definition 1.15. Let E be a vector bundle over X with frame bundle $P_{SO(E)}$. A $Spin^{\mathbb{C}}$-structure on E consist of a pair of principal bundles P_{U_1} and $P_{Spin^{\mathbb{C}}}$ with an $Spin^{\mathbb{C}}_n$-equivariant bundle map $\xi : P_{Spin^{\mathbb{C}}} \to P_{SO(E)} \times P_{U_1}$ ($\xi(pg) = \xi(p)\xi(g)$) such that the diagram below is commutative;

$$\begin{tikzcd} & P_{Spin^{\mathbb{C}}} \arrow[dl, "\xi"'] \arrow[d, "\pi'"] \\ P_{SO(E)} \times P_{U_1} \arrow[r, "\pi"] & X \end{tikzcd} \tag{1.2}$$

The short exact sequence 1.1 induces the exact sequence

$$H^1(X; Spin^{\mathbb{C}}) \xrightarrow{\xi} H^1(X; SO_n) \oplus H^1(X, U_1) \xrightarrow{w_2+\rho} H^2(X, \mathbb{Z}_2),$$

where $\rho : H^1(X,\mathbb{Z}) \to H^1(X,\mathbb{Z}_2)$ is *mod* 2 reduction. However, by the isomorphism $H^1(X,U_1) \simeq H^2(X,\mathbb{Z})$, the exact sequence becames

$$H^1(X; Spin^{\mathbb{C}}) \xrightarrow{\xi} H^1(X; SO_n) \oplus H^2(X, \mathbb{Z}) \xrightarrow{w_2+\rho} H^2(X, \mathbb{Z}_2). \tag{1.3}$$

Therefore, a $Spin^{\mathbb{C}}$-structure $\xi : P_{Spin^{\mathbb{C}}} \to P_{SO(E)} \times P_{U_1}$ exists iff there exists a integral class $u \in H^2(X,\mathbb{Z})$ such that $w_2(E) = \rho(u)$ *mod* 2.

remarks:

1. The letter "c" in the subscript of $Spin^{\mathbb{C}}$ corresponds to the class $c \in H^2(X,\mathbb{Z})$ defined as $c = c_1(P_{U_1})$; its called the *canonical class* of the $Spin^{\mathbb{C}}$-structure.
2. If TX carries a $Spin^{\mathbb{C}}$-structure we say that X is a $Spin^{\mathbb{C}}$-manifold.
3. As an obstruction, an interpretation to a $Spin^{\mathbb{C}}$-structures can be done in the same way as done to a *Spin*-structure. A $Spin^{\mathbb{C}}$-structure over an oriented vector bundle E is equivalent to a complex structure over the 2-skeleton that can be extended over the 3-skeleton. As a consequence, $W_3(E) = 0$ $w_3 = W_3$ *mod* 2).

Example 1.7. .

1. Let (X,J) be a almost complex manifold, so $w_2(TX) = c_1(TX)$ *mod* 2. Hence, (X,J) is a $Spin^{\mathbb{C}}$-manifold. The representation $j : U_n \to Spin^c_{2n}$ induces the associated bundle

$$P_{Spin}(TX) = P_{U_n} \times_j Spin^c_{2n},$$

which U_1-bundle is $P_{U_1} = P_{U_n} \times_{det} U_1$ and whose 1^{st} Chern class is $c_1(X)$ because $c_1(\Lambda^n TX) = c_1(X)$.

2. In particular, if X is spin, then it is also a $Spin^{\mathbb{C}}$-manifold. In this case, there is the bundle $P_{Spin} \times P^0_{U_1}$ where $P^0_{U_1}$ is the trivial principal U_1-bundle over X. A $Spin^{\mathbb{C}}$-structure on X is defined by taking bundle

$$P_{Spin^{\mathbb{C}}} = P_{Spin} \times_{\mathbb{Z}_2} P^0_{U_1}.$$

Anothers $Spin^{\mathbb{C}}$-structures can defined on X by replacing the bundle $P^0_{U_1}$ by $P^\alpha_{U_1}$, the frame bundle of the complex line bundle λ_α, where $c_1(\lambda_\alpha) = \alpha \in H^2(X, \mathbb{Z})$ satisfies the identity $w_2(X) = \alpha \ mod\ 2$. Thus, the new $Spin^{\mathbb{C}}$-structure is

$$P_{Spin^{\mathbb{C}}} = P_{Spin} \times_{\mathbb{Z}_2} P^\alpha_{U_1}, \tag{1.4}$$

and the principal U_1-bundle defining the $Spin^{\mathbb{C}}$-structure is $P^{2\alpha}_{U_1}$, where $P^{2\alpha}_{U_1} = P^\alpha_{U_1}/\mathbb{Z}_2$ is the square bundle [3] of $P^\alpha_{U_1}$.

Let's examine the $Spin^{\mathbb{C}}$-concept from the point of view of vector bundles. For this purpose we need to know what a $Spin^{\mathbb{C}}$-representation is;

Definition 1.16. Let X be a $Spin^{\mathbb{C}}$-manifold of dimesion n. By a complex spinor bundle for X we mean a vector bundle S associated to a representation of $Spin^{\mathbb{C}}$ by Clifford multiplication, i.e,

$$S(X) = P_{Spin^{\mathbb{C}}}(X) \times_{\triangle} V,$$

where V is a complex Cl_n-module and $\triangle : Spin^{\mathbb{C}} \to GL(V)$ is given by restriction of the Cl_n-representation to $Spin^{\mathbb{C}} \subset Cl_n \otimes \mathbb{C}$. If the representation of Cl_n is irredutible, we say that S is fundamental.

remarks: As in the remark 1.2 there is only one fundamental spinor bundle $S(X)$ for any $Spin^{\mathbb{C}}$-manifold X, since

1. when n is even, there exists only one fundamental representation $\triangle : \mathbb{C}l_{2n} \to GL(W)$ and W admits the decomposition $W = W^+ \oplus W^-$, where $W^\pm$ are $\mathbb{C}l^0_n$-invariant representation spaces. Once $Spin_n \subset \mathbb{C}l^0_n$, the representations $\triangle_\pm : Spin_n \subset \mathbb{C}l^0_n \to GL(W^\pm)$ are fundamental representation for $Spin_n$, however, they are equivalents under $\triangle$-representation. The arguments extend to the spinor bundle

$$S(X) = S^+(X) \oplus S^-(X).$$

[3] $\lambda^2_\alpha = \lambda_{2\alpha}$

2. when n is odd, there are two irredutible complex representations of Cl_n. However, they are equivalent when restricted to $Spin_n \subset \mathbb{C}l_n^0$.

Example 1.8. Let's return to the example of a spin manifold X. The $Spin^{\mathbb{C}}$-structure defined in **??** is equivalent to say that the complex spinor spinor bundle $S_\alpha(X)$ associated to the principal U_1-bundle $P_{U_1}^\alpha$ is

$$S_\alpha = S_0(X) \otimes \lambda_\alpha,$$

where $\lambda_{2\alpha}$ is square of the complex line bundle λ_α associated to $P_{U_1}^\alpha$, and so, $c_1(\lambda_\alpha) = \alpha$ and $c_1(\lambda_\alpha) = \alpha$. By the unicity of $S_0(X)$, it follows from **??** that the space $H^2(X, \mathbb{Z})$ acts on the space $Spin^{\mathbb{C}}(X)$. Since the space of spin structures on X is $H^1(X, \mathbb{Z}_2)$, it follows that for a spin manifold

$$Spin^c(X) = \{\alpha + \beta \in H^1(X, \mathbb{Z}_2) \oplus H^2(X, \mathbb{Z}) \mid 0 = \beta \ mod\ 2\}.$$

(observe that $0 = w_2 = c_1(\lambda_{2\alpha}) = 2\alpha\ mod\ 2$)

1.5.1 Local Description of a Fundamental Complex Spinor Bundle

As seen in the example 1.8, X being spin results that for each $\alpha \in H^2(X, \mathbb{Z})$ satisfying $w_2(X) = \alpha\ mod\ 2$, there is a complex spinor bundle $S_\alpha(X) = S_0(X) \otimes \lambda_\alpha$, where $S_0(X)$ is the fundamental spin bundle over X and λ_α is a complex line bundle over X with $c_1(\lambda_\alpha) = \alpha$. However, X not being spin implies that the bundle $S_0(X)$ doesn't exist. Let's see that, in the last case, the decomposition $S_\alpha(X) = S_0(X) \otimes \lambda_\alpha$ exists locally;

Proposition 1.17. *The obstruction to the existence of $S_0(X)$ is equal to the obstruction to the existence of λ_α^2.*

Demonstração. The existence of $S_0(X)$ is equivalent to the vanishing of the 2^{nd} Stiefel-Whitney class $w_2(X)$, as follows: the short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin_n \longrightarrow SO_n \longrightarrow 1$$

induces the exact sequence

$$H^1(X, \mathbb{Z}_2) \longrightarrow H^1(X, Spin_n) \xrightarrow{\overline{Ad}} H^1(X, SO_n) \xrightarrow{w_2(E)} H^2(X, \mathbb{Z}_2),$$

Assuming that $w_2 = c\ mod\ 2$, for some class $c \in H^2(X\mathbb{Z})$, we know that there is a $Spin^{\mathbb{C}}$-structure on X, which P_{U_1}-bundle is the frame bundle of a complex line bundle λ_c. In order to analyse the existence of the square root bundle of λ_c, let's consider the short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow S^1 \xrightarrow{\sigma} S^1 \longrightarrow 1,$$

where $\sigma(z) = z^2$. The sequence aboce induces the exact sequence

$$H^1(X,S^1) \xrightarrow{\sigma^*} H^1(X,S^1) \xrightarrow{w'} H^2(X,\mathbb{Z}_2). \tag{1.5}$$

Let $\{U_\nu\}_{\nu\in\Lambda}$ be a covering of X such that the finite intersections $\cap_{i=1}^m U_{\nu_i}$, $\{\nu_i\}\subset\Lambda$, is always contractible. Considering $\{\gamma_{\mu\nu} \mid \gamma_{\mu\nu} : U_\mu\cap U_\nu \to S^1$, the transition functions of λ_c, the bundle $\lambda_c^{1/2}$ exists iff its transition functions $\{\widetilde{\gamma}_{\mu\nu} = (\gamma_{\mu\nu})^{1/2}\}$ define a cocycle

$$w'([\widetilde{\gamma}_{\mu\nu}]) = \widetilde{\gamma}_{\mu\nu}\widetilde{\gamma}_{\nu\eta}\widetilde{\gamma}_{\eta\mu} : U_\mu\cap U_\nu\cap U_\eta \to \mathbb{Z}_2 = ker(\sigma^*).$$

From the commutative diagram below

$$\begin{array}{ccccc} H^1(X,S^1) & \xrightarrow{\sigma^*} & H^1(X,S^1) & \xrightarrow{w'} & H^2(X,\mathbb{Z}_2) \\ \simeq\downarrow & & \simeq\downarrow & & =\downarrow \\ H^2(X,\mathbb{Z}) & \xrightarrow{2} & H^2(X,\mathbb{Z}) & \xrightarrow{\rho} & H^2(X,\mathbb{Z}_2). \end{array} \tag{1.6}$$

it is clear that the obstructions agree because $[w'([\gamma_{\mu\nu}]) = \rho(c_1(\lambda_c)) = \rho(c) = w_2$. Therefore, since both bundles $S_0(X)$ and $\lambda_c^{1/2}$ exist locally, the computation above shows that their tensor product locally represent the $Spin^{\mathbb{C}}$-vector bundle having P_{U_1} bundle with Chern class $c_1(P_{U_1}) = \alpha$. □

Corollary 1.7. *The space of $Spin^c$-structures on a manifold X is*

$$Spin^c(X) = \{\alpha + c \in H^1(X,\mathbb{Z}_2)\otimes H^2(X,\mathbb{Z}) \mid w_2(X) = c \ mod \ 2\}$$

1.6 Cobordant $Spin^c$-Structures

A *Spin*-structure on an orientable 4-manifold X with boundary $Y = \partial X$ induces a Spin-structure on the boundary Y. In order to see this, we fix a local frame $\beta_4 = \{e_1, e_2, e_3, e_4\}$ on X, such that the frame $\beta_3 = \{e_1, e_2, e_3\}$ defines an orientation on Y and e_4 is orthogonal to Y. Thus, the Clifford Algebra isomorphism $Cl_3 \simeq Cl_4^0$ is given by $e_i \to e_1.e_4$, $i = 1, 2, 3$. We note that all 3-manifolds are spin.

The n-manifolds X_1 and X_2 are said to be cobordant iff there exists a (n+1)-manifold W such that $\partial W = X_1 \cup X_2$. Cobordance defines an equivalent relation and so classes $[X] = \{X' \mid X'$ is cobordant to $X\}$. The set $\Omega_n = \{[X] \mid X$ a n-manifold$\}$ is a abelian group under the operation defined by connected sum. Taking in account a spin sctructure s on X, we can define the class $[(X,s)]$ to be set of spin n-manifolds (X', s') such that X, X' are cobordant ($X\cup X' = \partial W$) and s, s' are also cobordants, where s is cobordant to s' iff there exits a spin structure S on W such that $s = S\mid_X$ and $s' = S\mid_{X'}$. Analogously, there is the abelian group $\Omega_n^{spin} = \{[(X,s)] \mid X$ a n-manifold, s a spin structure on $X\}$. The table below shows the structure of the Cobordism groups;

n	Ω_n^{spin}	Ω_n^{so}	$\Omega_n^{spin^c}$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$
1	$\mathbb{Z}_2$	0	$\mathbb{Z}$
2	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}$
3	0	0	$\mathbb{Z}$
4	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$

Capítulo 2

SW-Equations

2.1 Quadratic form

Let V be a complex vector space, V^* its dual and $End(V) = \{T : V \to V \mid T$ is $\mathbb{C}$-linear. By fixing a frame $\beta = \{e_1, \ldots, e_n\}$ in V and the corresponding co-frame $\beta^* = \{e^1, \ldots, e^n\}$ in V^*, there is the isomorphism $(.)^* : V \to V^*$ defined as the anti-linear extension of the map $(e_i)^* = e^i$; i.e,

$$(\sum_k v^k e_k)^* = \sum_k \overline{v}^k e^k.$$

Proposition 2.1.

$$End(V) = V \otimes V^*$$

Demonstração. Let $\beta = \{e_1, \ldots, e_n\}$ be a frame of V and $\beta^* = \{e^1, \ldots, e^n\}$ the co-frame associated to β. Let $T \in End(V)$ and $v \in V$;

$$T(v) = \sum_k v^k T(e_k) = \sum_k T(e_k) e^k(v) = \left(\sum_k T(e_k) e^k \right)(v).$$

In this way, $T = \sum_k T(e_k) e^k$ motivates the definition of $B : V \otimes V^* \to End(V)$ by

$$B(v \otimes w^*) = \sum_k \sum_l v^k \overline{w}^l e_k \otimes e^l, \quad w^* = \sum_l \overline{w}_l e^l.$$

Since $(e_k \otimes e^l)(u) = u^l e_k$, its associated matrix is $E_{kl} = (\delta_{kl})$. Hence, the explicit formula of the isomorphism $B : V \otimes V^* \to End(V)$, written in terms of the basis $\{E_{kl} \mid 1 \leq k, l \leq dim_{\mathbb{C}}(V)\}$ of V, is

$$B(v \otimes w^*) = \sum_{k,l} v^k \overline{w}^l E_{kl}. \tag{2.1}$$

$\square$

Thus, the isomorphism $B : V \otimes V^* \to End(V)$ is a hermitian bilinear form, which quadratic form is

$$q_B(v) = v \otimes v^* = \sum_{k,l} v^k \overline{v}^l E_{kl}.$$

For the purposes of defining later the $\mathcal{SW}$-equations, let $End_0(V) = \{T \in End(V) \mid tr(T) = 0\}$ and $\sigma : V \to End^0(V)$ defined by

$$\sigma(v) = v \otimes v^* - \frac{\mid v \mid^2}{2} I = \begin{pmatrix} \frac{|\phi_1|^2 - |\phi_2|^2}{2} & \phi_1\overline{\phi}_2 \\ \overline{\phi}_1\phi_2 & \frac{|\phi_2|^2 - |\phi_2|^2}{2} \end{pmatrix}. \tag{2.2}$$

The bilinear form associated to σ is $\sigma : V \times V \to End^0(V)$,

$$\sigma(v, w) = \frac{1}{2}\left\{v \otimes w^* + w \otimes v^* - Re\{< v, w >\}I\right\}. \tag{2.3}$$

It follows from the defintion that

$$\sigma(v + w) = \sigma(v) + \sigma(w) + 2\sigma(v, w). \tag{2.4}$$

Proposition 2.2. *If* $T \in \mathfrak{su}_2 = \{A \in M_2(\mathbb{C}) \mid A^* = -A,\ tr(A) = 0\}$ *and* $v, w \in V$*, then*

1. $< T(v), v >= 2 < \sigma(v), T >$.
2. $iIm\{< T(v), w >\} =< \sigma(v, w), T >$.
3. $\sigma(v).v = -\frac{|v|^2}{2} v$.
4. $\sigma(T(v), w) + \sigma(v, T(w)) = -Re(< v, w >)T$.
5. $\sigma(v, w) = 0$ *iff* $w = i\lambda v$ *for some* $\lambda \in \mathbb{R}$.

