Dispersionless Limit of Integrable Models

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Abstract

Nonlinear dispersionless equations arise as the dispersionless limit of well know integrable hierarchies of equations or by construction, such as the system of hydrodynamic type. Some of these equations are integrable in the Hamiltonian sense and appear in the study of topological minimal models. In the first part of the review we will give a brief introduction to integrable models, mainly its Lax representation. Then, we will introduce the dispersionless limit and show some of our results concerning the two-component hyperbolic system of equations such as the polytropic gas and Born-Infeld equations.

1. Introduction

The study of integrable models or solvable nonlinear partial differential equations is an active area of research since the discovery of the inverse scattering method [1-3]. These models are in a sense universal since they show up in many areas of physics such as solid state, nonlinear optics, hydrodynamics, field theory just to name a few. Also, integrable models are linked to many areas of mathematics (see the chart in `http://www.ma.hw.ac.uk/solitons/procs/bullough1/bullough1/bullough1.html`) and have beautiful structures behind them.

In this review we want to approach the dispersionless limit of some integrable models and describe some of our work on this subject [4-7]. This review is organized as follows: In Section 2 we review or at least introduce some basic facts on integrable models. We use the Korteweg-de Vries equation (KdV) as an example. In Section 3 we introduce the dispersionless limit of an integrable model using the KdV equation to obtain the corresponding Riemann equation. Section 4 reviews our work with a special class of dispersionless systems known as two-component hyperbolic systems. We show our results concerning the Hamiltonian structures for the Riemann equation [4] the dispersionless Lax representation for the polytropic gas dynamics [5] and Born-Infeld equation [6]. Finally, in Section 5, we conclude with some problems that deserve further investigations.

2. Integrable Models

2.1 Solitons

We are interested in nonlinear partial differential equations such as the sine-Gordon equation, nonlinear Schrödinger equation, Korteweg-de Vries equation (KdV), etc. These equations, as we will see, are very special since they are integrable. From now on we will illustrate the main results concerning integrability using the KdV equation.

The KdV equation has as solution what is called today a soliton. We can trace the discovery of the soliton back to 1834 with the Scott Russell's experiment [8] to generate solitary waves in water, i.e., localized single entity waves. A modern version of his experiment is shown in Figure 1 (see `http://www.ma.hw.ac.uk/~chris/scott_russel.html` for an attempt to recreate Scott Russell's soliton). Scott Russell found that the volume V of water wave is equal to the volume of water displaced and that the speed c of the solitary wave is related with its amplitude a, depth of water h and acceleration of gravity g by

$$c^2 = g(h+a) \tag{2.1}$$

This equation shows that higher waves travel faster. Attempts to obtain (2.1) theoretically were done by Boussinesq (1871) and Lord Rayleigh (1876) but an equation for $u(x,t)$ in the

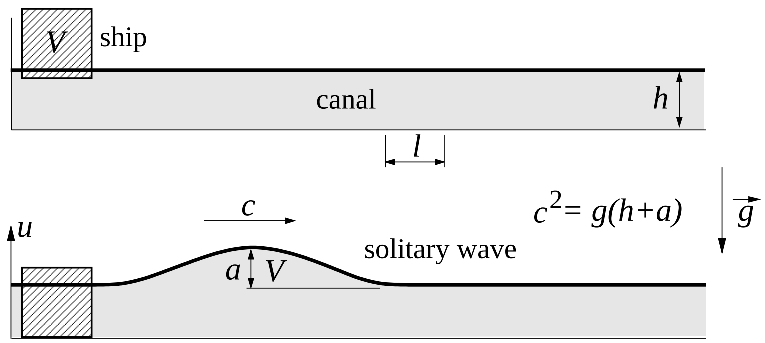

Figure 1. Generation of a solitary wave.

small amplitude ($h \gg a$) and in the long wave regime ($h \ll \ell$) was deduced by Korteweg-de Vries in 1895 [9]. This is the now famous KdV equation

$$\boxed{u_t = uu_x + u_{xxx}} \tag{2.2}$$

where $u(x,t)$ is the wave profile and $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}, \ldots$.

The interest in the KdV equation (2.2) was resumed after studies of Fermi, Pasta and Ulam in 1955 [10] on numerical models of phonons in non-linear lattices, which are models closely related with the discretisation of the KdV equation. Motivated by these results, Zabusky and Kruskal in 1965 [11] studied numerically equations like (2.2) with periodic boundary conditions and were led to introduce the concept of "soliton" solutions. In 1967 Gardner, Greene, Kruskal and Miura [12] solved equation (2.2) exactly, introducing the "Inverse Scattering Transform Method" (ISTM), and were able to obtain its analytic expression. The so called 1-soliton and 2-soliton solutions of the KdV equation (2.2), for rapidly decreasing boundary conditions

$$u(x,t) \to 0 \quad \text{for} \quad x \to \pm\infty\,,$$

are

$$\begin{aligned} u(x,t) =& \frac{1}{2}c^2 \text{sech}^2\left(\frac{1}{2}c(x+c^2t)\right) \quad \rightarrow \quad 1-\ \text{soliton} \\ u(x,t) =& 12\frac{3+4\cosh(2x-8t)+\cosh(4x-64t)}{\{3\cosh(x-28t)+\cos(3x-36t)\}^2} \quad \rightarrow \quad 2-\ \text{soliton} \end{aligned} \tag{2.3}$$

In Figure 2 we have pictures for the time evolution of the KdV solitons (2.3) (for some brief solitons movies see `http://www.ma.hw.ac.uk/solitons` and `http://www.physics.otago.ac.nz/Physics100/simulations/Gamelan/java/toda`). The 1-soliton solution in Figure 2 is the solitary wave obtained in the Scott Russell's experiment. Observe that as the time evolves the wave keeps its form. For the 2-soliton solution in