Demonstração. .

1. $< T(v), v >= 2 < \sigma(v), T >$;
   (a) $< T(v), v >= \sum_{i,j} v_i \overline{v}_j t_{ji}$.
   (b) $< \sigma(v), T >$;
   Since $< \sigma(v), T >= \frac{1}{2} tr\left[\sigma(v)^*.T\right]$, let's compute $\sigma(v)^*.T$;

$$\sigma(v) = \sum_{i,j} v_i \overline{v}_j E_{ij} - \sum_i \frac{\mid v \mid^2}{2} E_{ii} \quad \Rightarrow \quad \sigma(v)^* = \sum_{i,j} \overline{v}_i v_j E_{ji} - \sum_i \frac{\mid v \mid^2}{2} E_{ii}.$$

$$\sigma(v)^*.T = \left\{\sum_{i,j}\overline{v}_i v_j E_{ji} - \sum_i \frac{|\ v\ |^2}{2}E_{ii}\right\}.\left\{\sum_{k,l} t_{kl}E_{kl}\right\} =$$
$$= \sum_{i,j}\sum_{k,l}\overline{v}_i v_j t_{kl} E_{ji}E_{kl} - \sum_i\sum_{k,l} t_{kl}\frac{|\ v\ |^2}{2}E_{ii}E_{kl} =$$
$$= \sum_{i,j}\sum_{k,l}\overline{v}_i v_j t_{kl} E_{jl}\delta_{ik} - \sum_i\sum_{k,l} t_{kl}\frac{|\ v\ |^2}{2}E_{il}\delta_{ik} =$$
$$= \sum_{i,j}\sum_{l}\overline{v}_i v_j t_{il} E_{jl} - \sum_i\sum_{l} t_{il}\frac{|\ v\ |^2}{2}E_{il}$$

So,

$$tr\left(\sigma(v)^*.T\right) = \sum_{i,j}\sum_{l}\overline{v}_i v_j t_{il}\delta_{jl} - \sum_i\sum_{l} t_{il}\frac{|\ v\ |^2}{2}\delta_{il} =$$
$$= \sum_{i,j}\overline{v}_i v_j t_{ij} - \sum_i t_{ii}\frac{|\ v\ |^2}{2} = \sum_{i,j}\overline{v}_i v_j t_{ij} - \frac{|\ v\ |^2}{2}.tr(T).$$

Once $T \in \mathfrak{su}_2$, $tr(T) = 0$, therefore the expressions for $< T(v), v >$ and $2 < \sigma(v), T >$ are equal.

2. From the identities

$$< T(v+w), v+w > = < T(v), v > + < T(w), w > + 2iIm\{< T(v), w >\},$$
$$\sigma(v+w) = \sigma(v) + \sigma(w) + v\otimes w^* + w\otimes v^* - Re\{< v, w >\},$$

it follows that $iIm\{< T(v), w >\} = < \sigma(v, w), T >$.

3. the other itens are proved by straight computations using the explicit isomorphism in 2.1.

$\square$

2.2 The Quadratic Forms $\sigma_3 : \mathbb{C}^2 \to \Lambda^1\mathbb{R}^3$ and $\sigma_4 : \mathbb{C}^2 \to \Lambda^2_+\mathbb{R}^4$

From the classification of Clifford Algebras, we know that $Cl_3 = \mathbb{H}\oplus\mathbb{H}$ and, because $\mathbb{H}\otimes\mathbb{C} = M_2(\mathbb{C})$, it follows that $\mathbb{C}l_3 = M(2,\mathbb{C})\oplus M(2,\mathbb{C})$, where $\mathbb{C}l_3^{\pm} = M(2,\mathbb{C})$. Therefore, there are two inequivalent $\mathbb{C}l_3$ representations $\rho_\pm : \mathbb{C}l_3 \to M_2(\mathbb{C})$, each one characterized by the fact that $\rho_\pm(w) = \pm I$. The quadratic forms $\sigma_3 : \mathbb{C}^2 \to \Lambda^1\mathbb{R}^3$ and $\sigma_4 : \mathbb{C}^2 \to \Lambda^2_+\mathbb{R}^4$ are defined by describing explicitly the representation ρ_+, as shown in the follwing steps;

step 1: $Cl_3 \simeq \mathbb{H} \oplus \mathbb{H}$

In terms of its generators, $C_3^{\pm} \simeq \mathbb{H}$ is described as follows: let $\beta = \{e_1, e_2, e_3\}$ be a frame in $\mathbb{R}^3$, $\pi_+ : Cl_3 \to Cl_3^+$ and $w = e_1e_2e_3$;

$$\eta_1 = \pi_+(e_1) = \frac{e_1 - e_2e_3}{2}, \quad \eta_2 = \pi_+(e_2) = \frac{e_2 + e_1e_3}{2},$$
$$\eta_3 = \pi_+(e_3) = \frac{e_3 - e_1e_2}{2}$$

Thus, $< \frac{1+w}{2}, \eta_1, \eta_2, \eta_3 >$ is a basis for Cl_3^+. Due to the relations

$$\eta_1\eta_2 = -\eta_3, \quad \eta_2\eta_3 = -\eta_1, \quad \eta_3\eta_1 = -\eta_2,$$

the identification $\eta_1 \mapsto i$, $\eta_2 \mapsto j$ and $\eta_3 \mapsto -k$ is performed and extended linearly to define the isomorphism $Cl_3^+ \simeq \mathbb{H}$. The volume form $w_{\mathbb{C}}$ in the complex case of $\mathbb{C}l_3$ satisfies the identity $w_{\mathbb{C}} = -w$, as shown in equation 2. Therefore, the set $\{\frac{1-w}{2}, \zeta_1, \zeta_2, \zeta_3\}$ is a basis of $\mathbb{C}l_3^+$, where

$$\zeta_1 = \frac{e_1 + e_2e_3}{2}, \quad \zeta_2 = \frac{e_2 - e_1e_3}{2}, \quad \zeta_3 = \frac{e_3 + e_1e_2}{2},$$
$$\zeta_1\zeta_2 = -\zeta_3, \quad \zeta_2\zeta_3 = -\zeta_1, \quad \zeta_3\zeta_1 = -\zeta_2.$$

step 2: $\mathbb{H} \hookrightarrow M(2, \mathbb{C})$

There is the mononorphism

$$a + bj \to \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

sitting the quaternions within $M(2, \mathbb{C})$.

step 3: $\mathbb{C}l_3 = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$

$$Cl_3 \otimes \mathbb{C} \simeq (\mathbb{H} \oplus \mathbb{H}) \otimes \mathbb{C} = (\mathbb{H} \otimes \mathbb{C}) \oplus (\mathbb{H} \otimes \mathbb{C}),$$

and $\mathbb{H} \otimes \mathbb{C} = M_2(\mathbb{C})$. The generatros of $\mathbb{C}l_3^+$ are

$$\frac{1 + w_{\mathbb{C}}}{2} \longmapsto 1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.1}$$

$$\zeta_1 \longmapsto i \longmapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{2.2}$$

$$\zeta_2 \longmapsto j \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.3}$$

$$\zeta_3 \longmapsto -k \longmapsto \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{2.4}$$

step 4: $\rho_+ : \mathbb{C}l_3 \to M_2(\mathbb{C})$:

Considering the vector space isomorphism $\mathbb{C}l_3 \simeq \Lambda^*\mathbb{R}^3 \otimes \mathbb{C}$, the representation $\rho_+ : \mathbb{C}l_3 \to M_2(\mathbb{C})$ induces $\rho_+ : \Lambda^*\mathbb{R}^3 \to M_2(\mathbb{C})$. In order to represent the former isomorphism, let $\beta^* = \{e^1, e^2, e^3\}$ be a co-frame of $(\mathbb{R}^3)^*$, and

$$\rho_+(e^1) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \rho_+(e^2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho_+(e^3) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Thus, once $\mathfrak{su}_2 = \{A \in M_2(\mathbb{C}) \mid A^* = -A,\ tr(A) = 0\}$, there is the vector space isomorphism

$$\rho_+ : (\mathbb{R}^3)^* \to \mathfrak{su}_2, \quad \rho_+(a_1e^1 + a_2e^2 + a_3e^3) = \begin{pmatrix} -ia_3 & a_2 + ia_1 \\ -a_2 + ia_1 & ia_3 \end{pmatrix}. \tag{2.5}$$

Since $End_0(\mathbb{C}^2) = \mathfrak{su}_2 \otimes \mathbb{C}$ as a $\mathbb{C}$-linear space, there is the natural extension $\rho_+ : \Lambda^1\mathbb{R}^3 \otimes \mathbb{C} \to End_0(\mathbb{C}^2)$. Its inverse is $\rho_+^{-1} : End_0(\mathbb{C}^2) \to \Lambda^1\mathbb{R}^3 \otimes \mathbb{C}$,

$$\rho_+^{-1}(\begin{pmatrix} \alpha & z \\ w & -\alpha \end{pmatrix}) = \frac{1}{2i}(z+w)e^1 + \frac{1}{2}(z-w)e^2 - \frac{1}{i}\alpha e^3. \tag{2.6}$$

Besides, ρ_+ is an isometry.

step 5: definition of σ_3

Definition 2.1. The 3-dimensional $\mathcal{SW}$-quadratic form $\sigma_3 : \mathbb{C}^2 \to \Lambda^1\mathbb{R}^3 \oplus \mathbb{C}$ is

$$\sigma_3(v) = \rho_+^{-1}(\sigma(v)) \tag{2.7}$$

Proposition 2.3.

$$\sigma_3(v) = -\frac{1}{2}\sum_{i=1}^{3} < e_i.v, v > e^i. \tag{2.8}$$

Demonstração. In equation 2.6, consider $v = (v_1, v_2) \in \mathbb{C}^2$, $z = v_1\overline{v}_2$, $w = \overline{z}$ and $\alpha = \mid v_1 \mid^2 - \mid v_2 \mid^2$, so

$$\rho_+^{-1}(\begin{pmatrix} \alpha & v_1\overline{v}_2 \\ v_2\overline{v}_1 & -\alpha \end{pmatrix}) = i\left(-Re(v_1\overline{v}_2)e^1 + Im(v_1\overline{v}_2)e^2 + \alpha e^3\right). \tag{2.9}$$

Therefore, the identity 2.12 follows from the identities below:

$$< e_1.v, v > = -2iIm(v_1\overline{v}_2), \quad < e_2.v, v >= 2iRe(v_1\overline{v}_2),$$
$$< e_3.v, v > = i(| v_1 |^2 - | v_2 |^2).$$

□

step 6: definition of σ_4

It was proved in **??** that $(\mathbb{C}l_4^0)^+ \simeq (< \frac{1+w}{2} > \oplus \Lambda_2^+\mathbb{R}^4) \otimes \mathbb{C}$. Moreover,

1. $\mathbb{C}l_4^0 \simeq \mathbb{C}l_3$ and so $(\mathbb{C}l_4^0)^+ \simeq \mathbb{C}l_3^+ \simeq M_2(\mathbb{C})$
2. $End_0(\mathbb{C}^2) = \Lambda^2_+\mathbb{R}^4 \otimes \mathbb{C}$.

   In order to explicit the isomorphism above, let $\beta = \{e_1, e_2, e_3\}$ be a frame in $\mathbb{R}^4$;

   (a) the isomorphism $f : \mathbb{C}l_3 \to \mathbb{C}l_4^0$ is given by $f(e_i) = e_ie_4$. Therefore $f(e_1e_2e_3) = e_1e_2e_3e_4$, $f(\mathbb{C}l_3^+) = (\mathbb{C}l_4^0)^+$ and

   $$\widetilde{\zeta}_1 = f(\zeta_1) = \frac{e_1e_4 + e_2e_3}{2}, \quad \widetilde{\zeta}_2 = f(\zeta_2) = \frac{e_2e_4 - e_1e_3}{2},$$
   $$\widetilde{\zeta}_3 = f(\zeta_3) = \frac{e_3e_4 + e_1e_2}{2},$$

   The volume in $\mathbb{C}l_4$ is $w = -e_1e_2e_3e_4$ and a basis for $(\mathbb{C}l_4^0)^+$ is given by $< \frac{1-w}{2}, \widetilde{\zeta}_1, \widetilde{\zeta}_2, \widetilde{\zeta}_3 >$.

   (b) Let $\beta^* = \{e^1, e^2, e^3, e^4\}$ be a co-frame in $(\mathbb{R}^4)^*$. The set $\{\widetilde{\zeta}_1, \widetilde{\zeta}_2, \widetilde{\zeta}_3\}$ form a basis for $\Lambda^2_+\mathbb{R}^4$.

3. All the isomorphisms described so far define, through the sequence below, the isomorphism $\delta : \Lambda^2_+\mathbb{R}^4 \otimes \mathbb{C} \to End_0(\mathbb{C}^2)$,

$$\left(< \frac{1-w}{2} > \oplus \Lambda^2_+\mathbb{R}^4\right) \otimes \mathbb{C} \simeq (\mathbb{C}l_4^0)^+ \simeq \mathbb{C}l_3^+ \overset{\rho_+}{\simeq} M_2(\mathbb{C}) \simeq < I > \oplus End_0(\mathbb{C}^2). \tag{2.10}$$

Definition 2.2. The 4-dimensional $\mathcal{SW}$-quadratic form $\sigma_4 : \mathbb{C}^2 \to \Lambda^2_+\mathbb{R}^4 \oplus \mathbb{C}$ is

$$\sigma_4(v) = \delta^{-1}(\sigma(v)) \tag{2.11}$$

Proposition 2.4.

$$\sigma_4(v) = -\frac{1}{2}\sum_{i=1}^{3} < \widetilde{\zeta}_i.v, v > \widetilde{\zeta}^i. \tag{2.12}$$

2.3 SW-Equations on a 3-Manifold Y

A $Spin^c$ structure on Y is a class $\mathfrak{s}(\alpha) = \mathfrak{s} + \alpha \in H^1(X, \mathbb{Z}_2) \oplus H^2(X, \mathbb{Z})$ such that $\alpha = 0 \ mod\ 2$. For each class $\mathfrak{s}(\alpha) \in Spin^{\mathbb{C}}(X)$ there is a principal bundle $P_{Spin^{\mathbb{C}}}(X)$ as in definition 1.15. Since $Spin_3^{\mathbb{C}} = S^3 \times_{\mathbb{Z}_2} S^1 = U_2$, for each class $\mathfrak{s}(\alpha)$ it is associated the following vector bundles;

1. the complex spinor bundle

$$S_{\mathfrak{s}(\alpha)} = P_{Spin_3^{\mathbb{C}}}(X) \times_{\triangle} \mathbb{C}^2,$$

   where $\triangle : Spin_3^{\mathbb{C}} \to U_2$ is induced by the isomorphism $\mathbb{C}l_3^0 = Cl_3^0 \otimes \mathbb{C} \simeq M_2(\mathbb{C})$.

2. The determinant line bundle

$$L_\alpha = P_{Spin^{\mathbb{C}}} \times_{det} \mathbb{C}$$

   where $det : U_2 \to \mathbb{C}$ and $c_1(L_\alpha) = \alpha$.

Definition 2.3. On Y, the configuration space for the Seiberg-Witten Theory is $\mathcal{C}_{\mathfrak{s}(\alpha)} = \mathcal{A}_\alpha \times \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$, where $\mathcal{A}_\alpha$ is the space of U_1-connections defined on L_α and $\Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$ is the space of sections of $S_{\mathfrak{s}(\alpha)}$. The Seiberg-Witten map is

$$\mathcal{F}_{\mathfrak{s}(\alpha)} : \mathcal{C}_{\mathfrak{s}(\alpha)} \to \Omega^1(X, i\mathbb{R}) \times \Gamma(S_{\mathfrak{s}(\alpha)}) \tag{2.1}$$
$$(A, \phi) \longmapsto (*F_A - \sigma_3(\phi), D_A(\phi)) \tag{2.2}$$

The abelian gauge group $\mathcal{G}_{\mathfrak{s}(\alpha)} = Map(Y, U_1)$ acts on $\mathcal{C}_{\mathfrak{s}(\alpha)}$ through the action

$$\mathcal{G}_{\mathfrak{s}(\alpha)} \times \mathcal{C}_{\mathfrak{s}(\alpha)} \to \mathcal{C}_{\mathfrak{s}(\alpha)}, \quad g.(A, \phi) = (A + 2g^{-1}dg, g^{-1}\phi).$$

The fixed points of the $\mathcal{G}_{\mathfrak{s}(\alpha)}$-action are $(A, 0)$, $\forall A \in \mathcal{A}_\alpha$, which isotropy groups $G_{(A,0)}$ are isomorphic to $U_1 = \{g \in \mathcal{G}_{\mathfrak{s}(\alpha)} \mid g(y) = const\}$. Thus, the space $\mathcal{B}_{\mathfrak{s}(\alpha)} = \mathcal{C}_{\mathfrak{s}(\alpha)}/\mathcal{G}_{\mathfrak{s}(\alpha)}$ is a singular space. Instead, if the action is restricted to the free action of the group $\mathcal{G}^0_{\mathfrak{s}(\alpha)} = \{g \in \mathcal{G}_{\mathfrak{s}(\alpha)} \mid g(y_0) = I\}$, then the orbit space $\mathcal{B}^0_{\mathfrak{s}(\alpha)} = \mathcal{C}_{\mathfrak{s}(\alpha)}/\mathcal{G}^0_{\mathfrak{s}(\alpha)}$ is a manifold. The group $\mathcal{G}^0_{\mathfrak{s}(\alpha)}$ fits into the short exact sequence

$$1 \longrightarrow \mathcal{G}^0_{\mathfrak{s}(\alpha)} \longrightarrow \mathcal{G}_{\mathfrak{s}(\alpha)} \xrightarrow{e_0} U_1 \longrightarrow 1, \quad e_0(g) = g(y_0).$$

So far, due to the action, there are two categories of points to be considered in $\mathcal{C}_{\mathfrak{s}(\alpha)}$: (1) the reducibles $(A, 0) \in \mathcal{C}_{\mathfrak{s}(\alpha)}$ such that $G_{(A,0)} \simeq U_1$ and (2) the irreducibles $(A, \phi) \in \mathcal{C}_{\mathfrak{s}(\alpha)}$ such that $G_{(A,\phi)} = \{I\}$. Thus, an important space to be considered is the space of irreducibles

$$\mathcal{C}^*_{\mathfrak{s}(\alpha)} = \{(A,\phi) \in \mathcal{C}_{\mathfrak{s}(\alpha)} \mid G_{(A,\phi)} = \{I\}\},\ \mathcal{B}^*_{\mathfrak{s}(\alpha)}/\mathcal{G}_{\mathfrak{s}(\alpha)}.$$

In fact, the projection $\mathcal{C}^*_{\mathfrak{s}(\alpha)} \to \mathcal{B}^*_{\mathfrak{s}(\alpha)}$ defines an universal $\mathcal{G}_{\mathfrak{s}(\alpha)}$-principal bundle because the $\mathcal{G}_{\mathfrak{s}(\alpha)}$-action is free on the contractible space $\mathcal{C}^*_{\mathfrak{s}(\alpha)}$ (it is homotopic to the space $\Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})\backslash\{0\}$, and the former space has the homotopy type of S^∞). Also, there is the principal U_1-bundle $\mathfrak{b} : \mathcal{B}^0_{\mathfrak{s}(\alpha)} \to \mathcal{B}^*_{\mathfrak{s}(\alpha)}$.

The map $\mathcal{F}_{\mathfrak{s}(\alpha)}$ is $\mathcal{G}_{\mathfrak{s}(\alpha)}$-equivariant because

$$*F_{g(A)} - \sigma_3(g^{-1}\phi) = *F_A - \sigma_3(\phi) \quad \text{and} \quad D_{g(A)}(g^{-1}\phi) = g^{-1}D_A\phi.$$

Hence, $\mathcal{F}_{\mathfrak{s}(\alpha)}(g.(A,\phi)) = g.\mathcal{F}_{\mathfrak{s}(\alpha)}$. In this way, the Seiberg-Witten map defines a section $\mathcal{F}_{\mathfrak{s}(\alpha)} : \mathcal{B}^*_{\mathfrak{s}(\alpha)} \to \mathcal{E}_{\mathfrak{s}(\alpha)}$ of the associated vector bundle

$$\mathcal{E}_{\mathfrak{s}(\alpha)} = \mathcal{C}^*_{\mathfrak{s}(\alpha)} \times_{\mathcal{G}_{\mathfrak{s}(\alpha)}} \left(\Omega^1(Y, i\mathbb{R}) \times \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})\right).$$

By analogy with the finite dimensional case, the euler-class of $\mathcal{E}_{\mathfrak{s}(\alpha)}$ can be measured by the intersection number of $\mathcal{F}^{-1}_{\mathfrak{s}(\alpha)}(0)$ with the 0-section. However, it is not at all clear how one can define in general the euler class of an infinite dimensional vector bundle.

Definition 2.4. The Seiberg-Witten equation are

$$\begin{cases} *F_A = \sigma_3(\phi), \\ D_A\phi = 0. \end{cases} \quad \Leftrightarrow \quad \mathcal{F}_{\mathfrak{s}(\alpha)}(A,\phi) = 0 \tag{2.3}$$

The structure of the space of solutions to the $\mathcal{SW}$-equations is the main issue in this notes. As observed before, these equations are $\mathcal{G}_{\mathfrak{s}(\alpha)}$-invariant. A technical point to be overcome is the existence of reducible solutions. The reducible solutions are all of type $(A,0)$, otherwise ($\phi \neq 0$) they are irreducible. Note that if $(A,0)$ is a reducible solution, then $F_A = 0$ and the bundle $\mathcal{L}_\alpha$ is trivial since its 1^{st}-Chern class is $c_1(\mathcal{L}_\alpha) = \frac{1}{2\pi i}F_A$.

Proposition 2.5. *The space of reducible solutions is diffeomorphic to the Jacobian Torus*

$$T^{b_1(Y)} = \frac{H^1(Y,\mathbb{R})}{H^1(Y,\mathbb{Z})} \in \mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha). \tag{2.4}$$

Demonstração. Let $(A,0)$ and $(B,0)$ be reducible solutions. Consider $b \in \Omega^1(Y, i\mathbb{R})$ such that $B = A + b$. A, B being flat connections imply that $db = 0$ and so $b \in H^1(Y,\mathbb{R})$. If A and B are gauge equivalent, then $b = 2g^{-1}dg$ and $b \in H^1(Y,\mathbb{Z})$ because $b([\gamma]) = \int_\gamma g^{-1}dg \in \mathbb{Z}$, for all $[\gamma] \in H_1(Y,\mathbb{R})$. So, the map $[(a,0)] \to [a] \in T^{b_1(Y)}$ defines the diffeomorphism. □

Definition 2.5. The $\mathcal{SW}_{\mathfrak{s}(\alpha)}$-monopole space associated to the $Spin^{\mathbb{C}}$ structure $\mathfrak{s}(\alpha) \in Spin^{\mathbb{C}}(Y)$ is the space

$$\mathcal{M}_{\mathfrak{s}(\alpha)} = \{(A,\phi) \in \mathcal{C}_{\mathfrak{s}(\alpha)} \mid \mathcal{F}_{\mathfrak{s}(\alpha)}(A,\phi) = 0\}/\mathcal{G}_{\mathfrak{s}(\alpha)}.$$

As it well be clear along the notes, is the amazingly rich topological structure of $\mathcal{M}_{\mathfrak{s}(\alpha)}$ which allows the many applications. The backbone of all this is the following estimates;

Lemma 2.1. *Let $k_g : Y \to \mathbb{R}$ be the scalar curvature function of (Y,g) and $k_g = \max_{y\in Y} k_g(y)$. If $(A,\phi) \in \mathcal{M}_{\mathfrak{s}(\alpha)}$, then*

$$\| \phi \|_0 \leq \max\{0, -k_g\} \tag{2.5}$$

Proposition 2.6. *The irreducible solutions exist only for a finite number of classes $\mathfrak{s}(\alpha) \in Spin^{\mathbb{C}}(X)$.*

Demonstração. Let (A,ϕ) be an irreducible monopole solution, so the norm of $\frac{1}{2\pi i}F_A$ is bounded in $H^2(Y,\mathbb{R})$ and, consequently, $\alpha = \frac{1}{2\pi i}F_A$ lies inside of the finite set $H^1(Y,\mathbb{Z}) \cap H^1(Y,\mathbb{R})$. Hence, there exists irreducible monopole solutions only for a finite number of $\mathfrak{s}(\alpha) \in Spin^{\mathbb{C}}(X)$. $\square$

One of the very surprising and useful fact about the topological structure of $\mathcal{M}_{\mathfrak{s}(\alpha)}$ is its compactness;

Theorem 2.1. *$\mathcal{M}_{\mathfrak{s}(\alpha)}$ is a compact set.*

Originally, the $\mathcal{SW}$-equations were not of calculus of variation nature. It is remarkable that there exists a variational formulation for them, as it is shown next. Before procceding to a variational setting, let's stabilish the Sobolev structure on the spaces $\mathcal{C}_{\mathfrak{s}(\alpha)}$ and $\mathcal{G}_{\mathfrak{s}(\alpha)}$. First of all, by fixing a connection $A_0 \in \mathcal{A}_\alpha$, the space $\mathcal{A}_\alpha$ becames a vector space isomorphic to $\Omega^1(ad(\mathfrak{u}_1)) = \Omega^1(Y, i\mathbb{R})$ and inheritages the metric structure;

1. $\mathcal{A}_\alpha = L^{1,2}(\mathcal{A}_\alpha)$, $\Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}) = L^{1,2}(\Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}))$ and $\mathcal{C}_{\mathfrak{s}(\alpha)} = \mathcal{A}_\alpha \times \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$. The metric on $\mathcal{A}_\alpha$ is induced by the inner product on $\Omega^1(Y,\mathbb{R})$. The inner product on $\Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$ is defined by using the hermitian form on $\Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$ and integrating its real part;

$$< \phi, \psi >= \int_Y \mathfrak{Re}(< \phi, \psi >). \tag{2.6}$$

2. $\mathcal{G}_{\mathfrak{s}(\alpha)} = L^{2,2}(\mathcal{G}_{\mathfrak{s}(\alpha)})$.

3. the tangent space of $\mathcal{C}_{\mathfrak{s}(\alpha)}$ at (A,ϕ) is

$$T_{(A,\phi)}\mathcal{C}_{\mathfrak{s}(\alpha)} = \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}). \tag{2.7}$$

4. For the purpose of deducting the $\mathcal{SW}$-equations as the Euler-Lagrange equations of a functional, the Hilbert structure to be considered on the bundle $T\mathcal{C}_{\mathfrak{s}(\alpha)}$ is

$$< \theta + V, \Lambda + W >= \int_Y < \theta, \Lambda > dv_g + \int_Y \mathfrak{Re}(< V, W >) dv_g.$$

Remark 1. The Hodge star operator $\hat{*} : \Omega^p(Y, i\mathbb{R}) \to \Omega^{3-p}(Y, i\mathbb{R})$ satisfies the following properties;

1. Let $* : \Omega^p(Y, \mathbb{R}) \to \Omega^{3-p}(Y, \mathbb{R})$ be the usual Hodge star operator and $\hat{*} = -* : \Omega^p(Y, i\mathbb{R}) \to \Omega^{3-p}(Y, i\mathbb{R})$. Consider $\widetilde{\omega} = i\omega,\ \widetilde{\eta} = i\eta \in \Omega^1(Y, i\mathbb{R}) = \Omega^1(Y) \otimes i$, so

$$\widetilde{\omega} \wedge \hat{*}\widetilde{\eta} = (i\omega) \wedge \hat{*}(i\eta) = \omega \wedge (-\hat{*}\eta) = \omega \wedge *\eta =< \omega, \eta > dv_g. \tag{2.8}$$

2. Besides, from the computation above, it follows that

$$\widetilde{\omega} \wedge \hat{*}\widetilde{\eta} =< \widetilde{\omega}, \widetilde{\eta} > dv_g. \tag{2.9}$$

3. $\hat{*}^2 = *^2 = (-1)^{p(3-p)}$.

4. The adjoint operator $d^* : \Omega^p(Y, i\mathbb{R}) \to \Omega^{p-1}(Y, i\mathbb{R})$ associated to the exterior derivative $d : \Omega^{p-1}(Y, i\mathbb{R}) \to \Omega^p(Y, i\mathbb{R})$, by the riemannian metric g on Y, is

$$d^* = (-1)^{3p} \hat{*} d \hat{*} = (-1)^{3p} * d * . \tag{2.10}$$

Definition 2.6. The Chern-Simons-Dirac functional $\Upsilon : \mathcal{C}_{\mathfrak{s}(\alpha)} \to \mathbb{R}$ is

$$\Upsilon(A, \phi) = \int_Y \left\{ \frac{1}{2}(A - A_0) \wedge (F_A + F_0) + < D_A\phi, \phi > \right\} dx,$$

where $A_0 \in \mathcal{A}_\alpha$ is a fixed connection and $F_0 = F_{A_0}$.

In case we consider another fixed connection A_1, the difference among the functionals is a constant term, and so the fixed connection is irrelevant for the theory.

Before going further to obtain the Euler-Lagrange equations of the functional Υ, let's prove an identity which is important to perform many computations;

Lemma 2.2. *The L^2-adjoint of the linear operator $T_\phi : \Omega^1(Y, i\mathbb{R}) \to \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$, $T_\phi(\theta) = \frac{1}{2}\theta \bullet \phi$, is $T^*_\phi : \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}) \to \Omega^1(Y, i\mathbb{R})$, where*

$$T^*_\phi(W) = \sigma(\phi, W). \tag{2.11}$$

Demonstração. The prove is divided into two steps which main issue is to prove the identity

$$\int_Y Re\left(< \frac{1}{2}\theta \bullet \phi, W >\right) dv = \int_Y < \theta, \sigma(\phi, W) > dv.$$

step 1: $\int_Y Re\left(< \frac{1}{2}\theta \bullet \phi, \phi >\right) dv = \int_Y < \theta, \sigma(\phi) > dv.$

Applying the identity 2.28, it follows that

$$i\int_Y Im\left(< \frac{1}{2}\theta \bullet \phi, W >\right) dv = \int_Y < \sigma(\phi), \theta > dv.$$

By $\mathbb{C}$-linear extending to an operator $T_\phi : \Omega^1(Y, i\mathbb{R}) \to \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$, $T_\phi(i\theta) = i\theta$, the identity above becames

$$\begin{aligned}\int_Y < \sigma(\phi), \theta > dv &= i\int_Y Im\left(-i < \frac{1}{2}i\theta \bullet \phi, \phi >\right) dv = \\ &= -i\int_Y Re\left(< \frac{1}{2}i\theta \bullet \phi, \phi >\right) dv.\end{aligned}$$

Hence,

$$\int_Y < \sigma(\phi), i\theta > dv = \int_Y Re\left(< \frac{1}{2}i\theta \bullet \phi, \phi >\right) dv.$$

step 2: By the 1^{st}-step,

$$\int_Y < \sigma(\phi + W), i\theta > dv = \int_Y Re\left(< \frac{1}{2}i\theta \bullet (\phi + W), \phi + W >\right) dv.$$

Therefore,

$$\int_Y < \sigma(\phi, W), i\theta > dv = \int_Y Re\left(< \frac{1}{2}i\theta \bullet \phi, W >\right) dv.$$

Hence, $T^*_\phi(W) = \sigma(\phi, W)$.