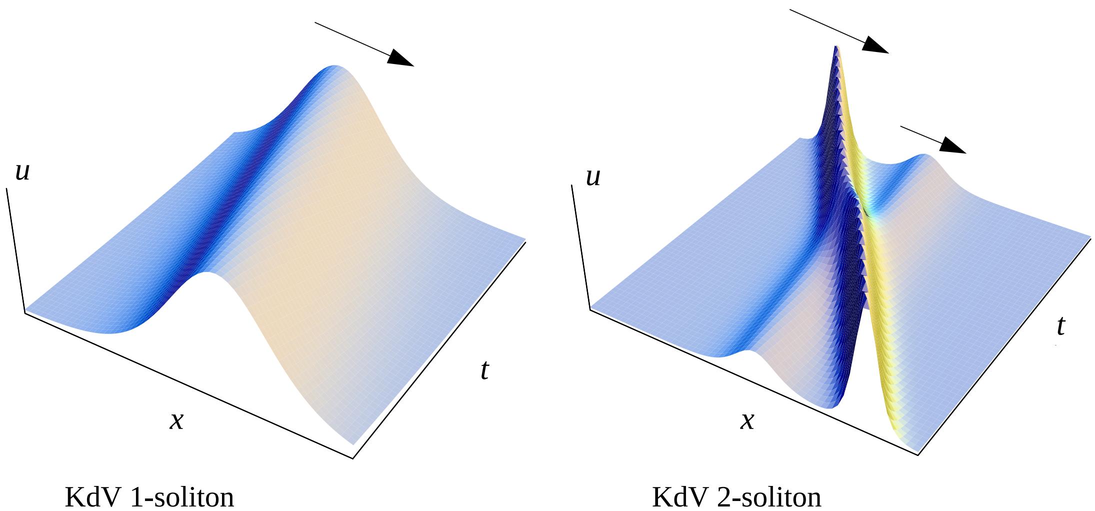

Figure 2. Time evolution for the solitons of the KdV equation.

Figure 2, since the taller the soliton the faster it moves, the two solitons will interact nonlinearly when they meet. But, the amazing fact is that the two solitons will almost keep their initial form after interaction, there will be only a shift in their positions. This particle-like character and ability to retain its identity after interactions is what characterize a soliton solution of a nonlinear equation such as the KdV one.

2.2 Inverse Scattering

The next breakthrough in the soliton thread came in 1968 with the Lax [13] discovery about the meaning of the ISTM. His observation is that the KdV equation has the representation

$$\boxed{\frac{\partial L}{\partial t} = [B, L]} \tag{2.4}$$

where

$$\begin{aligned} L =& \partial^2 + \frac{1}{6}u \\ B =& 4\,\partial^3 + \frac{1}{2}(\partial u + u\partial) \end{aligned} \tag{2.5}$$

are operators. Here $\partial \equiv \frac{\partial}{\partial x}$ satisfies $\partial f = f_x + f\partial$. We call L the Lax operator and in some sense we can find a Lax representation such as (2.4) for any integrable system. In this way, starting from (2.4), we can apply the ISTM for other nonlinear equations.

We can write the following eigenvalue problem for the Lax operator L

$$L\psi = -\lambda\psi \tag{2.6}$$

It is easy to see that since L evolves in time as (2.4) we have $\lambda_t = 0$, i.e., the eigenvalue problem is isospectral. For the KdV equation (2.6) assumes the form

$$\frac{\partial^2 \psi}{\partial x^2} + \left(\frac{1}{6}u(x,t) + \lambda\right)\psi = 0 \tag{2.7}$$

which is the time-independent Schrödinger equation and where t is a parameter (not the time in the Schrödinger equation). Now we can obtain a solution $u(x,t)$ as follows: For some given initial condition $u(x,0)$ we solve (2.7) and obtain the scattering data $S(t=0)$, since u satisfies the KdV equation we can obtain the scattering data for any t, so from $S(t)$ we use the inverse scattering (as we usually do in quantum mechanics) to find the "potential" $u(x,t)$ from the scattering data $S(t)$. This is the ISTM routine and the main steps are illustrated in the diagram bellow.

$$
\begin{array}{cccc}
\text{initial condition} & & \text{scattering data can} & \\
\text{(given)} & & \text{be calculated} & \\
\Uparrow & & \Uparrow & \\
u(x,0) & \longrightarrow & S(t=0) & \\
\uparrow & & \downarrow & \Rightarrow \text{KdV} \\
\boxed{u(x,t)} & \underset{\substack{\text{inverse}\\ \text{scattering}}}{\longleftarrow} & S(t) &
\end{array}
$$

Inverse Scattering Transform Method

2.3 Hamiltonian Systems

In 1970 Gardner [14] showed that the KdV equation is a Hamiltonian integrable system. Then, Faddeev and Zakharov in 1971 [15] were able to interpret the ISTM as a change of variables to the action angle variables. In fact, the representation of integrable models as integrable Hamiltonian systems is the starting point to the "Quantum Inverse Scattering Method". Before we see how the KdV equation can be expressed in Hamiltonian form let us review the symplectic formalism for Hamiltonian systems. A Hamiltonian system is described by a phase space q_i, p_i, with $i = 1, \ldots, N$, and a Hamiltonian function

$H(p_i, q_i)$. The equations of motion are then given by the Hamilton's equations

$$\begin{aligned}\dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}\end{aligned} \tag{2.8}$$

Alternatively, we can describe a Hamiltonian system using Poisson brackets, for the dynamical variables $A(q,p)$ and $B(q,p)$, defined by

$$\{A, B\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \tag{2.9}$$

which is skew-symmetric and satisfies the Jacobi identity. The variables of phase space satisfy the canonical relations $\{q_i, q_j\} = \{p_i, p_j\} = 0$ and $\{q_i, p_j\} = \delta_{ij}$. The Hamilton's equations (2.8) assume the form

$$\begin{aligned}\dot{q}_i =& \{q_i, H\} \\ \dot{p}_i =& \{p_i, H\}\end{aligned} \tag{2.10}$$

Putting the variables q_i and p_i in an $2N$ dimension column z the equations (2.8) assume the form

$$\frac{d}{dt}\underbrace{\begin{pmatrix} q_1 \\ \vdots \\ q_N \\ p_1 \\ \vdots \\ p_N \end{pmatrix}}_{\equiv z} = \underbrace{\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}}_{\equiv J} \underbrace{\begin{pmatrix} \partial/\partial q_1 \\ \vdots \\ \partial/\partial q_N \\ \partial/\partial p_1 \\ \vdots \\ \partial/\partial p_N \end{pmatrix}}_{\equiv \vec{\nabla}} H \tag{2.11}$$

or

$$\dot{z}^a = J^{ab}\partial_b H \qquad a, b = 1, \ldots, 2N \tag{2.12}$$

and even in a more compact form as

$$\boxed{\dot{z} = J\vec{\nabla}H} \tag{2.13}$$

This is the symplectic formalism for Hamiltonian systems. The Poisson brackets can be written as

$$\{A, B\} = \left(\vec{\nabla}A\right)^t J \left(\vec{\nabla}B\right) \tag{2.14}$$

where $J^{ab} = -J^{ba}$ and $\sum \left(J^{ab}\partial_d J^{bc} + \text{cyclic}\right) = 0$. The canonical relations are given by $\{z, z\} = J$ and (2.10) by

$$\boxed{\dot{z} = \{z, H\}} \tag{2.15}$$

We can perform some generalizations, allowing J to depend on z, $J(z)$, and going from a discret sympletic space, of dimension $2N$, to the continuum where we have now a field $u(x,t)$ instead of $z(t)$. Then, we have the following "dictionary"