□

Proposition 2.7. *The L^2-gradient of Υ is*

$$\nabla\Upsilon(A, \phi) = (- * F_A + \sigma_3(\phi), D_A\phi). \tag{2.12}$$

Demonstração. First of all, let's observe that for $A \in \mathcal{A}_\alpha$ and $\Theta = i\theta \in \Omega^1(Y, i\mathbb{R})$,

1. $F_{A+t\Theta} = F_A + td\Theta$;
2. $D_{A+t\Theta}\phi = \sum_i e_i \bullet \nabla_i^{A+t\Theta}\phi = \sum_i e_i \bullet \{\nabla_i^A \phi + t\Theta(e_i).\phi\} = D_A\phi + \frac{t}{2}\Theta \bullet \phi$. (the factor 1/2 in the last expression is due to the Clifford multiplication, [5] pg 42, lemma 3.3.2)

The total derivative of Υ is $d\Upsilon = (\partial_A \Upsilon)dA + (\partial_\phi \Upsilon)d\phi$, where :

1. $\partial_A \Upsilon(A, \phi) = \int_X \{< - * F_A + \sigma_3(\phi), \Theta >\} \, dv_g$.

$$
\begin{aligned}
\partial_A \Upsilon(A, \phi) &= \lim_{t \to 0} \frac{1}{t} \left\{ \int_Y \left[\frac{1}{2} F_{A+t\Theta} + F_0) \wedge (A - A_0 + t\Theta) + < D_{A+t\Theta}\phi, \phi > \right] dv_g - \Upsilon(A, \phi) \right\} = \\
&= \int_Y \{(d\Theta \wedge (A - A_0) + (F_A + F_0) \wedge \Theta\} \, dv_g + \frac{1}{2} \int_Y \left\{ < \frac{1}{2} \Theta \bullet \phi, \phi > \right\} dv_g = \\
&= \int_Y \{F_A \wedge \Theta + < \sigma_3(\phi), \Theta >\} \, dv_g = \int_Y \{- < *F_A, \Theta > + < \sigma_3(\phi), \Theta >\} \, dv_g = \\
&= \int_Y \{< - * F_A + \sigma_3(\phi), \Theta >\} \, dv_g.
\end{aligned}
$$

2. $\partial_\phi \Upsilon(A, \phi) = \int_X \mathfrak{Re}(< D_A\phi, V >) dv_g$.

$$
\begin{aligned}
\partial_\phi \Upsilon(A, \phi) &= \lim_{t \to 0} \frac{1}{2t} \int_Y \{(F_A + F_0) \wedge (A - A_0) + < D_A(\phi + tV), \phi + tV >\} \, dv_g = \\
&= \frac{1}{2} \int_Y \{< D_A\phi, V > + < D_A V, \phi >\} \, dv_g = \int_Y \mathfrak{Re}(< D_A\phi, V >) dv_g
\end{aligned}
$$

Therefore,

$$
\nabla \Upsilon(A, \phi) = (- * F_A + \sigma_3(\phi), D_A\phi).
$$

□

Remark 2. Once $\mathcal{F}_{\mathfrak{s}(\alpha)} = grad(\Upsilon) : \mathcal{C}_{\mathfrak{s}(\alpha)} \to T\mathcal{C}_{\mathfrak{s}(\alpha)}$ is a $\mathcal{G}_{\mathfrak{s}(\alpha)}$-equivariant section, it induces the section $\mathcal{F}_{\mathfrak{s}(\alpha)} : \mathcal{B}_{\mathfrak{s}(\alpha)} \to T\mathcal{B}_{\mathfrak{s}(\alpha)}$, $T\mathcal{B}^*_{\mathfrak{s}(\alpha)} = \mathcal{E}_{\mathfrak{s}(\alpha)}$.

Although the $\mathcal{SW}$-equations are $\mathcal{G}_{\mathfrak{s}(\alpha)}$-invariant, the functional Υ is not, as shown next;

Proposition 2.8. *Let $\alpha = c_1(\mathcal{L}_\alpha)$, $g \in \mathcal{G}_{\mathfrak{s}(\alpha)}$ and $H^1(U_1, \mathbb{Z}) =< \mu >$. Thus,*

$$
\Upsilon(g.(A, \phi)) = \Upsilon(A, \phi) - 8\pi^2 \{g^*(\mu) \cup \alpha\} ([Y]),
$$

where $g^* : H^1(U_1, \mathbb{Z}) \to H^1(Y, \mathbb{Z})$.

Demonstração. Since $[F_A] = [F_0] = 2\pi i c_1(\mathcal{L}_\alpha)$, the computations yield

$$\Upsilon(g.(A,\phi)) - \Upsilon(A,\phi) = \frac{1}{2}\int_Y g^{-1}dg \wedge (F_A + F_0) = -8\pi^2 \left\{g^*(\mu) \cup c_1(\mathcal{L}_\alpha)\right\}([Y]),$$

where $\mu = \frac{1}{2\pi}d\theta$. From $g(y) = e^{\theta(y)}$ we get $dg = ie^{i\theta}d\theta$ and so $-ig^{-1}dg = d\theta$. Finally, $\frac{1}{2\pi i}g^{-1}dg = \frac{1}{2\pi}d\theta$.

□

From the last result, it follows that the functional Υ is not gauge invariant. In case we fix the identity component $\mathcal{G}^0_{\mathfrak{s}(\alpha)} \subset \mathcal{G}_{\mathfrak{s}(\alpha)}$, then Υ becames $\mathcal{G}^0_{\mathfrak{s}(\alpha)}$-invariant because, for all $g \in \mathcal{G}^0_{\mathfrak{s}(\alpha)}$, $g^*\mu = 0$.

Definition 2.7.

$$d(\mathfrak{s}(\alpha)) = g.c.d\{< c_1(\mathfrak{s}(\alpha)), \tau >\},\ \forall \tau \in H^2(Y,\mathbb{Z})/torsion.$$

In fact, Υ descends to a map $\mathcal{B}_{\mathfrak{s}(\alpha)} \to \mathbb{R}/d(\mathfrak{s}(\alpha))$.

2.3.1 Slice for $\mathcal{B}_{\mathfrak{s}(\alpha)} = \mathcal{C}_{\mathfrak{s}(\alpha)}/\mathcal{G}_{\mathfrak{s}(\alpha)}$

The tangent space to the $\mathcal{G}_{\mathfrak{s}(\alpha)}$ action at (A,ϕ) is $T_{(A,\phi)}\left[\mathcal{G}_{\mathfrak{s}(\alpha)}.(A,\phi)\right]$. In order to describe it, let $g_t : (-\epsilon,\epsilon) \to \mathcal{G}_{\mathfrak{s}(\alpha)}$ be a curve such that $g(0) = I$ and $g'(0) = f$ (recall that $\mathcal{G}_{\mathfrak{s}(\alpha)}$ is a Lie group and its Lie algebra is $\mathfrak{g} = Map(Y, i\mathbb{R}) = \Omega^0(Y, i\mathbb{R})$). So,

$$\begin{aligned} g_t.(A,\phi) &= (A + 2g_t^{-1}dg_t, g_t^{-1}\phi) \\ \frac{d}{dt}(g_t.(A,\phi))\mid_{t=0} &= (2df, -f.\phi) \in \Omega^1(X, i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}). \end{aligned}$$

Let $G_{(A,\phi)} : \Omega^0(Y, i\mathbb{R}) \to \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$ be the map

$$G_{(A,\phi)}(f) = (2df, -f.\phi), \quad Imag(G_{(A,\phi)}) = T_{(A,\phi)}\mathcal{G}_{\mathfrak{s}(\alpha)}.(A,\phi). \tag{2.13}$$

Once

$$\Omega^1(Y, i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}) = Imag(G_{(A,\phi)}) \oplus ker(G^*_{(A,\phi)}),$$

the slice is locally described by $Ker(G^*_{(A,\phi)})$. First of all, recall that $\Omega^1(X, i\mathbb{R}) = Imag(d) \oplus ker(d^*)$. Next, we consider the map $t_\phi : \Omega^0(Y, i\mathbb{R}) \to \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$, $t_\phi(f) = f.\phi$ and its dual map $t^*_\phi : \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}) \to \Omega^0(Y, i\mathbb{R})$;

$$\begin{aligned}< t_\phi(f), V > &= \int_Y < t_\phi(f)(y), V(y) > dv_g = \int_Y f(y)[-\overline{< \phi(y), V(y) >}]dv_g = \\ &=< f, - < \phi, V >> .\end{aligned}$$

Thus, $t_\phi^*(V) = - < \phi, V >$ is a map $t_\phi^* : \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}) \to \Omega^0(Y, \mathbb{C})$, which projection on $\Omega^0(Y, i\mathbb{R})$ gives the desired map $t_\phi^*(V) = -iIm(< \phi, V >)$. Finally,

$$G^*_{(A,\phi)}(\Theta, V) = 2d^*\theta + iIm(< \phi, V >) \tag{2.14}$$

and so $(T_{g.(A,\phi)})^\perp = Ker(G^*_{(A,\phi)}) = Ker(d^*) \oplus Ker(iIm(< \phi, . >))$. If $\phi = 0$, then $(T_{g.(A,0)})^\perp = Ker(G^*_{(A,0)}) = Ker(d^*) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$.

The derivative $dg_{(A,\phi)} : T_{(A,\phi)}\mathcal{C}_{\mathfrak{s}(\alpha)} \to T_{g.(A,\phi)}\mathcal{C}_{\mathfrak{s}(\alpha)}$ of the map $g : \mathcal{C}_{\mathfrak{s}(\alpha)} \to \mathcal{C}_{\mathfrak{s}(\alpha)}$ at (A, ϕ) induces a map $dg_{(A,\phi)} : Ker(G^*_{(A,\phi)}) \to Ker(G^*_{g.(A,\phi)})$, where (i) if $\phi \neq 0$, then $dg_{(A,\phi)}.(\theta, V) = (\theta, g^{-1}V)$, (ii) if $\phi = 0$, then $dg_{(A,\phi)}.(\theta, V) = (\theta, V)$.

In order to obtain a local chart for $\mathcal{B}_{\mathfrak{s}(\alpha)}$, consider the C^∞-map

$$\Phi_{(A,\phi)} : Ker(G^*_{(A,\phi)}) \times \mathcal{G}_{\mathfrak{s}(\alpha)} \to \mathcal{C}_{\mathfrak{s}(\alpha)}, \tag{2.15}$$

$$((\theta, V), g) \longmapsto g.(\theta, V) = (\theta + 2g^{-1}dg, g^{-1}V). \tag{2.16}$$

and the following two cases:

1. $\phi \neq 0$;

   The derivative at (A, ϕ) is

   $$d(\Phi_{(A,\phi)})_{((\theta,V),I)}((\omega, W), f) = (\omega + 2df, -fV + W),\ d^*w = 0,\ W \in V^\perp.$$

   By construction, if (A, ϕ) is an irreducible point, then $d\Phi_{(A,\phi)}$ is onto. Hence, by the Inverse Function Theorem there exists a neighbourhood $\mathcal{U} \subset Ker(G^*_{(A,\phi)}) \times \mathcal{G}_{\mathfrak{s}(\alpha)}$ of $((A, \phi), I)$ such that $\Phi_{(A,\phi)} : \mathcal{U} \to \mathcal{C}_{\mathfrak{s}(\alpha)}$ is a diffeomorphism. Therefore, a neighbourhood of $[(A, \phi)] \in \mathcal{B}_{\mathfrak{s}(\alpha)}$ is diffeomorphic to a neighbourhood of $(0, 0) \in Ker(G^*_{(A,\phi)})$.

2. $\phi = 0$; At a redutible point $(A, 0)$, the derivative is no longer onto because $Ker(G^*_{(A,0)}) = Ker(d^*) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$ is $G_{(A,0)}$-invariant. In order to describe the link of a singular point $(A, 0)$, let's consider $\epsilon > 0$ and $\mathcal{W}$ a neighbourhood of the origin in $\Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$, such that $S^\infty_\epsilon = \{V \in \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})\ ;\ | V |= \epsilon\} \subset \mathcal{W}$. So,

   $$\begin{aligned}&\Big\{Ker(G^*_{(A,0)}) \cap \{Ker(d^*) \oplus (\mathcal{W} - \{0\})\}\Big\} \diagup G_{(A,0)} = \\ &= \{Ker(d^*) \oplus (\mathcal{W} - \{0\}) \diagup U_1)\} \overset{htpy}{\sim} S^\infty_\epsilon \diagup U_1 = \mathbb{C}P^\infty.\end{aligned}$$

Hence, a neighbourhood of a reducible point $(A,0) \in \mathcal{C}_{\mathfrak{s}(\alpha)}$ is homotopic to a cone over $\mathbb{C}P^{\infty}$.

2.3.2 Homotopy Aspects

Thus $\mathcal{C}^*_{\mathfrak{s}(\alpha)}$ is an universal bundle and the base space $\mathcal{B}^*_{\mathfrak{s}(\alpha)}$ is the classifying space for $\mathcal{G}_{\mathfrak{s}(\alpha)}$-vector bundles. In this way, for each $\mathfrak{s}(\alpha) \in Spin^{\mathbb{C}}$, the vector bundles $\mathcal{E}_{\mathfrak{s}(\alpha)}$ are classified by the homotopy classes $[\mathcal{B}^*_{\mathfrak{s}(\alpha)}, \mathcal{B}^*_{\mathfrak{s}(\alpha)}]$. The space $\mathcal{B}^*_{\mathfrak{s}(\alpha)}$ has the homotopy type of $T^{b_1}(Y) \times \mathbb{C}P^{\infty}$, where $T^{b_1}(Y) = H^1(Y,\mathbb{R})/H^1(Y,\mathbb{Z})$ and $b_1 = dim H^1(Y,\mathbb{R})$. So,

$$\begin{aligned}
[\mathcal{B}^*_{\mathfrak{s}(\alpha)}, \mathcal{B}^*_{\mathfrak{s}(\alpha)}] &= H^2(\mathcal{B}^*_{\mathfrak{s}(\alpha)},\mathbb{Z}) \oplus (H^1(\mathcal{B}^*_{\mathfrak{s}(\alpha)},\mathbb{Z}))^{b_1} = \\
&= \Big[H^2(\mathbb{C}P^{\infty},\mathbb{Z}) \oplus H^2(T^{b_1},\mathbb{Z})\Big] \oplus [H^1(T^{b_1},\mathbb{Z})]^{b_1} = \\
&= \mathbb{Z} \oplus \mathbb{Z}^{\frac{b_1(b_1-1)}{2}} \oplus \mathbb{Z}^{b_1^2}.
\end{aligned}$$

If $b_1(Y) = 0$, then the euler class $\chi(\mathcal{E}_{\mathfrak{s}(\alpha)}) \in H^2(\mathbb{C}P^{\infty},\mathbb{Z}) \simeq \mathbb{Z}$.

Later, when studying the deformed $\mathcal{SW}$-equations, it will become clear that the $H^2(\mathbb{C}P^{\infty},\mathbb{Z})$ contribution is the one which matters to define the Seiberg-Witten invariant, since the others contribution will vanish by the ausence of reducible solutions. In this way, we consider the heuristic euler class $\mu_{\mathfrak{s}(\alpha)} = \chi(\mathcal{E}_{\mathfrak{s}(\alpha)}) \in H^2(\mathcal{B}^*_{\mathfrak{s}(\alpha)},\mathbb{Z})$. Also, by using the 1^{st}-Chern class of the principal U_1-bundle $\mathfrak{b} : \mathcal{B}^0_{\mathfrak{s}(\alpha)} \to \mathcal{B}^*_{\mathfrak{s}(\alpha)}$, the $\mathcal{SW}$-invariant is defined as $\mu_{\mathfrak{s}(\alpha)} = c_1(\mathfrak{b}) \in H^2(\mathbb{C}P^{\infty},\mathbb{Z}) \simeq \mathbb{Z}$.