$z(t)$	$\rightarrow$	$u(x,t)$
$H(z)$	$\rightarrow$	$H[u]$ functional
$\vec{\nabla} H$	$\rightarrow$	$\dfrac{\delta H}{\delta u}$ functional derivative
$J(z)$ skew-symmetric matrix	$\rightarrow$	$\mathcal{D}(u)$ skew-adjoint operator
$\dot{z} = J\vec{\nabla} H$	$\rightarrow$	$\dot{u} = \mathcal{D}\dfrac{\delta H[u]}{\delta u}$
$\{z, z\} = J(z)$	$\rightarrow$	$\{u(x), u(x')\} = \mathcal{D}\delta(x - x')$
$\{A, B\} = \left(\vec{\nabla} A\right)^t J \left(\vec{\nabla} B\right)$	$\rightarrow$	$\{A[u], B[u]\} = \int dx \dfrac{\delta A}{\delta u}\mathcal{D}\dfrac{\delta B}{\delta u}$

If there is a J^{-1} we say that we are in a symplectic manifold, otherwise we are in a more general situation of a Poisson manifold. Note that the functional derivative $\frac{\delta H[u]}{\delta u}$ is defined as

$$\frac{\delta H[u(x)]}{\delta u(y)} = \lim_{\epsilon \rightarrow 0} \frac{H[u(x) + \epsilon\delta(x-y)] - H[u(x)]}{\epsilon} \tag{2.16}$$

which for $H[u] = u(x)$ yields

$$\frac{\delta H[u(x)]}{\delta u(y)} = \delta(x-y)$$

and for $H[u] = \int dx\, h(x, u, u_x, u_{xx}, \ldots)$

$$\frac{\delta H[u(x)]}{\delta u(y)} = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial x}\frac{\partial}{\partial u_x} + \frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial u_{xx}^2} + \ldots\right) h$$

where the right hand side is just the Euler-Lagrange operator acting on h.

Now, let us return to the KdV equation (2.2) and observe that it can be rewritten as

$$\begin{aligned} u_t =& u u_x + u_{xxx} \\ =& \frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + u_{xx}\right) \end{aligned} \tag{2.17}$$

Introducing the Hamiltonian

$$H_2 = \int dx \left(\frac{1}{3!} u^3 - \frac{1}{2} u_x^2 \right) \tag{2.18}$$

we see that $\frac{\delta H_2}{\delta u} = \frac{1}{2} u^2 + u_{xx}$ and $\frac{dH_2}{dt} = 0$. The operator

$$\mathcal{D}_1 = \frac{\partial}{\partial x} \tag{2.19}$$

is skew-adjoint and satisfies the Jacobi identity. So, (2.17) can be written in Hamiltonian form as

$$u_t = \mathcal{D}_1 \frac{\delta H_2}{\delta u} = \{u(x), H_2\}_1 \tag{2.20}$$

where

$$\{u(x), u(y)\}_1 = \mathcal{D}_1 \delta(x - y) \tag{2.21}$$

and we are omitting the explicit dependence on t.

Besides (2.18) the KdV equation (2.2) has an infinite number of conserved charges

$$\begin{aligned}
H_0 &= \int dx\, u \\
H_1 &= \int dx\, \frac{1}{2} u^2 \\
H_2 &= \int dx\, \left(\frac{1}{3!} u^3 - \frac{1}{2} u_x^2 \right) \\
H_3 &= \int dx\, \left(\frac{1}{4} u^4 - 3 u u_x + \frac{9}{5} u_{xx}^2 \right) \\
H_4 &= \int dx\, \left(\frac{1}{5} u^5 - 6 u^2 u_x^2 + \frac{36}{5} u u_{xx}^2 - \frac{108}{35} u_{xxx}^2 \right) \\
&\vdots
\end{aligned} \tag{2.22}$$

and it can be shown that these charges are in involution, i.e.,

$$\{H_n, H_m\}_1 = 0 \tag{2.23}$$

making the KdV equation integrable in Lioville's sense.

In 1978 Magri [16] discovered that equations like KdV have a second Hamiltonian structure. The operator

$$\mathcal{D}_2 = \frac{\partial^3}{\partial x^3} + \frac{1}{3} \left(\frac{\partial}{\partial x} u + u \frac{\partial}{\partial x} \right) \tag{2.24}$$

is skew-adjoint and satisfies Jacobi identity, and the KdV equation can be written in the alternative Hamiltonian form

$$u_t = \mathcal{D}_2 \frac{\delta H_1}{\delta u} = \{u(x), H_1\}_2 \tag{2.25}$$

where

$$\{u(x), u(y)\}_2 = \mathcal{D}_2 \delta(x-y) \tag{2.26}$$

These charges (2.22) are also in involution with respect to this second Hamiltonian structure

$$\{H_n, H_m\}_2 = 0 \tag{2.27}$$

We say that the KdV equation is a bi-Hamiltonian system. In general we say that a system is bi-hamiltonian if there are Hamiltonian operators $\mathcal{D}_1$ and $\mathcal{D}_2$ which are compatible, i.e., such that $\mathcal{D}_1$, $\mathcal{D}_2$ and $\lambda_1\mathcal{D}_1 + \lambda_2\mathcal{D}_2$ satisfy the Jacobi identity. It can be shown [16] that if a system is bi-Hamiltonian it is integrable in Lioville's sense.

Starting with the works of Gel'fand and Dickey in 1975 [17], Adler in 1979 [18] and many others, algebraic developments started to take place. The key role played by the Lax operator L, in obtaining the conserved charges H_n, the Hamiltonian structures, the hierarchy of equations that share H_n was then revealed. In the next sections we will introduce and apply some of these techniques in the dispersionless situation.