2.3.3 The Moduli Space of $\mathcal{SW}_{\mathfrak{s}(\alpha)}$-Monopoles

The local description of the $\mathcal{SW}_{\mathfrak{s}(\alpha)}$-monopole space $\mathcal{M}_{\mathfrak{s}(\alpha)} = \mathcal{F}^{-1}_{\mathfrak{s}(\alpha)}(0)/\mathcal{G}_{\mathfrak{s}(\alpha)}$ depends on its linear approximation. Since $\mathcal{F}_{\mathfrak{s}(\alpha)} : \mathcal{C}_{\mathfrak{s}(\alpha)} \to \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}) \oplus \Omega^1(Y,i\mathbb{R})$ is C^{∞}, its derivative[1] $(d\mathcal{F}_{\mathfrak{s}(\alpha)})_{(A,\phi)} : Ker(G^*_{(A,\phi)}) \to \Omega^1(Y,i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$ restricted to the slice is

$$\begin{aligned}
&(d\mathcal{F}_{\mathfrak{s}(\alpha)})_{(A,\phi)}.(\Theta,V) = \partial_A\mathcal{F}_{\mathfrak{s}(\alpha)}(A,\phi)\Theta + \partial_\phi\mathcal{F}_{\mathfrak{s}(\alpha)}(A,\phi)V, \\
&= \left(*d\Theta + \sigma_3(\phi,V), D_A(V) + \frac{1}{2}\Theta.\phi\right) = \begin{pmatrix} *d & \sigma_3(\phi,.) \\ \frac{1}{2}(.)\phi & D_A \end{pmatrix} . \begin{pmatrix} \Theta \\ V \end{pmatrix}.
\end{aligned}$$

The term $\Theta \bullet \phi$ is defined by $\Theta \bullet \phi = \sum_i \Theta(e_i)e_i \bullet \phi$. Whenever (A,ϕ) is a solution to the $\mathcal{SW}_{\mathfrak{s}(\alpha)}$-equations, then the restriction to $Imag(G_{(A,\phi)})$ is null, i.e.,

$$d\mathcal{F}_{\mathfrak{s}(\alpha)} \circ G_{(A,\phi)}(f) = (d\mathcal{F}_{\mathfrak{s}(\alpha)})_{(A,\phi)}.(2df, -f.\phi) = 0. \tag{2.17}$$

Remark 3. The operator $(d\mathcal{F}_{\mathfrak{s}(\alpha)})_{(A,\phi)}$ is self-adjoint and is also the hessian of the functional Υ at (A,ϕ);

[1] recall the relation $\hat{*} = -*$ among the Hodge star opertaors. From now on, the $\hat{}$ is being ignored.

1. For the sake of simplicity, let $L_{(A,\phi)} = (d\mathcal{F}_{\mathfrak{s}(\alpha)})_{(A,\phi)}$.

2. If $(A,\phi) \in \mathcal{F}^{-1}_{\mathfrak{s}(\alpha)}(0)$, then by equation 2.17 $L_{(A,\phi)} \circ G_{(A,\phi)} = 0$.

3. At (A,ϕ), the linearization of $\mathcal{F}_{\mathfrak{s}(\alpha)}$ yields the sequence

$$\Omega^0(Y, i\mathbb{R}) \stackrel{G_{(A,\phi)}}{\longrightarrow} \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}) \stackrel{L_{(A,\phi)}}{\longrightarrow} \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}), \tag{2.18}$$

   which is a complex if (A,ϕ) is a $\mathcal{SW}$-monopole and exact if $Ker(L_{(A,\phi)}) = Im(G_{(A,\phi)})$. In analogy with Hodge Theory, the introduction of the vector spaces

$$H^0_{(A,\phi)} = Ker(G_{(A,\phi)}), \quad H^1_{(A,\phi)} = Ker(L_{(A,\phi)}) \cap Ker(G^*_{(A,\phi)}), \tag{2.19}$$

$$H^2_{(A,\phi)} = \left\{\Omega^1(Y, i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})\right\} / Imag(L_{(A,\phi)}), \tag{2.20}$$

   leads to the following useful interpretations;

   (a) $H^0_{(A,\phi)} \neq 0 \Leftrightarrow (A,\phi)$ is reducible.

   (b) $H^2_{(A,\phi)} = 0 \Leftrightarrow L_{(A,\phi)}$ is onto. It is also equivalent to the transversality of the section $\mathcal{F}_{\mathfrak{s}(\alpha)} : \mathcal{B}_{\mathfrak{s}(\alpha)} \to \mathcal{E}_{\mathfrak{s}(\alpha)}$.

   (c) if $H^0_{(A,\phi)} = 0$ and $H^2_{(A,\phi)} = 0$, then $\mathcal{M}_{\mathfrak{s}(\alpha)}$ is a manifold and $H^1_{(A,\phi)} = T_{(A,\phi)}\mathcal{M}_{\mathfrak{s}(\alpha)}$.

4. At $(A,0)$,

$$L_{(A,0)} = \begin{pmatrix} *d & 0 \\ 0 & D_A \end{pmatrix} \cdot \begin{pmatrix} \Theta \\ V \end{pmatrix}.$$

5. let $H_\Upsilon(A,\phi) : T_{(A,\phi}\mathcal{C}_{\mathfrak{s}(\alpha)} \times T_{(A,\phi)}\mathcal{C}_{\mathfrak{s}(\alpha)} \to \mathbb{R}$ be the bilinear map associated to the hessian of the functional Υ; thus,

$$H_\Upsilon((\Theta, V), (\Lambda, W)) = \overline{< (\Theta, V), L_{(A,\phi)}(\Lambda, W) >}.$$

Lemma 2.3. *If $(A,\phi) \in \mathcal{M}_{\mathfrak{s}(\alpha)}$, then*

$$L_{(A,\phi)} : Ker(G^*_{(A,\phi)}) \to Ker(G^*_{(A,\phi)})$$

Demonstração. Let $(A,\phi) \in \mathcal{M}_{\mathfrak{s}(\alpha)}$ and $(\theta, V) \in Ker(G^*_{(A,\phi)})$.

$$G^*_{(A,\phi)} \circ L_{(A,\phi)}(\theta, V) = d^*(-*d\theta + \sigma_3(\phi, V)) + iIm\left(< \phi, D_A V + \frac{1}{2}\theta \bullet V >\right) =$$
$$= d^*(\sigma_3(\phi, V)) + iIm\left(< \phi, \frac{1}{2}\phi \bullet V >\right)$$

By lemma 2.2, $iIm\left(< \phi, \frac{1}{2}\phi \bullet V >\right) = 0$. The computation of term $d^*(\sigma_3(\phi, V))$ is performed assuming the identity

$$d^*(\sigma_3(\phi)) = iIm\left(< D_A\phi, \phi >\right),$$

and applying it to $\phi + V$. Comparing the terms in the expressions below,

$d^*(\sigma_3(\phi + V)) = d^*(\sigma_3(\phi)) + d^*(\sigma_3(V)) + 2d^*(\sigma_3(\phi, V))$,
$Im\left(< D_A(\phi + V), \phi + V >\right) = Im\left(< D_A\phi, \phi >\right) + Im\left(< D_AV, V >\right) + 2\mathfrak{Re}(< D_A\phi, V >)$,

it follows that $d^*(\sigma_3(\phi, V)) = 0$.

Claim: $d^*(\sigma_3(\phi)) = iIm\left(< D_A\phi, \phi >\right)$;

In order to prove it, consider at y_0 a normal frame $\beta = \{e_1, e_2, e_3\}$ and its coframe $\beta^* = \{e^1.e^2, e^3\}$ ($e^i \wedge *e^j = \delta^{ij} dv_g$), such that $(\nabla^A_{e_j} e_i)(y_0) = 0$. So,

$$\nabla^A_{e_j}(e_i \bullet \phi)(y_0) = (e_i \bullet \nabla^A_{e_j}\phi)(y_0).$$

Since $A \in \mathfrak{su}_2$ and $d^* = -*d*$,

$$d^*(\sigma_3(\phi)) = -\frac{1}{2}\sum_{i=1}^{3} d^*(< e_i \bullet \phi, \phi > e^i) = \frac{1}{2}\sum_{i=1}^{3} *d*(< e_i \bullet \phi, \phi > e^i) =$$
$$= \frac{1}{2}\sum_{i=1}^{3} *\left(< e_i \bullet \nabla^A_{e_j}\phi, \phi > + < e_i \bullet \phi, \nabla^A_{e_j}\phi >\right) *(e^i \wedge *e^i) =$$
$$= \frac{1}{2}\sum_{i=1}^{3} *\left(< e_i \bullet \nabla^A_{e_j}\phi, \phi > - < \phi, e_i \bullet \nabla^A_{e_j}\phi >\right) *(e^i \wedge *e^i) =$$
$$= \frac{1}{2}\left(< D_A\phi, \phi > - < \phi, D_A\phi >\right) * dv_g = iIm(< D_A\phi, \phi >).$$

□

The analysis becames more neat by introducing the self-adjoint operator $\mathcal{T}_{(A,\phi)}$: $(\Omega^1(Y,i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})) \oplus \Omega^0(X,i\mathbb{R}) \to (\Omega^1(Y,i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})) \oplus \Omega^0(Y,i\mathbb{R})$, defined by

$$\begin{aligned}\mathcal{T}_{(A,\phi)}((\Theta,V),f) &= (L_{(A,\phi)}(\Theta,V) + G_{(A,\phi)}(f), G^*_{(A,\phi)}(\Theta,V)) = \\ &= \begin{pmatrix} L_{(A,\phi)} & G_{(A,\phi)} \\ G^*_{(A,\phi)} & 0 \end{pmatrix} \cdot \begin{pmatrix} \begin{pmatrix} \Theta \\ V \end{pmatrix} \\ f \end{pmatrix}.\end{aligned}$$

Note that $\mathcal{T}_{(A,\phi)}$ may have a chance of being an isomorphism (this analysis will be carried out later in order to achieve the surjectivity). It is important to keep track of the term $Ker(\mathcal{T}_{(A,\phi)}\mid_{\Omega^0}) = H^0(Y,i\mathbb{R})$ introduced along with the $\Omega^0(Y,i\mathbb{R})$ direct summand because this sort of solution, named virtual solution, do not belongs to the monopole space .

Now, the whole of the information of the complex 2.18 is incoded into the kernel of the operator $\mathcal{T}_{(A,\phi)}$ as follows; assume $(A,\phi) \in (\mathcal{F}_{\mathfrak{s}(\alpha)})^{-1}(0)$, so

$$((\Theta,V),f) \in Ker(\mathcal{T}_{(A,\phi)}) \;\Leftrightarrow\; \begin{cases} (i)\ L_{(A,\phi)}(\Theta,V) = 0, \\ (ii)\ G^*_{(A,\phi)}(\Theta,V) = 0 \\ (iii)\ G_{(A,\phi)}(f) = 0. \end{cases}$$

Hence, $Ker(\mathcal{T}_{(A,\phi)}) = H^0_{(A,\phi)} \oplus H^1_{(A,\phi)}$. The vector space $H^2_{(A,\phi)}$ is the obstruction to the surjectivity of $\mathcal{T}_{(A,\phi)}$. In this set up, the cohomology groups defined in 2.19 are described as follows:

$$\begin{aligned}&H^0_{(A,\phi)} = Ker(G_{(A,\phi)}), \quad H^1_{(A,\phi)} = Ker(L_{(A,\phi)}) \cap Ker(G^*_{(A,\phi)}) = Ker(\mathcal{T}_{(A,\phi)}), \\ &H^2_{(A,\phi)} = \left\{\Omega^1(Y,i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})\right\}/Imag(\mathcal{T}_{(A,\phi)}).\end{aligned}$$

Besides,

1. $H^0_{(A,\phi)} \neq 0 \Leftrightarrow (A,\phi)$ is reducible.
2. $H^2_{(A,\phi)} = 0 \Leftrightarrow \mathcal{T}_{(A,\phi)}$ is onto.
3. if $H^0_{(A,\phi)} = 0$ and $H^2_{(A,\phi)} = 0$, then $\mathcal{M}_{\mathfrak{s}(\alpha)}$ is a manifold and $H^1_{(A,\phi)} = T_{(A,\phi)}\mathcal{M}_{\mathfrak{s}(\alpha)} = Ker(\mathcal{T}_{(A,\phi)})$.

The space $\mathcal{M}^*_{\mathfrak{s}(\alpha)}$ seen as the intersection in $\mathcal{E}_{\mathfrak{s}(\alpha)}$ of sections $\mathcal{F}_{\mathfrak{s}(\alpha)} : \mathcal{B}^*_{\mathfrak{s}(\alpha)} \to \mathcal{E}_{\mathfrak{s}(\alpha)}$ and 0-section is a manifold. However, the transversality condition may not occur and to handle the lack of transversality a perturbation will be performed later.

The operator $\mathcal{T}_{(A,\phi)}$ described in coordinates is

$$\mathcal{T}_{(A,\phi)} = \begin{pmatrix} L_{(A,\phi)} & G_{(A,\phi)} \\ G^*_{(A,\phi)} & 0 \end{pmatrix} . \left(\begin{pmatrix} \Theta \\ V \end{pmatrix} \right) = \begin{pmatrix} *d & -\sigma_3(\phi,.) & 2d \\ \frac{1}{2}(.).\phi & D_A & -(.).\phi \\ 2d^* & iIm(<\phi,.>) & 0 \end{pmatrix} . \begin{pmatrix} \Theta \\ V \\ f \end{pmatrix} =$$

$$= \begin{pmatrix} *d & 0 & 2d \\ 0 & D_A & 0 \\ 2d^* & 0 & 0 \end{pmatrix} . \begin{pmatrix} \Theta \\ V \\ f \end{pmatrix} + \begin{pmatrix} 0 & -\sigma_3(\phi,.) & 0 \\ \frac{1}{2}(.).\phi & 0 & -(.)\phi \\ 0 & iIm(<\phi,.>) & 0 \end{pmatrix} . \begin{pmatrix} \Theta \\ V \\ f \end{pmatrix} .$$

It can be decomposed as $\mathcal{T}_{(A,\phi)} = \mathcal{P}_{(A,\phi)} \oplus \mathcal{Q}_{(A,\phi)} + K_{(A,\phi)}$, where

$$\mathcal{P}_{(A,\phi)} = \begin{pmatrix} *d & 0 & 2d \\ 0 & 0 & 0 \\ 2d^* & 0 & 0 \end{pmatrix}, \quad \mathcal{Q}_{(A,\phi)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & D_A & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are self-adjoint elliptic operators and

$$K_{(A,\phi)} = \begin{pmatrix} 0 & -\sigma_3(\phi,.) & 0 \\ \frac{1}{2}(.).\phi & 0 & -(.)\phi \\ 0 & iIm(<\phi,.>) & 0 \end{pmatrix} . \begin{pmatrix} \Theta \\ V \\ f \end{pmatrix} .$$

is a compact operator (the resolvent). The operator $\mathcal{P}_{(A,\phi)} : \Omega^1(Y,i\mathbb{R}) \oplus \Omega^0(Y,i\mathbb{R}) \to \Omega^1(Y,i\mathbb{R}) \oplus \Omega^0(Y,i\mathbb{R})$ is the rolled-up operator obtained by the composition

$$\Omega^1 \oplus \Omega^0 \xrightarrow{\begin{pmatrix} * & 0 \\ 0 & 2.I \end{pmatrix}} \Omega^2 \oplus \Omega^0 \xrightarrow{\begin{pmatrix} d^* & d \\ d & 0 \end{pmatrix}} \Omega^1 \oplus \Omega^3 \xrightarrow{\begin{pmatrix} I & 0 \\ 0 & 2* \end{pmatrix}} \Omega^1 \oplus \Omega^0$$

Remark 4. At a reducible solution $(A,0)$,

1. $H^0_{(A,0)} = H^0(Y,\mathbb{R})$,
2. $H^1_{(A,0)} = Ker(\mathcal{T}_{(A,0)}) = H^0(Y,\mathbb{R}) \oplus H^1(Y,\mathbb{R}) \oplus Ker(D_A)$. The $H^0(Y,i\mathbb{R})$ summand correspond to the virtual solutions and the $H^1(Y,i\mathbb{R})$ corresponds to the tangent space to the Jacobian torus $T^{b_1(Y)}$.
3. The self-adjointness of $\mathcal{P}_{(A,0)}$ and $\mathcal{Q}_{(A,0)}$ yields $H^2_{(A,0)} = H^1_{(A,0)}$.
4. For later purposes: if $b_1(Y) = 0$, then $H^1_{(A,0)} = H^2_{(A,0)} = Ker(D_A)$.

Theorem 2.2. *The operator $\mathcal{T}_{(A,\phi)}$ is a self-adjoint operator with domain the vector space $(\Omega^1(Y,i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})) \oplus \Omega^0(X,i\mathbb{R})$ endowed with a $L^{1,2}$ Sobolev structure and image in the vector space $(\Omega^1(Y,i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})) \oplus \Omega^0(X,i\mathbb{R})$ endowed with a L^2 Sobolev structure. Moreover, $\mathcal{T}_{(A,\phi)}$ has compact resolvent and thus discrete spectrum. In particular, it is a Fredholm operator.*

Demonstração. Its symbol defines a isomorphism, so $\mathcal{T}_{(A,\phi)}$ is an elliptic operator over a compact manifold, hence it is an Fredholm operator. □

Definition 2.8. A monopole $(A,\phi) \in \mathcal{M}_{\mathfrak{s}(\alpha)}$ is non-degenerated if $H^1_{(A,\phi)} = 0$.

The non-degenerency means that, up to gauge equivalence, $Ker(H_\Upsilon) = \{0\}$ at (A,ϕ) and H_Υ is surjective. If $(A,0)$ is a reducible solution and $b_1(Y) > 0$, then the non-degenerecency is never achieved because $H^2_{(A,0)} = H^1(Y,\mathbb{R}) \oplus Ker(D_A)$, by remark 4.3. If $b_1(Y) = 0$, then the obstruction to achieve the transversality is $H^2_{(A,0)} = Ker(D_A)$.

Proposition 2.9. *The non-degenerated points are isolated.*

Demonstração. Let $(A,\phi) \in \mathcal{M}_{\mathfrak{s}(\alpha)}$ be a non-degenerated point. Thus, the linear map $(d\mathcal{F}_{\mathfrak{s}(\alpha)})_{(A,\phi)} : Ker(G^*_{(A,\phi)}) \to \Omega^1 \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$ $((d\mathcal{F}_{\mathfrak{s}(\alpha)})_{(A,\phi)} = L_{(A,\phi)})$ is non singular, or equivalently $Ker(L_{(A,\phi)})/\mathcal{G}_{\mathfrak{s}(\alpha)} = \{0\}$. Hence, $\mathcal{F}$ is an immersion at (A,ϕ). By the Inverse Function Theorem, there exists a neighbourhood U of (A,ϕ) such that $\mathcal{F} : U \to \mathcal{F}(U)$ is a difeomorphism. If (A,ϕ) were not isolated it would exist a sequence of points $(A_n,\phi_n) \in \mathcal{M}_{\mathfrak{s}(\alpha)}$ and $n_0 \in \mathbb{N}$ such that, $\forall n > n_0$, $(A_n,\phi_n) \in U$, which is a contradiction with the fact that $\mathcal{F}\mid_U$ is a difeomorphism. □

Thanks to the compacity of $\mathcal{M}_{\mathfrak{s}(\alpha)}$, whenever the non-degenerecency is satisfied for all $(A,\phi) \in \mathcal{M}^*_{\mathfrak{s}(\alpha)}$, then $\mathcal{M}^*_{\mathfrak{s}(\alpha)}$ is a 0-dimensional manifold, hence a finite set of points.

2.3.4 Perturbed $\mathcal{SW}$-Equations

In order to achieve the transversal condition $H^2_{(A,\phi)} = 0$ a perturbation is performed on the functional Υ. Let $Z^2(Y,i\mathbb{R}) = \{\nu \in \Omega^2(Y,i\mathbb{R}) \mid d\nu = 0\}$ be the space of closed 2-forms.

Definition 2.9. Fix $A_0 \in \mathcal{A}_\alpha$ and $\nu \in Z^2(Y,i\mathbb{R})$. Consider $\Upsilon_\nu : \mathcal{C}_{\mathfrak{s}(\alpha)} \to \mathbb{R}$ as

$$\Upsilon_\nu = \frac{1}{2}\int_Y \{(A - A_0) \wedge (F_A + F_0 + 2\nu) + < D_A\phi, \phi >\}$$

Remark 5. .

1. Υ_ν is $\mathcal{G}_{\mathfrak{s}(\alpha)}$-invariant.
2. The formula 2 becames
$$\Upsilon_\nu(g.(A,\phi)) = \Upsilon_\nu(A,\phi) - \{(4\pi g^*(\mu) + [\nu])) \cup 2\pi c_1(L))\,([Y]).$$
3. The L^2-gradient of Υ_ν is
$$grad(\Upsilon_\nu)(A,\phi) = (- * F_A + \sigma_3(\phi) + *\nu, D_A\phi). \tag{2.21}$$

The map $\nu \mapsto grad(\Upsilon_\nu)$ defines a section of $\mathcal{F}^\nu_{\mathfrak{s}(\alpha)} : \mathcal{B}_{\mathfrak{s}(\alpha)} \to \mathcal{E}_{\mathfrak{s}(\alpha)}$.

Definition 2.10. Let $\nu \in \Omega^2(Y, i\mathbb{R})$ be a closed 2-form. The ν-perturbed $\mathcal{SW}$-equations are

$$- * F_A + \sigma_3(\phi) + *\nu = 0, \qquad D_A\phi = 0. \tag{2.22}$$

The ν-monopole space is $\mathcal{M}_{\mathfrak{s}(\alpha)}(\nu) = (\mathcal{F}^{\nu}_{\mathfrak{s}(\alpha)})^{-1}(0)$.

Remark 6. .

1. Consider the map

$$\mathcal{F} : Z^2(Y, i\mathbb{R}) \times \Omega^1(Y, i\mathbb{R}) \times \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}) \to \Omega^1(Y, i\mathbb{R}) \times \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}), \tag{2.23}$$

$$\mathcal{F}(\nu, A, \phi) = \mathcal{F}^{\nu}(A, \phi) = (- * F_A + \sigma_3(\phi) + *\nu, D_A\phi). \tag{2.24}$$

Its derivative is the linear operator

$$L^{\nu}_{(A,\phi)} : Z^2(Y, i\mathbb{R}) \oplus \Omega^1(X, i\mathbb{R}) \oplus Ker(G^*_{(A,\phi)}) \to Ker(G^*_{(A,\phi)}),$$
$$L^{\nu}_{(A,\phi)}((\zeta, (\Theta, V))) = L_{(A,\phi)}(\Theta, V) + (*\zeta, 0).$$

2. Fixed ν and suppose that the equation $F_A = \nu$ admits a solution A_0; recall that a necessary condition for the existence of A_0 is $\frac{\nu}{2\pi i} = c_1(\mathcal{L}_\alpha) \in H^1(Y, \mathbb{Z})$. Thus, $(A_0, 0)$ is a reducible solution for the ν-perturbed $\mathcal{SW}$-equation. Whenever $a \in \Omega^1(Y, i\mathbb{R})$ is closed, $(A_0 + a, 0)$ is also a reducible solution. Besides, A_0 and $A_0 + a$ are gauge equivalent iff $[a] \in H^1(Y, \mathbb{Z})$, where $H^1(Y, \mathbb{Z})$ is a lattice within $H^1(Y, \mathbb{R})$. So, if the space of reducible solutions is not empty, then it is diffeomorphic to the Jacobian torus $T^{b_1(Y)} = H^1(Y, \mathbb{R})/H^1(Y, \mathbb{Z})$. There are three cases to be analysed;

   $b_2(Y) > 1$: The space $H^2(Y, \mathbb{R}) - H^2(Y, \mathbb{Z})$ is arc connected. Therefore, the space of closed 2-forms ν not admiting reducible solutions is connected.

   $b_2(Y) = 1$: Once $dim(H^2(Y, \mathbb{R})) = 1$, the space $H^2(Y, \mathbb{R}) - H^2(Y, \mathbb{Z})$ has many arc connect components. In his case, the space of closed 2-forms not admiting reducible solutions has also many arc connected components.

   $b_2(Y) = 0$: In this case, for every closed 2-form ν there exist a reducible solution $(A, 0)$ of $F_A = \nu$, and it is unique up to gauge equivalence. To construct such solution it is enough to observe that ν being exact yields $\nu = d\mu$, for some $\mu \in \Omega^1(Y, i\mathbb{R})$, and also that the bundle $\mathcal{L}_\alpha$ admits a flat connection A_0. So, $(A_0 + \mu, 0)$ is a ν-reducible solution, which is unique up to the $\mathcal{G}_{\mathfrak{s}(\alpha)}$-action.

Theorem 2.3. *Consider the map* $\mathcal{F}_{\mathfrak{s}(\alpha)} : Z^2(Y, i\mathbb{R}) \times \mathcal{C}_{\mathfrak{s}(\alpha)} \to \Omega^1(X, i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$, *defined by* $\mathcal{F}_{\mathfrak{s}(\alpha)}(A, \phi, \nu) = \mathcal{F}^{\nu}_{\mathfrak{s}(\alpha)}(A, \phi)$. *The following claims are true:*

1. *There is a Baire subset of 2-forms $\mathfrak{F}_2 \subset Z^2(Y, i\mathbb{R})$ such that, for all $\nu \in \mathfrak{F}_2$, $\mathcal{F}^{\nu}_{\mathfrak{s}(\alpha)}$ is tranversal to the 0-section at $(A,\phi) \in (\mathcal{F}^{\nu}_{\mathfrak{s}(\alpha)})^{-1}(0)$ ($H^2_{(A,\phi)}(\nu) = 0$).*
2. *If $b_1(Y) = 0$, then there is a Baire subset of 2-forms $\mathfrak{F}_2 \subset Z^2(Y, i\mathbb{R})$ such that, for all $\nu \in \mathfrak{F}_2$, $\mathcal{F}^{\nu}_{\mathfrak{s}(\alpha)}$ is tranversal to the 0-section at $(A,0) \in (\mathcal{F}^{\nu}_{\mathfrak{s}(\alpha)})^{-1}(0)$ ($H^2_{(A,0)}(\nu) = 0$).*

Demonstração. By considering the map

$$\mathcal{F} : Z^2(Y, i\mathbb{R}) \times \Omega^1(Y, i\mathbb{R}) \times \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}) \to \Omega^1(Y, i\mathbb{R}) \times \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}), \tag{2.25}$$
$$\mathcal{F}(\nu, A, \phi) = \mathcal{F}^{\nu}(A,\phi) = (- * F_A + \sigma_3(\phi) + *\nu, D_A\phi), \tag{2.26}$$

the first step is to prove the surjectivity of the linear operator $L = d\mathcal{F}_{(\nu,A,\phi)}$;

$$L : Z^2(Y, i\mathbb{R}) \oplus Ker(G^*_{(A,\phi)}) \to \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}),$$
$$L(\zeta, \Theta, V) = L_{(A,\phi)}(\Theta, V) + (*\zeta, 0).$$

By the decomposition $\Omega^1(Y, i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}) = Imag(L) \oplus Ker(L^*)$, the surjectivity of L is verified by proving that $Ker(L^*) = \{0\}$. In order to compute L^*, let

$$\begin{aligned} < L(\alpha,\theta,V),(\zeta,W) > &= < *\alpha, \zeta > \oplus \{< *d\theta - \frac{1}{2}\sigma_3(\phi,V), \zeta > + < D_AV + \frac{1}{2}\theta.\phi, W >\} = \\ &= < \alpha, *\zeta > \oplus \{< \theta, *d\zeta + \frac{1}{2}\sigma_3(\phi,W) > + < V, D_AW - \frac{1}{2}\zeta.\phi >\} \end{aligned}$$

Thus,

$$L^*(\zeta, W) = (*d\zeta + \frac{1}{2}\sigma_3(\phi,W), D_AW - \frac{1}{2}\zeta.\phi, *\zeta). \tag{2.27}$$

Therefore, if $(\zeta, W) \in Ker(L^*)$, then

$$(i)\ \zeta = 0, \qquad (ii)\ \sigma_3(\phi, W) = 0 \tag{2.28}$$
$$(iii)\ D_AW - \frac{1}{2}\zeta.\phi = 0 \qquad (iv)\ d^*\zeta + Im(< \phi, W >) = 0. \tag{2.29}$$

A solution (ζ, W) of these equations is C^∞. Let's consider the following cases:

1. (A,ϕ) is irreducible ($\phi \neq 0$);
   The equation (ii) implies that $W = ir\phi$, where $r \in Map(X, \mathbb{R})$. So, the equation (iii) implies that $dr = 0$. Therefore, from equation (iv) it follows that

$$Im(-ir \mid \phi \mid^2) = 0 \quad \Leftrightarrow \quad \phi = 0.$$

Hence, $\phi = 0$ and $\zeta = 0$ yields the surjectivity of L. By Sard's theorem, there is a Baire set $\mathcal{F}\in \in Z^2(Y, i\mathbb{R})$ such that for all $\nu \in \mathcal{F}\in$ $\mathcal{F}^\nu$ is transversal to the 0-section.

2. $(A, 0)$ is a ν-reducible solution, so $F_A = \nu$. If $\mu \in Z^1(Y, i\mathbb{R})$, then $(A + \mu, 0)$ is a solution of $F_A = \nu$. At $(A + \mu, 0)$, it follows from **??** that $Ker(L^*) = H^1(Y, \mathbb{R}) \oplus Ker(D_{A+\mu})$, so the transversality can be achieved only by assuming $b_1(Y) = 0$. Consider the map $s : \Omega^1(Y, i\mathbb{R}) \times \mathit{\Gamma}(\mathcal{S}_{\mathfrak{s}(\alpha)}) \to \mathcal{V}$, $s(\mu, w) = D_{A+\mu}(w)$, where $\mathcal{V}$ is the vector bundle $\mathcal{V} \to \mathit{\Gamma}(\mathcal{S}_{\mathfrak{s}(\alpha)})$, which fiber over $\phi \neq 0$ is the vector space $\mathcal{V}_\phi = Ker(Re < i\phi, . >) = (Span_{\mathbb{R}}(i\phi))^\perp$. The definition of $\mathcal{V}$ yields from the fact that, for all $w \in \mathit{\Gamma}(\mathcal{S}_{\mathfrak{s}(\alpha)})$, $D_{A+\mu}w \in \mathcal{V}_V$, since

$$< D_{A+\mu}w, iw >= -\overline{< D_{A+\mu}w, iw >} \;\Rightarrow\; Re(< D_{A+\mu}w, iw >) = 0.$$

Because $ind(D) = 0$ and $\mathcal{V}$ is a codimension 1 subspace of $\mathit{\Gamma}(\mathcal{S}_{\mathfrak{s}(\alpha)})$, it follows that $ind_{\mathbb{R}}(ds_{(\mu,w)}) = 1$, for all $(\mu, w) \in \Omega^1(Y, i\mathbb{R}) \times \mathit{\Gamma}(\mathcal{S}_{\mathfrak{s}(\alpha)})$. The derivative $ds_{(\nu,\phi)} : \Omega^1(Y, i\mathbb{R}) \times \mathit{\Gamma}(\mathcal{S}_{\mathfrak{s}(\alpha)}) \to \Gamma(\mathcal{V})$ is,

$$ds_{(\nu,w)}(\lambda, u) = \lambda.w + D_{A+\mu}u \tag{2.30}$$

Suppose that $\exists \psi \in \mathit{\Gamma}(\mathcal{S}_{\mathfrak{s}(\alpha)})$ such that $\psi \perp Imag(ds_{\mu,w}) \subset \{iw\}^\perp$. So,

(a) for all $\lambda \in \Omega^1(Y, i\mathbb{R})$, $< \psi, \lambda.w >= 0 \Rightarrow \sigma_3(\psi, w) = 0$ and $\psi = irw$

(b) for all $u \in \mathit{\Gamma}(\mathcal{S}_{\mathfrak{s}(\alpha)})$, $< \psi, D_{A+\mu}u >= 0 \Rightarrow D_{A+\mu}\psi = 0$, hence $dr = 0 \Rightarrow$ r is constant.