3. Dispersionless Limit

We have seen that solitons preserve their shape and speed after collision. The soliton solution has a nondispersive nature. This is so not because dispersion effects are absent but because there is a compensation by the nonlinearities of the system. Let us look at the KdV equation (2.2) more closelly. If we eliminate the nonlinear term in (2.2) we get the linear dispersive equation

$$u_t = u_{xxx} \tag{3.1}$$

which admits the solution

$$u(x,t) = \int dk\, A(k) \mathrm{e}^{i(kx - w(k)t)} \tag{3.2}$$

This is a pure dispersive solution. In Figure 3 we see that a initial configuration at $t = 0$ will disperse as time goes on. Eliminating the dispersive term we get the pure nonlinear equation

$$u_t = u u_x \tag{3.3}$$

It can be easily checked by substitution that

$$u(x,t) = f(x - ut) \tag{3.4}$$

with f arbitrary, satisfies (3.3). From this solution we conclude that the velocity of a point of the wave, with constant amplitude u, is proportional to its amplitude leading to the "breaking" of the wave, as shown in Figure 3. The wave also develops discontinuities (indicated by the vertical dashed line in Figure 3) in its evolution. The "miracle" of the soliton solution is due to a balance between the dispersion and the breaking of the wave, both phenonema placed together lead to the wave profile to propagate without changing its shape.

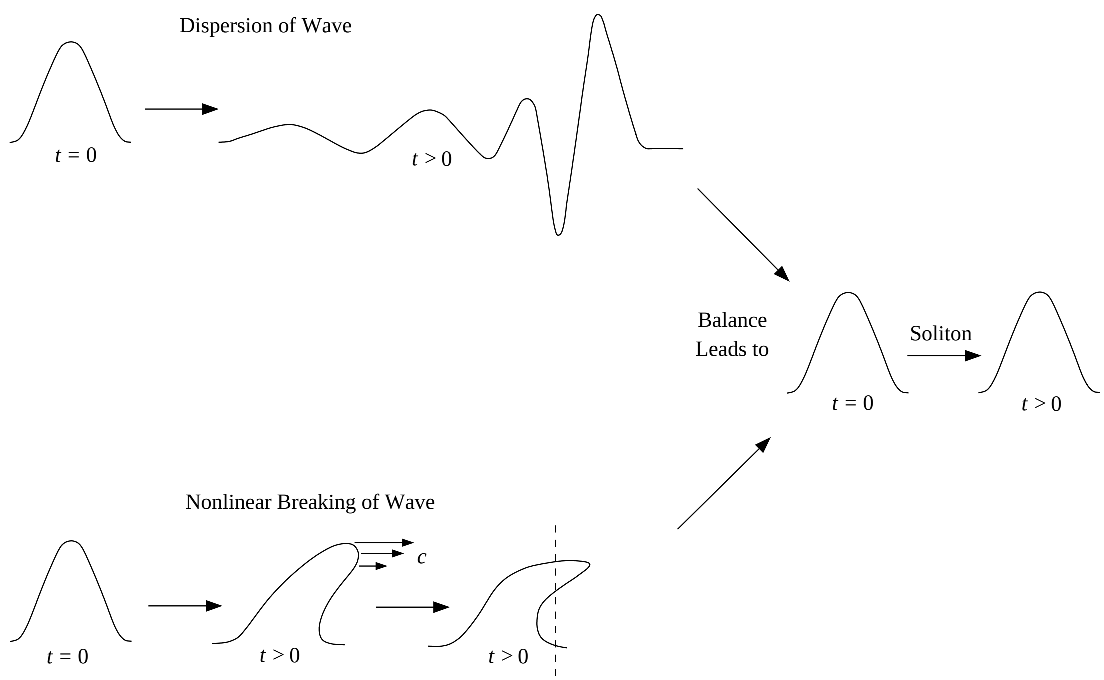

Figure 3. The balance effects of dispersion and breaking in a soliton.

Equation (3.3) is called the dispersionless KdV or Riemann equation [19]. The interesting fact is that this equation is a integrable Hamiltonian system. We will return to study this equation in the next section but for the moment let us analyse how we get dispersionless equations. Dispersionless equations can be obtained by construction or as a quasi-classical limit of integrable ones [20] . In the latter case we make the scaling $\frac{\partial}{\partial t} \to \alpha \frac{\partial}{\partial t}$, $\frac{\partial}{\partial x} \to \alpha \frac{\partial}{\partial x}$ and take the limit $\alpha \to 0$. For the KdV equation (2.2) (we will change the constant factors on it, so instead of (2.5) we have $L = \partial^2 + u$ and $B = \partial^3 + \frac{3}{4}(\partial u + u\partial)$)

$$\begin{array}{ccc} 4u_t = u_{xxx} + 6uu_x & \Rightarrow & 4\alpha u_t = \alpha^3 \cancelto{0}{u_{xxx}} + 6\alpha u_x u \\ \Downarrow & & u_t = \frac{3}{2} u u_x \\ \text{KdV} & & \Downarrow \\ & & \text{Riemann} \end{array}$$

This is like the WKB aproximation in quantum mechanics and we will use it as our guideline [20].

Dispersionless integrable systems were introduced by Lebedev and Manin [21] and Zakharov [22], and although interesting on their own started to appear recently in developments in low-dimensional quantum field theory. It has been shown that there is a connection between 2-dimensional field theories and integrable equations of hydrodynamical type [23-25] (which are dispersionless systems). In 2-dimensional topological field theories [26] we are interested in calculating, from the partition function

$$Z_M = \int [d\phi]\, \mathrm{e}^{-S[\phi]} \tag{3.5}$$

the correlation functions

$$\langle \phi_\alpha(x)\phi_\beta(y)\cdots\rangle_M = \langle \phi_\alpha\phi_\beta\cdots\rangle_M \tag{3.6}$$

which depend only on the topology of the manifold M. The 2-point and 3-point correlation functions are given respectively by [27]

$$\begin{aligned} \langle \phi_\alpha\phi_\beta\rangle &= \eta_{\alpha\beta} = \text{nondegenerate constant} \\ \langle \phi_\alpha\phi_\beta\phi_\gamma\rangle &= c_{\alpha\beta\gamma}(t) = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \qquad \alpha,\beta,\gamma = 1,2,\ldots,n \end{aligned} \tag{3.7}$$

where $t = (t^1, t^2, \ldots, t^n)$ are the coupling constants and $F(t)$ is the free energy. The correlations (3.7) define a commutative and associative algebra (with an identity)

$$e_\alpha \circ e_\beta = c^\gamma_{\alpha\beta} e_\gamma \tag{3.8}$$

with e_α defining a basis for the algebra. The associativity of the algebra, $(e_\alpha \circ e_\beta)\circ e_\gamma = e_\alpha \circ (e_\beta \circ e_\gamma)$, gives

$$\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\lambda}\, \eta^{\lambda\mu}\, \frac{\partial^3 F(t)}{\partial t^\gamma \partial t^\delta \partial t^\mu} = \frac{\partial^3 F(t)}{\partial t^\gamma \partial t^\beta \partial t^\lambda}\, \eta^{\lambda\mu}\, \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\delta \partial t^\mu} \tag{3.9}$$

These are the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations [26,27] and can be identified with equations of hydrodynamic type. So, solutions of hydrodynamic equation can be identified with particular solutions of the topological field theory [25].