Once $\psi \perp iw$ in L^2, it follows that $r = 0$. Consequently, the map $ds_{(\mu,w)}$: $\Omega^1(Y, i\mathbb{R}) \oplus \mathit{\Gamma}(\mathcal{S}_{\mathfrak{s}(\alpha)}) \to \mathit{\Gamma}(\mathcal{S}_{\mathfrak{s}(\alpha)})$ is surjective and so, the map s is transversal. Agian, by Sard's theorem there exists a Baire subset of 1-forms $\mathfrak{F}\mathfrak{1} \subset Z^1(Y, i\mathbb{R})$ such that, for all $\mu \in \mathfrak{F}\mathfrak{1}$, $s_\mu = D_{A+\mu} : \mathit{\Gamma}(\mathcal{S}_{\mathfrak{s}(\alpha)}) \to \mathcal{V}$ is transversal to the 0-section. Moreover, $dim_{\mathbb{R}}(s_\mu^{-1}(0)) = 1$. Neverthless, $D_{A+\mu}$ is a $\mathbb{C}$-linear operator, so $dim_{\mathbb{R}}(Ker(D_{A+\mu}))$ must be an even number. Hence, for all $\mu \in \mathfrak{F}\mathfrak{1}$ $s_\mu^{-1}(0) = Ker(D_{A+\mu}) = \{0\}$. As a by-product, the transversality is settled in case $b_1(Y) = 0$.

□

Corollary 2.1. *There is a Baire set of forms $\mathfrak{F}\mathfrak{2} \subset \Omega^2(Y, i\mathbb{R})$ such that, for all $\nu \in \mathfrak{F}\mathfrak{2}$, the space $\mathcal{M}^*_{\mathfrak{s}(\alpha)}(\nu)$ is a compact, 0-dimensional manifold.*

Remark 7. .

1. The transversality attained in the last theorem does not take in account the quotient by the $\mathcal{G}_{\mathfrak{s}(\alpha)}$-action. If the analysis is carried on to the quotient, then it is not defined at reducible solutions.

2. For later applications, it is very important to understand the existence of reducible solutions of the ν-perturbed $\mathcal{SW}$-equation whenever $b_1(Y) \leq 1$. A necessary condition to the existence of a solution $(A, 0)$ of $F_A = \nu$ is that $c_1(\mathcal{L}_\alpha) = \frac{\nu}{2\pi i}$, otherwise there is no such solution and the ν-perturbed $\mathcal{SW}$-equation is free of reducible connections. Let's consider the following cases;

   (i) $b_1(Y) = 1$: let A_0 be a solution of $F_A = \nu$ and $\theta_0 \in H^1(Y, \mathbb{R})$ such that $H^1(Y, \mathbb{R}) \diagup H^1(Y, \mathbb{Z}) =< \theta_0 >$. Thus, the space of ν-reducible solutions is diffeomorphic to S^1 and parametrized by $\mathcal{M}^{red}_{\mathfrak{s}(\alpha)}(\nu) = \{A_0 + t\theta_0 \mid t \in [0, 1]\}$.

   (ii) $b_1(Y) = 0$: in this case, the equation $F_A = \nu$ implies that there exists $a \in \Omega^1(Y, i\mathbb{R})$ such that $da = \nu$. By considering A_0 a flat connection (the bundle $\mathcal{L}_\alpha$ is trivial), thus $(A_0 + a, 0)$ is the unique reducible solution of the ν-perturbed $\mathcal{SW}$-invariant, therefore, $\mathcal{M}^{red}_{\mathfrak{s}(\alpha)}(\nu) = \{A_0 + a\}$.

2.3.5 Spectral Flow of the Dirac Operator

For later purposes, it is important to understand the spectrum behavior of a family of Dirac operators. A C^∞ curve $\mu : [0, 1] \to \Omega^1(Y, i\mathbb{R})$ induces a curve $D_t : [0, 1] \to Fred^0(\mathit{\Gamma}(\mathcal{S}_{\mathfrak{s}(\alpha)})) \in \Omega^1(Y, i\mathbb{R})$, $D_t(w) = D_{A+\mu(t)}$, where $Fred^0(\mathit{\Gamma}(\mathcal{S}_{\mathfrak{s}(\alpha)}))$ is the space of Fredholm operators with index 0.

In this way, the spectrum varies smoothly with t, besides it can be assumed that the eigenvalues are distincts. The spectrum is the subset

$$\bigcup_{t \in [0,1]} \{t\} \times Spec(D_t) \subset [0, 1] \times \mathbb{R}.$$

The interesting aspect of $Spec(D_t)$ is the change of signs of the eignevalues along the path. If they don't change, nothing different turns up, however if one eigenvalue λ changes its sign, then at some t_0 $\lambda(t_0) = 0$.

Proposition 2.10. *Let $\lambda_t \in Spec(D_t)$ and v_t an unitary λ_t-eigenvector. So,*

$$\frac{d\lambda_t}{dt} =< \frac{dD_t}{dt} v_t, v_t > .$$

Demonstração. Once $D_t v_t = \lambda_t v_t$, it follows that $D_t' v_t + D_t v_t' = \lambda_t' v + \lambda_t v_t'$. Since $< v_t, v_t' >= 0$, by taking the L^2-inner product with v_t,

$$< D_t' v_t, v_t > + < D_t v_t', v_t >= \lambda_t'$$

Besides, the self-adjointness of D_t implies that

$$< D_t v_t', v_t >=< v_t', D_t v_t >= \overline{\lambda}_t < v_t', v_t >= 0.$$

Therefore, $\lambda_t' =< D_t' v_t, v_t >$. □

Definition 2.11. The spectral flow of a path $D_t : [0,1] \to Fred^0(\Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}))$ is

$$SF(D_t) = \sigma_1 - \sigma_0, \text{ where } \sigma_i \text{ is the signature of } D_i.$$

All this topic concerning the Spectral flow is relevant to study the case $b_1(Y) \leq 1$. Let $A_t = A+\mu+t\mu_0$, $t \in [0,1]$ be a 1-parameter family of reducible solutions of $F_{A_t} = \nu$, $\nu = d\mu$. The corresponding 1-parameter family of Dirac operators $D_t = D_{A+\mu+t\mu_0} \in Fred^0(\Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}))$ may go through the 0-line in $[0,1] \times \mathbb{R}$.

Proposition 2.11. *Consider that for* $t_0 \in [0,1]$, $0 \in Spec(D_{A+\mu+t_0\mu_0})$. *Let* $\lambda_t \in Spec(D_t)$ *be the eigenvalue such that* $\lambda_{t_0} = 0$. *So,* $\lambda_{t_0}' > 0$ *meaning that the curve* $(t, \lambda_t) \subset [0,1] \times \mathbb{R}$ *across transversaly the axis* $[0,1] \times \{0\}$.

Demonstração. Assume that $\mu \in \mathfrak{F}\mathfrak{1}$, as in the theorem **??**. So, the map $s_\mu : [0,1] \times \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}) \to \mathcal{V}$, given by $s_\mu(t,w) = D_{A+\mu+t\mu_0}(w)$, is transversal to the 0-section yielding the surjectivity of $(ds_\mu)_{(t_0,w)} : \mathbb{R} \times \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}) \to \mathcal{V}$, where

$$(ds_\mu)_{(t_0,w)}(1,u) = \frac{d}{dt}(D_{A+\mu+t\mu_0}(w)) \mid_{t=t_0} + D_{A+\mu+t_0\mu_0} u.$$

Now, let's consider $u_0 \in \mathcal{V}$ an unitary harmonic spinor of $D_{A+\mu+t_0\mu_0}$. Due to the surjectivity of $(ds_\mu)_{(t_0,w)}$, there exists $v_0 \in \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$ such that

$$\frac{d}{ds}(D_{A+\mu+s\mu_0}(w)) \mid_{s=0} + D_{A+\mu+t_0\mu_0} v_0 = u_0.$$

Therefore, by taking the inner product with u_0 and using the self-adjointness od $D_{A+\mu+t_0\mu_0}$, it follows that

$$\frac{d}{dt}(\lambda_t) \mid_{t=t_0} =< \frac{d}{dt}(D_{A+\mu+t\mu_0}) \mid_{t=t_0}, v_0 > > = 0.$$

This is the required condition to (t, λ_t) be transversal to the axis $[0,1] \times \{0\}$. □

2.3.6 $\mathcal{M}_{\mathfrak{s}(\alpha)}$ is compact for all $\mathfrak{s}(\alpha) \in Spin^c(X)$

Lemma 2.4. *Let* (A, ϕ) *be an irreducible solution of 2.3. Thus,*

$$\| \phi \|_\infty \leq \max_{y \in Y}\{0, -k_g(y)\} \tag{2.31}$$

2.3.7 $\mathcal{M}_{\mathfrak{s}(\alpha)}$ is orientable for all $\mathfrak{s}(\alpha) \in Spin^c(X)$

Let $\nu \in \mathfrak{F}$ as in theorem **??**. Therefore, the monopole space $\mathcal{M}_{\mathfrak{s}(\alpha)}(\nu)$ is a 0-dimensional compact manifold, thus it is a finite set. Let's describe a general procedure to orient, at once, all $\mathcal{M}_{\mathfrak{s}(\alpha)}(\nu) = (\mathcal{F}^{\nu}_{\mathfrak{s}(\alpha)})^{-1}(0)$. For the sake of simplicity, let's consider $\mathcal{F} = \mathcal{F}^{\nu}_{\mathfrak{s}(\alpha)}$.

As seen before, the map $\mathcal{F}$ is a Fredholm map. Under the hypothesis of $\phi \neq 0$ and $\nu \in \mathfrak{F}2$, $\mathcal{M}_{\mathfrak{s}(\alpha)}(\nu)$ is a manifold and the tangent space at (A,ϕ) is $T_{(A,\phi)}\mathcal{M}_{\mathfrak{s}(\alpha)}(\nu) = Ker(\mathcal{T}_{(A,\phi)}) = H^1_{(A,\phi)}$. Thus, $\mathcal{M}_{\mathfrak{s}(\alpha)}(\nu)$ is orientable if the vector space $\Lambda^{max}(Ker(\mathcal{T}_{(A,\phi)}))$ is a trivial bundle ($\Lambda^{max}V$ stands for the highest exterior power of V). The index of $\mathcal{T}_{(A,\phi)}$ is

$$ind(\mathcal{T}_{(A,\phi)}) = dim(Ker(\mathcal{T}_{(A,\phi)})) - dim(CoKer(\mathcal{T}_{(A,\phi)})), \tag{2.32}$$

which corresponds to the dimension of the virtual bundle

$$[Ker(\mathcal{T}_{(A,\phi)})] - [CoKer(\mathcal{T}_{(A,\phi)})].$$

In order to achieve the triviality of $\Lambda^{max}(Ker(\mathcal{T}_{(A,\phi)}))$, we consider the determinant line bundle associated to a Fredholm operator;

Definition 2.12. The determinant line bundle of a Fredholm operator $\mathcal{T}(A,\phi)$ is the line bundle

$$det(\mathcal{T}_{(A,\phi)}) = \Lambda^{max}(Ker(\mathcal{T}_{(A,\phi)})) \otimes \left[\Lambda^{max}(CoKer(\mathcal{T}_{(A,\phi)}))\right]^* .$$

The determinant line bundle of a family of Fredholm operators $\{\mathcal{T}(A,\phi) \mid (A,\phi) \in \mathcal{C}_\alpha\}$ is the line bundle

$$det(\mathcal{T}) = \bigcup_{(A,\phi)\in\mathcal{C}_{\mathfrak{s}(\alpha)}} det(\mathcal{T}_{(A,\phi)}).$$

Remark 8. .

1. Consider $\mathcal{F}(V,W)$ the space of Fredholm operators $F : V \to W$. The index defined in 2.32 is invariant by a homotopy performed in $\mathcal{F}(V,W)$. Thus, $ind(T_1) = ind(T_2)$ whenever $T_1, T_2 \in \mathcal{F}(V,W)$ are connected by a continuous path in $\mathcal{F}(V,W)$. Moreover, $det(T_1) = det(T_2)$.

2. Although the dimensions of the vector spaces $Ker(\mathcal{T}_{(A,\phi)})$ and $CoKer(\mathcal{T}_{(A,\phi)})$ may jump, the index doesn't and $det(\mathcal{T})$ is a complex line bundle over $\mathcal{C}_{\mathfrak{s}(\alpha)}$. Once these spaces are all gauge invariant, it turns out that $det(\mathcal{T})$ is a line bundle over $\mathcal{B}_{\mathfrak{s}(\alpha)}$.

By considering a connected path $\gamma : [0,1] \to \mathcal{C}_{\mathfrak{s}(\alpha)}$, $\gamma(t) = (1-t)(A,\phi) + t(A,0)$, the index of the operator $\mathcal{T}_{(A,\phi)}$, as in 2.28, is equal to the index of the elliptic operator $\mathcal{T}_{(A,0)} = \mathcal{P} \oplus \mathcal{Q}_A : \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}) \oplus \Omega^0(Y, i\mathbb{R}) \to \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}) \oplus \Omega^0(Y, i\mathbb{R})$, where

$$\mathcal{P} = \begin{pmatrix} *d & 0 & 2d \\ 0 & 0 & 0 \\ 2d^* & 0 & 0 \end{pmatrix} : \Omega^1(Y, i\mathbb{R}) \oplus \Omega^0(Y, i\mathbb{R}) \to \Omega^1(Y, i\mathbb{R}) \oplus \Omega^0(Y, i\mathbb{R}),$$
$$\mathcal{Q}_A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & D_A & 0 \\ 0 & 0 & 0 \end{pmatrix} : \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}) \to \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}).$$

As observed before, $ind(T) = ind(\mathcal{P}) + ind(\mathcal{Q}_A)$, where $\mathcal{P}$ and $\mathcal{Q}_A$ are both self-adjoint with kernel

$$Ker(\mathcal{P}) = H^0(Y, \mathbb{R}) \oplus H^1(Y, \mathbb{R}), \qquad Ker(\mathcal{Q}_A) = Ker(D_A).$$

Therefore, $Ker(T_{(A,0)}) = H^0(Y, \mathbb{R}) \oplus H^1(Y, \mathbb{R}) \oplus Kerd(D_A)$. Moreover, the bundle $det(\mathcal{T}_\gamma) \to [0,1]$ is trivial, so $det(\mathcal{T}_{\gamma(0)})$ and $det(\mathcal{T}_{\gamma(1)})$ are isomorphics.

Theorem 2.4. *The line bundle $det(\mathcal{T}) \to \mathcal{B}_{\mathfrak{s}(\alpha)}$ is trivial. Moreover, $det(\mathcal{T})$ is orientable and an orientation is fixed by choosing a orientation of*

$$\Lambda^{max} H^0(Y, \mathbb{R}) \oplus \Lambda^{max} H^1(Y, \mathbb{R}).$$

Demonstração. Since $det(\mathcal{T}_{(A,\phi)})$ is isomorphic to $det(\mathcal{T}_{(A,0)})$, it follows that the fibers of $det(\mathcal{T}_{(A,\phi)})$ are isomorphic to $V_1(A) \oplus V_2(A)$, where $V_1(A) = \Lambda^{max} H^0(Y, \mathbb{R}) \oplus \Lambda^{max} H^1(Y, \mathbb{R})$ and $V_2(A) = \Lambda^{max} Ker(D_A)$. The sub-bundle V_1, which fiber at A is $V_1(A)$, is trivial because its fibers independ on A. Also, the sub-bundle V_2 is trivial because $Ker(D) \to \mathcal{A}_\alpha$ is a complex vector bundle, hence orientable. Besides, the $\mathcal{G}_{\mathfrak{s}(\alpha)}$-action preserves the complex structure and all the decompositions in the setting.

□

Corollary 2.2. *The manifold $\mathcal{M}_{\mathfrak{s}(\alpha)}$ is orientable and an orientation is induced by orienting the vector spaces $H^0(Y, \mathbb{R})$ and $H^1(Y, \mathbb{R})$.*

In this way, if $\mathcal{M}_{\mathfrak{s}(\alpha)} = \{p_1, \ldots, p_n\}$, then for each $p_i \in \mathcal{M}_{\mathfrak{s}(\alpha)}$, $i = 1, \ldots, n$, we can associate either $n_i = +1$ or $n_i = -1$.

2.3.8 Seiberg-Witten Invariants of Y^3

As seen in the sections before, under the hypothesis that $\nu \in \mathcal{F}\in$ and $b_1(Y) > 1$, there are a finite number of classes $\mathfrak{s} \in Spin^{\mathbb{C}}(X)$ (basic classes of Y) such that $\mathcal{M}_{\mathfrak{s}(\alpha)}(\nu) \neq \varnothing$ and is an orientable, compact 0-dimensional manifold.