4. Two-Component Hyperbolic Systems

In a series of papers [28-31] Nutku and collaborators started to study dispersionless systems of equations that are in-between the simple Riemann equation [19] and the more general equations of hydrodynamic type [24]

Riemann

Two Component Hyperbolic

Equations of Hydrodynamic Type

$$u^i_t = V^{ij}(u) u^j_x \quad i, j = 1, \ldots, n$$

In Figure 4 we can find a chart with the main equations, of the two-component hyperbolic system type, studied on these papers.

A wealth of results concerning the integrability of these systems were revealled. Infinitely many conservation laws and multi-Hamiltonian structures were obtained. In this section we will be interested in reproduce some of these results from an algebraic point of view. In order to achive this goal we must understand the Lax representation for these systems.

4.1 Riemann Equation

The Riemann equation

$$\boxed{u_t = \tfrac{3}{2} u u_x} \tag{4.1}$$

is the prototype for the hyperbolic systems. We address the following question: Is there a Lax representation for (4.1)? Yes, and we can obtain it performing the semiclassical limit [20] explained in Section 3. So, if the KdV equation goes to the Riemann equation (4.1) in the semiclassical limit, the Lax operator $L = \partial^2 + u$ and $B = \partial^3 + \frac{3}{4}(\partial u + u\partial)$ goes to the polynomials in the variable p

$$\begin{aligned} E &= p^2 + u \\ M &= p^3 + \frac{3}{2} u p \end{aligned} \tag{4.2}$$

and the Lax representation (2.4) goes to

$$\frac{\partial E}{\partial t} = \{M, E\} \tag{4.3}$$

called dispersionless Lax representation (note the resemblance when we pass from quantum to classical mechanics doing $\partial \to p$ and $[\ ,\ ] \to \{\ ,\ \}$). Here

$$\boxed{\{A(x,p), B(x,p)\} = \frac{\partial A}{\partial p}\frac{\partial B}{\partial x} - \frac{\partial B}{\partial p}\frac{\partial A}{\partial x}} \tag{4.4}$$

is the dispersionless Poisson bracket [21,22]. So, if we substitute (4.2) in (4.3) we get (4.1).

Two-Component Hyperbolic

$$\boxed{\begin{aligned} u_t &= H_{uv} u_x + H_{vv} v_x \\ v_t &= H_{uu} u_x + H_{uv} v_x \end{aligned}}^{\,*}$$

$$\longrightarrow \quad u_t = \frac{3}{2} u u_x \;^{\dagger}$$

Riemann

$\swarrow$ $\qquad\downarrow\qquad$ $\searrow$

$H(u,v) = \frac{u}{v} + \frac{v}{u}$

$\downarrow$

$$\boxed{\begin{aligned} u_t &= \left(\frac{1}{u^2} + \frac{1}{v^2}\right) u_x - \frac{2u}{v^3} v_x \\ v_t &= \left(\frac{1}{v^2} + \frac{1}{u^2}\right) v_x - \frac{2v}{u^3} u_x \end{aligned}}^{\,\ddagger}$$

Born−Infeld

$\downarrow$

$\widetilde{u} = -(u^{-2} + v^{-2})$

$\widetilde{v} = \frac{1}{2} uv$

Verosky Transformation

$\downarrow$

$$\gamma = -1 \qquad \longleftarrow$$

Chaplygin Gas

$H(u,v) = -\left(\frac{1}{2} u^2 v + F(v)\right)$

$f(v) = F''(v)$

$\downarrow$

$$\boxed{\begin{aligned} u_t + u u_x + f(v) v_x &= 0 \\ v_t + (uv)_x &= 0 \end{aligned}}$$

Gas Dynamics

$\downarrow$

$F(v) = \frac{v^\gamma}{\gamma(\gamma-1)} \qquad \gamma \geq 2$

$\downarrow$

$$\boxed{\begin{aligned} u_t + u u_x + v^{\gamma-2} v_x &= 0 \\ v_t + (uv)_x &= 0 \end{aligned}}^{\,\dagger}$$

Polytropic Gas

$\downarrow$

$$\gamma = 2$$

Shallow Water
(Dispersionless TB)

$H(u,v) = \frac{1}{2} u^2 + \widetilde{F}(v)$

$\sigma(v) = \widetilde{F}'(v)$

$\downarrow$

$$\boxed{\begin{aligned} u_t &= [\sigma(v)]_x \\ v_t &= u_x \end{aligned}}$$

Elastic Media

$\downarrow$

$\widetilde{F}(v) = \frac{v^{\gamma-1}}{(\gamma-1)(\gamma-2)}$

$\downarrow$

$$\boxed{\begin{aligned} u_t &= v^{\gamma-3} v_x \\ v_t &= u_x \end{aligned}}^{\,\dagger}$$

Polytropic Elastic
Media

$*$ bi−Hamiltonian if $\frac{H_{uu}}{H_{vv}} = \frac{\lambda(u)}{\mu(v)}$

† quadri−Hamiltonian

‡ six−Hamiltonian

Figure 4. Two-component hyperbolic equations.

From now on we will apply some of the techniques described in [17] and [18] in a very informal way, since we want to give only a flavor of how the "machinery" works.

Let us calculate the square root of E in (4.2). So, we write the Laurent polynomial

$$E^{1/2} = p + a_0 + a_1 p^{-1} + a_2 p^{-2} + a_3 p^{-3} + \cdots \tag{4.5}$$

and from $E = E^{1/2}E^{1/2}$ we obtain $a_0, a_1, a_2, a_3, \ldots$, or equivalently, we perform a series expansion for $p \to \infty$

$$E^{1/2} = \left[p^2\left(1+\frac{u}{p^2}\right)\right]^{1/2} \overset{p\to\infty}{=} p + \frac{1}{2}up^{-1} - \frac{1}{8}u^2p^{-3} + \frac{1}{16}u^3p^{-5} - \frac{5}{128}u^4p^{-7} + \cdots \tag{4.6}$$

Now we calculate $E^{3/2} = E^{1/2}E$, $E^{5/2} = E^{1/2}E^2$ and so on

$$\begin{aligned} E^{3/2} =& p^3 + \frac{3}{2}up + \frac{3}{8}u^2p^{-1} + \ldots \\ E^{5/2} =& p^5 + \frac{5}{2}up^3 + \frac{15}{8}u^2p + \frac{5}{16}u^3p^{-1} + \cdots \\ &\vdots \end{aligned} \tag{4.7}$$

The set of general Laurent polynomial $A = \sum_{i=-\infty}^{+\infty} a_i p^i$ gives rise to an associative algebra $g = \{A\}$. This algebra can be written as a direct sum $g = g_+ \oplus g_-$, where $g_+ = \{A_+\}$ and $g_- = \{A_-\}$ with $A_+ = \sum_{i\geq 0} a_i p^i$ and $A_- = \sum_{i<0} a_i p^i$, respectively. We can recognize M in (4.2) as

$$M = (E^{3/2})_+ \tag{4.8}$$

In fact, from (4.3) we are motivated to write

$$\begin{aligned} \frac{\partial E}{\partial t} &= \{(E^{1/2})_+, E\} \quad \Rightarrow \qquad u_t = u_x \\ \frac{\partial E}{\partial t} &= \{(E^{3/2})_+, E\} \quad \Rightarrow \quad u_t = \frac{3}{2}uu_x \\ \frac{\partial E}{\partial t} &= \{(E^{5/2})_+, E\} \quad \Rightarrow \quad u_t = \frac{15}{8}u^2u_x \\ &\vdots \end{aligned} \tag{4.9}$$

and we have a hierarchy of equations. We call it dispersionless KdV (or Riemann) hierarchy and we write

$$\frac{\partial E}{\partial t_k} = \{(E^{\frac{2k+1}{2}})_+, E\}, \quad k = 0, 1, 2, 3, \ldots \tag{4.10}$$

treating u as a function of $k+1$ variables

$$u = u(x, t_0, t_1, t_2, \ldots) \tag{4.11}$$

For each t_k we have what is called a flow and it can be shown that they commute

$$\frac{\partial^2 E}{\partial t_\ell \partial t_k} = \frac{\partial^2 E}{\partial t_k \partial t_\ell} \tag{4.12}$$

consequently, the whole set of equations (4.9) is integrable since, as we have already pointed out, the Riemann equation is an integrable Hamiltonian system (all the equations in (4.9) share the same set of conserved charges).

The Riemann equation can be put in the form

$$u_t = \frac{3}{2} u u_x = \frac{3}{4} (u^2)_x \tag{4.13}$$

It follows that the quantity $H \propto \int dx\, u^2$ is conserved. In fact $\int dx\, u^n$ are conserved as we can show explicitly. These conserved charges can also be obtained from E. Let be A any general Laurent polynomial

$$A = \cdots + a_{-1} p^{-1} + \cdots \tag{4.14}$$

following [18] we introduce the Adler's trace as

$$\mathrm{Tr} A = \int dx\, \mathrm{Res} A = \int dx\, a_{-1} \tag{4.15}$$

which satisfies the usual relation $\mathrm{Tr}AB = \mathrm{Tr}BA$. From (4.6) and (4.7) we see that

$$\begin{aligned}
\mathrm{Tr}E^{1/2} =& \frac{1}{2} \int dx\, u \\
\mathrm{Tr}E^{3/2} =& \frac{3}{8} \int dx\, u^2 \\
\mathrm{Tr}E^{5/2} =& \frac{5}{16} \int dx\, u^3 \\
& \vdots
\end{aligned}$$

and we have

$$H_n = \frac{2}{n} \underbrace{\mathrm{Tr}E^{n/2}}_{p \to \infty} \tag{4.16}$$

whit $\dot{H}_n = 0$. From a Hamiltonian point of view the Riemann equation is a quadri-Hamiltonian system [30]. There are Hamiltonian operators $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ which are compatible and another Hamiltonian operator $\mathcal{E}$ which is compatible only with $\mathcal{D}_1$. We can write

$$\boxed{u_t = \mathcal{D}_1 \frac{\delta H_5}{\delta u} = \mathcal{D}_2 \frac{\delta H_3}{\delta u} = \frac{3}{4} \mathcal{D}_3 \frac{\delta H_1}{\delta u} = \frac{35}{8} \mathcal{E} \frac{\delta H_9}{\delta u}} \tag{4.17}$$

where

$$
\begin{aligned}
H_1 &= \int dx\, u\,, & \mathcal{D}_1 &= 2\partial \\
H_3 &= \frac{1}{4}\int dx\, u^2\,, & \mathcal{D}_2 &= u\partial + \partial u \\
H_5 &= \frac{1}{8}\int dx\, u^3\,, & \mathcal{D}_3 &= u^2\partial + \partial u^2 \\
H_9 &= \frac{7}{128}\int dx\, u^5\,, & \mathcal{E} &= \partial \tfrac{1}{u_x}\partial \tfrac{1}{u_x}\partial
\end{aligned}
\tag{4.18}
$$

Hamiltonian structures can also be obtained from the Lax operator L (E in the dispersionless case). They are the symplectic structures of Kostant-Kirillov [32] on the orbits of the coadjoint representation of Lie groups [18,33]. For dispersionless equations the corresponding Lie algebra is given by the associative algebra of Laurent polynomials endowed with the bracket (4.4). For the KdV equation the Lie algebra is given by the algebra of the pseudo-differential operators with the usual commutator. Following this scheme the Hamiltonian structures $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ can be derived (see [4] for details) while we were not able to obtain $\mathcal{E}$ from this scheme.

4.2 Polytropic Gas Equation

We will try to apply the results of the last Section to some others dispersionless equations, such as the ones in the chart of Figure 4. The polytropic gas dynamics equation

$$
\boxed{\begin{aligned} u_t + uu_x + v^{\gamma-2}v_x &= 0 \\ v_t + (uv)_x &= 0 \end{aligned}} \qquad \gamma \geq 2 \tag{4.19}
$$

was studied from a Hamiltonian point of view in [30]. In (4.19) u is the velocity of the fluid, v is its density, $f = v^{\gamma-2}$ and is related to the pressure ($f(v) = \frac{p'(v)}{v}$) and γ is the ratio of specific heats (we call an ideal gas polytropic if the specific heats are constant over a large range of temperature).