Definition 2.13. Let (Y, g) be a riemannian structure on Y. The Seiberg-Witten invariant of (Y, g) is

$$\mathcal{SW} : Spin^{\mathbb{C}} \to \mathbb{Z} \tag{2.33}$$

$$\mathfrak{s}(\alpha) \longrightarrow \mathcal{SW}(\mathfrak{s}(\alpha)) = \begin{cases} \sum_i n_i, & \text{if } \mathfrak{s}(\alpha) \text{ is a basic class,} \\ 0, & \text{otherwise} \end{cases} \tag{2.34}$$

where $n_i = \pm 1$, according with the orientation given at $p_i \in \mathcal{M}_{\mathfrak{s}(\alpha)}$.

Another way of looking at the $\mathcal{SW}$-invariant is observing that it is the euler class of the vector bundle $\mathcal{E}_{\mathfrak{s}(\alpha)} \to \mathcal{B}_{\mathfrak{s}(\alpha)}$.

The hard work next is to deal with the cases $b_1 \leq 1$.

2.4 Metric Invariance of $\mathcal{SW}$-Invariant

The definition of the $\mathcal{SW}$-invariant requires a rimannian metric g on Y and also a 2-form $\nu \in \mathfrak{F}2$ to guarantee that $\mathcal{M}_{\mathfrak{s}(\alpha)}$ is a smooth, orientable manifold. Consider $\mathfrak{RM}_Y$ the space of riemannian metrics defined on Y. The metric dependence of is stressed in the cases considered next;

2.4.1 Case $b_1(Y) > 1$

Let $g_t : [0, 1] \to \mathcal{M}_Y$ be a smooth path connecting g_0 to g_1, and $\nu_t : [0, 1] \to \mathcal{F}\in \subset \Omega^2(Y, i\mathbb{R})$ be a smooth path of 2-forms connecting ν_0 to ν_1. Since $b_1(Y) > 1$, it can be assumed that the class $[\nu_t] \neq [\frac{F_A}{2\pi i}]$, $\forall t \in [0, 1]$. Next, by fixing a class $\mathfrak{s}(\alpha) \in Spin^{\mathbb{C}}(Y)$, we may consider the map $\widehat{\mathcal{F}} : [0, 1] \times \mathcal{C}_\alpha \times \Omega^2(Y\, i\mathbb{R}) \to \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$.

$$\widehat{\mathcal{F}}(t, A, \phi, \nu) = (*_t F_A - \sigma_3(\phi) - *\nu_t, D^t_A \phi), \tag{2.1}$$

where $*_t$ and D^t_A are the operators associated to g_t. The Chern-Simons-Dirac functional $\widehat{\Upsilon} : [0, 1] \times \mathcal{C}_\alpha \to \mathbb{R}$ has non-degenerated critical points since $\nu_t \subset \mathfrak{F}2$, for all $t \in [0, 1]$, and the linear map $d\widehat{\mathcal{F}} : \mathbb{R} \oplus \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}) \oplus \Omega^2(Y, i\mathbb{R}) \to \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$ is surjective. Hence, $\widehat{\mathcal{M}_{\mathfrak{s}(\alpha)}} = \widehat{\mathcal{F}}^{-1}(0)$ is a manifold. By the same arguments, the moduli space $\widehat{\mathcal{M}_{\mathfrak{s}(\alpha)}}$ of solutions of 2.1 is a compact, oriented manifold which is either empty or 1-dimensional; in the former case it is a set of arcs. From the construction above, the map $\pi : \widehat{\mathcal{M}_{\mathfrak{s}(\alpha)}} \to [0, 1]$ given by $\pi^{-1}(t) = \mathcal{M}^t_{\mathfrak{s}(\alpha)}$ is a fibration. As a manner of fact, if $b_1(Y) > 1$, then $\widehat{\mathcal{M}_{\mathfrak{s}(\alpha)}} = \mathcal{M}^0_{\mathfrak{s}(\alpha)} \times [0, 1]$.

Theorem 2.5. *Let $\mathfrak{s}(\alpha) \in Spin^{\mathbb{C}}(X)$ and consider $\mathcal{SW}^0(\mathfrak{s}(\alpha))$ and $\mathcal{SW}^1(\mathfrak{s}(\alpha))$ the invariants associated to the spaces $\mathcal{M}^0_{\mathfrak{s}(\alpha)}$ and $\mathcal{M}^1_{\mathfrak{s}(\alpha)}$, respectively. If $b_1(Y) > 1$, then*

$$\mathcal{SW}^0(\mathfrak{s}(\alpha)) = \mathcal{SW}^1(\mathfrak{s}(\alpha)).$$

Demonstração. From the construction, the space $\widehat{\mathcal{M}_{\mathfrak{s}(\alpha)}}$ is a cobordism among $\mathcal{M}^0_{\mathfrak{s}(\alpha)}$ and $\mathcal{M}^1_{\mathfrak{s}(\alpha)}$. However, for each $t \in [0,1]$, the invariant $\mathcal{SW}^t(\mathfrak{s}(\alpha))$ can be written as

$$\mathcal{SW}^t(\mathfrak{s}(\alpha)) = \int_{\mathcal{M}^t_{\mathfrak{s}(\alpha)}} 1,$$

where μ is the Lebesgue measure defined on $\mathcal{M}^t_{\mathfrak{s}(\alpha)}$. So, by Stoke's theorem,

$$\mathcal{SW}^0(\mathfrak{s}(\alpha)) - \mathcal{SW}^1(\mathfrak{s}(\alpha)) = \int_{\widehat{\mathcal{M}_{\mathfrak{s}(\alpha)}}} d(1) = 0.$$

□

2.4.2 Case $b_1(Y) = 1$

This case is particularly more delicate since the condition $b_1(Y) = 1$ means that $H^1(Y, \mathbb{R})$ is 1-dimensional, and so, along a variation $\nu_t : [a,b] \to \mathfrak{F}_2$ it may occur that ν_0 and ν_1 are in different connected component of $H^2(Y, \mathbb{R}) - \{\frac{F_A}{2\pi i}\}$. Therefore, the fibration $\pi : \widehat{\mathcal{M}_{\mathfrak{s}(\alpha)}} \to [0,1]$ has a singular fiber at $t = c$ because of existing a reducible solution. Thus, the reducible solution $(A, 0)$ has to be taken in account and no longer $\widehat{\mathcal{M}_{\mathfrak{s}(\alpha)}}$ is a manifold.

At $t = c$, the space of reducible solutions is the 1-sphere $S^1 = H^1(Y, \mathbb{R})/H^1(Y, \mathbb{Z})$.

In order to understand the invariant, let's consider the the projection $\Omega^2(Y, i\mathbb{R}) \to H^2(Y, \mathbb{R})$, $\nu \to [\nu]$, and the 1-codimension *wall*

$$\mathcal{W} = \{\nu \in \Omega^2(Y, i\mathbb{R}) \mid [\nu] = 2\pi i c_1(\mathcal{L}_\alpha)\}.$$

The wall splits $\Omega^2(Y, i\mathbb{R})$ into two connected components, named the chambers;

$$\begin{aligned}
\mathcal{W}^+ &= \{\nu \in \Omega^2(Y, i\mathbb{R});\ 2\pi i c_1(\mathcal{L}_\alpha)([\nu]) > 0\}, \\
\mathcal{W}^- &= \{\nu \in \Omega^2(Y, i\mathbb{R});\ 2\pi i c_1(\mathcal{L}_\alpha)([\nu]) < 0\}
\end{aligned}$$

As before, let's assume that the path $\nu_t : [0,1] \to \Omega^2(Y, i\mathbb{R})$ has not reducible solutions, but at $t = c$. The linear map $d\widehat{\mathcal{F}} : \mathbb{R} \oplus \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)}) \oplus \Omega^2(Y, i\mathbb{R}) \to \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(\mathcal{S}_{\mathfrak{s}(\alpha)})$ is surjective, however the $\mathcal{G}_{\mathfrak{s}(\alpha)}$-action is not free. Thus, the fibration $\pi : \widehat{\mathcal{M}_{\mathfrak{s}(\alpha)}} \to [0,1]$ has a singular fiber at $t = c$ because the space $\pi^{-1}(t) = \mathcal{M}^t_{\mathfrak{s}(\alpha)}$ miss to be a manifold. By cutting off the singular set $\mathcal{S}_c \subset \mathcal{M}^c_{\mathfrak{s}(\alpha)}$, the moduli space $\widehat{\mathcal{M}_{\mathfrak{s}(\alpha)}}$ defines an oriented cobordism among $\mathcal{M}^0_{\mathfrak{s}(\alpha)}$ and $\mathcal{M}^c_{\mathfrak{s}(\alpha)} - \mathcal{S}_c$ and another one among $\mathcal{M}^1_{\mathfrak{s}(\alpha)}$ and $\mathcal{M}^c_{\mathfrak{s}(\alpha)} - \mathcal{S}_c$.

For each $\mathfrak{s}(\alpha) \in Spin^{\mathbb{C}}(X)$, consider the monopole moduli spaces $\mathcal{M}^{\pm}_{\mathfrak{s}(\alpha)}$ corresponding to the solutions of the perturbed $\mathcal{SW}$-equations restricted to $\nu \in \mathcal{W}^{\pm}$, respectively.

Definition 2.14. The invariants $\mathcal{SW}_{\pm} : Spin^{\mathbb{C}}(Y) \to \mathbb{Z}$ are defined by

$$\mathcal{SW}^{+}(\mathfrak{s}(\alpha)) = \int_{\mathcal{M}^{+}_{\mathfrak{s}(\alpha)}} 1, \quad \mathcal{SW}^{-}(\mathfrak{s}(\alpha)) = \int_{\mathcal{M}^{-}_{\mathfrak{s}(\alpha)}} 1.$$

Theorem 2.6. *Let $b_1(Y) = 1$. The wall crossing formula is given by*

$$\mathcal{SW}^{+}(\mathfrak{s}(\alpha)) - \mathcal{SW}^{-}(\mathfrak{s}(\alpha)) = \int_{S^1} d\mu$$

Demonstração. It follows from the remark that the moduli space $\widehat{\mathcal{M}_{\mathfrak{s}(\alpha)}} - \mathcal{S}_c$ defines an oriented cobordism among $\mathcal{M}^0_{\mathfrak{s}(\alpha)}$ and $\mathcal{M}^1_{\mathfrak{s}(\alpha)}$. So,

$$\partial(\widehat{\mathcal{M}_{\mathfrak{s}(\alpha)}}) = \mathcal{M}^0_{\mathfrak{s}(\alpha)} \sqcup \mathcal{M}^1_{\mathfrak{s}(\alpha)} \sqcup S^1$$

□

2.4.3 Case $b_1(Y) = 0$

In general, the existence of a reducible solution $(A, 0)$ of the $\mathcal{SW}$-equations means that $F_A = 0$, which corresponds to a representation $\rho_A : \pi_1(Y) \to U_1$, hence an element $\rho^*_A \in H^1(Y, U_1)$.

The case $b_1(Y) = 0$ is restricted to the $\mathbb{Q}$-homology spheres $(H^*(Y, \mathbb{Q}) = H^*(S^3, \mathbb{Q}))$.

1. If Y is a $\mathbb{Z}$-homology sphere, then $H^1(Y, \mathbb{Z}_2) = H^2(Y, \mathbb{Z}) = 0$, so $Spin^{\mathbb{C}}(Y) = \{0\}$. In this case, $H^1(Y, U_1) = H^1(Y, \mathbb{Z}) \otimes U_1 = 0$, so the only representation is the trivial one.

2. If $H^1(Y, \mathbb{Z})$ is a torsion group, then $Spin^{\mathbb{C}}(Y) = H^1(Y, \mathbb{Z}_2)$ is finite. In this case, it may exist non-trivial representations $\rho^*_A \in H^1(Y, U_1)$.

In both cases, there is no way of getting rid of the reducible solution of the perturbed $\mathcal{SW}$-equations. The map $\mathcal{SW} : Spin^{\mathbb{C}}(Y) \to \mathbb{Z}$ is no longer a smooth invariant, it depends on the metric on Y and also on the 2-form ν, whenever a perturbation has been considered.

For all $\nu \in \Omega^2(Y)$, the pertubed $\mathcal{SW}$-equation admits only one reducible solution, up to the gauge invariance. Let A_0 be a flat connection and $\theta \in \Omega^1(Y, i\mathbb{R})$ the only 1-form satisfying $d\theta = \nu$ and $d^*\theta = 0$, so $A = A_0 + \theta$ is a solution of $F_A = \nu$. In this case, the space $\mathcal{M}_{\mathfrak{s}(\alpha)}(\nu)$ is always singular and there is no path turning around the singularity as in the case $b_2(Y) > 1$. Therefore, if we fix a class $s(\alpha) \in Spin^c(Y)$, then the moduli space $\mathcal{M}_{\mathfrak{s}(\alpha)}(g, \nu)$ depends on the metric g and on ν. Hence, for each pair (g, ν) we can associate the integer $\mathcal{SW}(\mathfrak{s}(\alpha); (g, \nu))$. In order to obtain a smooth invariant, let's introduce a curve $\sigma : [0, 1] \to \mathfrak{M}_Y \times \Omega^2(Y, i\mathbb{R})$ connecting the pairs (g_0, ν_0) and (g_1, ν_1). Also consider the moduli space $\mathcal{M}_{\mathfrak{s}(\alpha)}(\sigma)$ and the fibration $\pi : \mathcal{M}_{\mathfrak{s}(\alpha)}(\sigma) \to [0, 1]$, where

$\pi^{-1}(t) = \mathcal{M}_{\mathfrak{s}(\alpha)}(g_t, \nu_t)$. ALTHOUGH, FOR EACH $t \in [0,1]$, $\pi^{-1}(t)$ IS A MANIFOLD WHEN RETRICTED TO THE IRREDUCIBLES, IT CAN NOT BE GUARANTEE THAT THE SPECTRAL FLOW OF THE DIRAC OPERATOR FAMILY $D(\sigma)$ DOES NOT JUMPS ALONG σ. WHENERVER IT JUMPS THE NUMBER $\mathcal{SW}(g(t), \nu(t))$ changes, since

$$\mathcal{SW}(g(1), \nu(1)) - \mathcal{SW}(g(0), \nu(0)) = SF(D(\sigma)).$$

The spectral flow can also be computed via the Atiyah-Patodi-Singer index theorem. It is known from **??** that a spin manifold (Y, s_Y) bounds a 4-manifold (X, s_X) with only one 0-handle and finite many 2-handles ($b_1(X) = 0$). The following objects can be extended over X;

1. the $Spin^{\mathbb{C}}$-structure s_Y over Y extends to $s_X \in Spin^{\mathbb{C}}(X)$ over X,
2. the U_1-bundle $\mathcal{L}_\alpha$ over Y to the U_1-bundle $\widehat{\mathcal{L}_\alpha}$ over X,
3. the unique flat connection θ on $\mathcal{L}_\alpha$ to a connection Θ on $\widehat{\mathcal{L}_\alpha}$,
4. $\nu \in \Omega^2(i, \mathbb{R})$ to $\widehat{\nu} \in \Omega^2(X, i\mathbb{R})$.

Thus, by the Atiyah-Patodi-Singer index theorem, the spectral flow $SF(D(\sigma))$ is computed by the formula

$$\zeta(\sigma(1)) - \zeta(\sigma(0)) = SF(D(\sigma)),$$

where $\zeta(\sigma(t))$ is defined as follows;

Definition 2.15. Consider $b_1(Y) = 0$ and fix $(g, \nu) \in \mathfrak{M}(Y) \times \Omega^2(Y, i\mathbb{R})$. Let θ be the unique flat connection (up to gauge) on $\mathcal{L}_\alpha$ and let $a \in \Omega^1(Y, i\mathbb{R})$ be the unique 1-form satisfying the equations $d^*a = 0$ and $da = \nu$. Define

$$\zeta(g, \nu) = \frac{1}{8}\hat{\eta}(\delta, g) + \frac{1}{2}\left(dim_{\mathbb{C}} Ker(D_{\theta+A}) + \hat{\eta}(D_{\theta+A}) + \frac{1}{32\pi^2}\int_Y (A \wedge dA) \qquad (2.2)$$

where $\delta = d + d^* : \Omega^{even}(X) \to \Omega^{odd}(X)$.

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