The first step will be to derive a Lax representation for (4.19). We get a hint if we consider $\gamma = 2$ in (4.19). In this case we have the shallow water equation [19] also known as the irrotational Benney equation [34]. Even though we do not know the dispersive system which originates (4.19) for any γ we do know it for the case $\gamma = 2$. This is the dispersive shallow water [35] equation, also called the two boson equation in field theory

$$
\begin{aligned}
\frac{\partial J_0}{\partial t} &= (2J_1 + J_0^2 + J_0')' \\
\frac{\partial J_1}{\partial t} &= (2J_0 J_1 + J_1')'
\end{aligned}
\tag{4.20}
$$

This equation has the following nonstandard Lax representation [36,37]

$$\begin{aligned} L =&\partial - J_0 + \partial^{-1} J_1 \\ \frac{\partial L}{\partial t} =&[L, (L^2)_{\geq 1}] \end{aligned} \tag{4.21}$$

where $(L^2)_{\geq 1}$ stands for the purely nonnegative (without p^0 terms) part of the polynomial in p and $J_0 \propto u$, $J_1 \propto v$ are the two bosons fields. Now, if we perform the semiclassical limit and do the appropriate identifications (4.20) yields (4.19) for $\gamma = 2$ and from (4.21) we get the following dispersionless Lax representation

$$\begin{aligned} L =&p + u + vp^{-1} \\ \frac{\partial L}{\partial t} =&\frac{1}{2}\{(L^2)_{\geq 1}, L\} \end{aligned} \tag{4.22}$$

For any γ we can use (4.22) as an ansatz to obtain the dispersionless Lax representation for (4.19) and it reads [5]

$$\boxed{\begin{aligned} L =&p^{\gamma-1} + u + \frac{v^{\gamma-1}}{(\gamma-1)^2} p^{-(\gamma-1)} \\ \frac{\partial L}{\partial t} =&\frac{(\gamma-1)}{\gamma}\left\{\left(L^{\frac{\gamma}{\gamma-1}}\right)_{\geq 1}, L\right\} \end{aligned}} \tag{4.23}$$

In [30] two sets of conserved charges were derived for (4.19) when $\gamma \neq 2$. So, if (4.23) is really the correct Lax pair it must somehow provide both sets accordingly to the algebraic scheme described in the last section. In fact, since L has singularities in $p = 0$ and $p = \infty$ we can expand $L^{\frac{1}{\gamma-1}}$ in powers of p in the two following ways

$$\begin{aligned} L^{\frac{1}{\gamma-1}} = p\Bigg\{ & 1 + \frac{1}{\gamma-1}\left[up^{-(\gamma-1)} + \frac{v^{\gamma-1}}{(\gamma-1)^2} p^{-2(\gamma-1)}\right] + \frac{(2-\gamma)}{2(\gamma-1)^2}\Big[\cdots\Big]^2 + \\ & + \frac{(2-\gamma)(3-2\gamma)}{6(\gamma-1)^3}\Big[\cdots\Big]^3 + \cdots \Bigg\} \qquad p \to \infty \end{aligned} \tag{4.24a}$$

$$\begin{aligned} L^{\frac{1}{\gamma-1}} = \frac{vp^{-1}}{(\gamma-1)^{\frac{2}{\gamma-1}}}\Bigg\{ & 1 + \frac{1}{\gamma-1}\left[(\gamma-1)^2 v^{-(\gamma-1)}\left(up^{(\gamma-1)} + p^{2(\gamma-1)}\right)\right] + \\ & + \frac{(2-\gamma)}{2(\gamma-1)^2}\Big[\cdots\Big]^2 + \frac{(2-\gamma)(3-2\gamma)}{6(\gamma-1)^3}\Big[\cdots\Big]^3 + \cdots \Bigg\} \qquad p \to 0 \end{aligned} \tag{4.24b}$$

So, the first set of charges follows from

$$\boxed{\overline{\mathcal{H}}_n = \underbrace{\mathrm{Tr}\, L^{n+\frac{\gamma-2}{\gamma-1}}}_{p\to\infty} = \int dx\, \overline{H}_n} \qquad n = 0, 1, 2, 3, \ldots \tag{4.25}$$

where the first densities are

$$\begin{aligned}
\overline{H}_0 =& \frac{(\gamma-2)}{(\gamma-1)} u \\
\overline{H}_1 =& \frac{(2\gamma-3)(\gamma-2)}{(\gamma-1)^2} \left(\frac{1}{2!} u^2 + \frac{1}{(\gamma-1)(\gamma-2)} v^{\gamma-1} \right) \\
\overline{H}_2 =& \frac{(3\gamma-4)(2\gamma-3)(\gamma-2)}{(\gamma-1)^3} \left(\frac{1}{3!} u^3 + \frac{1}{(\gamma-1)(\gamma-2)} u v^{\gamma-1} \right) \\
&\vdots \\
\overline{H}_n =& (n+1)! \mathrm{C}_{n+1}^{\frac{(n+1)(\gamma-1)-1}{(\gamma-1)}} H_{n+1}
\end{aligned} \tag{4.26}$$

and

$$H_n = \sum_{m=0}^{\left[\frac{n}{2}\right]} \left(-\prod_{k=0}^{m} \frac{1}{k(\gamma-1)-1} \right) \frac{u^{n-2m}}{m!(n-2m)!} \frac{v^{m(\gamma-1)}}{(\gamma-1)^m} \tag{4.27}$$

which are the first set of charges obtained in [30]. The second set follows from

$$\boxed{\overline{\widetilde{\mathcal{H}}}_n = \underbrace{\mathrm{Tr}\, L^{n+\frac{1}{\gamma-1}}}_{p\to 0} = \int dx\, \overline{\widetilde{H}}_n} \qquad n = 0, 1, 2, 3, \ldots \tag{4.28}$$

and the first densities are

$$\begin{aligned}
\overline{\widetilde{H}}_0 =& (\gamma-1)^{-\frac{2}{\gamma-1}} v \\
\overline{\widetilde{H}}_1 =& (\gamma-1)^{-\frac{2}{\gamma-1}} \frac{\gamma}{(\gamma-1)} u v \\
\overline{\widetilde{H}}_2 =& (\gamma-1)^{-\frac{2}{\gamma-1}} \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} \left(\frac{1}{2!} u^2 v + \frac{v^\gamma}{\gamma(\gamma-1)} \right) \\
&\vdots \\
\overline{\widetilde{H}}_n =& \frac{n!}{(\gamma-1)^{\frac{2}{\gamma-1}}} \mathrm{C}_n^{\frac{n(\gamma-1)+1}{(\gamma-1)}} \widetilde{H}_n
\end{aligned} \tag{4.29}$$

where

$$\widetilde{H}_n = \sum_{m=0}^{\left[\frac{n}{2}\right]} \left(\prod_{k=0}^{m} \frac{1}{k(\gamma-1)+1} \right) \frac{u^{n-2m}}{m!(n-2m)!} \frac{v^{m(\gamma-1)+1}}{(\gamma-1)^m} \tag{4.30}$$

is the second set of charges obtained in [30]

In (4.23) $L^{\frac{1}{\gamma-1}}$ was expanded in $p=\infty$, a expansion around $p=0$ provides a second consistent dispersionless Lax equation

$$\frac{\partial L}{\partial t} = \left\{ \left(L^{\frac{\gamma-2}{\gamma-1}} \right)_{\leq 0}, L \right\} \tag{4.31}$$

which yields (with the proper rescaling) the equations

$$\begin{aligned} u_t &= v^{\gamma-3} v_x \\ v_t &= u_x \end{aligned} \tag{4.32}$$

From the chart in Figure 4 we recognize this equations as the polytropic elastic media equation.

4.3 Born-Infeld Equation

With the Lax representation for the polytropic gas, obtained in the last section, we can get a Lax representation for the Born-Infeld equation given in the chart of Figure 4

$$\boxed{\begin{aligned} u_t &= \left(\frac{1}{u^2} + \frac{1}{v^2} \right) u_x - \frac{2u}{v^3} v_x \\ v_t &= \left(\frac{1}{v^2} + \frac{1}{u^2} \right) v_x - \frac{2v}{u^3} u_x \end{aligned}} \tag{4.33}$$

In (4.33) the Born-Infeld equation is expressed in the so called null coordinates version [31]. If we perform the transformation

$$\begin{aligned} u &= \phi_x \\ v &= -\frac{\phi_x}{\sqrt{1+\phi_x\phi_t}} \end{aligned} \tag{4.34}$$

we obtain the Born-Infeld equation written as a second-order equation in null coordinates

$$\phi_x^2 \phi_{tt} + \phi_t^2 \phi_{xx} - (4 + 2\phi_x\phi_t)\phi_{xt} = 0 \tag{4.35}$$

A Lax representation for (4.33) can be obtained as follows [6]. In the first place if we do the change of variables

$$\begin{aligned} \widetilde{u} &= -(u^2+v^2) \\ \widetilde{v} &= \frac{1}{2} uv \end{aligned} \tag{4.36}$$

called Verosky transformation [31], we will end up with the equation

$$\begin{aligned} \widetilde{u}_t + \widetilde{u}\widetilde{u}_x + \frac{\widetilde{v}_x}{\widetilde{v}^3} = 0 \\ \widetilde{v}_t + (\widetilde{u}\widetilde{v})_x = 0 \end{aligned} \tag{4.37}$$

known as the Chaplygin gas. In view of this it would be desirable to first obtain a Lax description of the Chaplygin gas like equations

$$\begin{aligned} \widetilde{u}_t + \widetilde{u}\widetilde{u}_x + \frac{\widetilde{v}_x}{\widetilde{v}^{\alpha+2}} =& 0\,, \quad \alpha \geq 1 \\ \widetilde{v}_t + (\widetilde{u}\widetilde{v})_x =& 0 \end{aligned} \tag{4.38}$$

This is indeed possible if we set $\gamma \to -\alpha$, $\alpha \geq 1$ in (4.23), so (4.38) can be obtained from

$$\begin{aligned} L =& p^{-(\alpha+1)} + \widetilde{u} + \frac{\widetilde{v}^{-(\alpha+1)}}{(\alpha+1)^2} p^{\alpha+1} \\ \frac{\partial L}{\partial t} =& \frac{(\alpha+1)}{\alpha} \left\{ \left(L^{\frac{\alpha}{\alpha+1}}\right)_{\leq 1}, L \right\} \end{aligned} \tag{4.39}$$

where $L^{\frac{1}{\alpha+1}}$ is expanded around $p = 0$ and $\left(L^{\frac{\alpha}{\alpha+1}}\right)_{\leq 1}$ is the polynomial in p that produces consistent equations, instead of the purely nonnegative polynomial used in (4.23). For $\alpha = 1$ the Lax operator

$$\boxed{\begin{aligned} L =& p^{-2} - \left(\frac{1}{u^2} + \frac{1}{v^2}\right) + \frac{1}{u^2 v^2} p^2 \\ \frac{\partial L}{\partial t} =& 2 \left\{ \left(L^{\frac{1}{2}}\right)_{\leq 1}, L \right\} \end{aligned}} \tag{4.40}$$

reproduces (4.33). Again, conserved charges follows from

$$\widetilde{\mathcal{H}}_n = \underbrace{\mathrm{Tr}\, L^{n-\frac{1}{2}}}_{p\to\infty} = \int dx\, \widetilde{H}_n \quad n = 0, 1, 2, 3, \ldots \tag{4.41}$$

and the first Born-Infeld charges are

$$\begin{aligned} \widetilde{H}_0 =& -uv \\ \widetilde{H}_1 =& \frac{1}{2}\left(\frac{u}{v} + \frac{v}{u}\right) \\ \widetilde{H}_2 =& -\frac{3}{4}\left(\frac{u}{2v^3} + \frac{3}{uv} + \frac{v}{2u^3}\right) \\ & \vdots \end{aligned} \tag{4.42}$$

and these are exactly the charges derived in [31]. Another set is obtained from

$$\mathcal{H}_n = \underbrace{\mathrm{Tr}\, L^{n+\frac{3}{2}}}_{p\to 0} = \int dx\, H_n \quad n = 0, 1, 2, 3, \ldots \tag{4.43}$$

and the first ones are

$$\begin{aligned} H_0 =& -\frac{3}{2}\left(\frac{1}{u^2} + \frac{1}{v^2}\right) \\ H_1 =& \frac{15}{8}\left(\frac{1}{u^4} + \frac{11}{6}\frac{1}{u^2v^2} + \frac{1}{v^4}\right) \\ H_2 =& -\frac{35}{16}\left(\frac{1}{u^6} - \frac{1}{u^4v^2} - \frac{1}{u^2v^4} + \frac{1}{v^6}\right) \\ &\vdots \end{aligned} \tag{4.44}$$

This is a new set of conserved charges, for the Born-Infeld equation (4.33), not found previously in [31].

5. Conclusions

We believe, from the results of the Section 4, that the study of dispersionless systems via a Lax representation is worthwhile. So, the search for a dispersionless Lax representation for the equations in the upper part of the chart in Figure 4 is being pursued. Also, the derivation of the multi-Hamiltonian structures of these systems, as described in [28-31], is under investigation following the coadjoint orbit method [32,33]. Another question that comes to mind is the dispersive generalization of these equations. Attempts in this direction can be found in [38].

Some topological equations are also related with the systems discussed here. For instance, the hyperbolic Monge-Ampère equation

$$U_{tt}U_{xx} - (U_{tx})^2 = -1 \tag{5.1}$$

may be related with the Born-Infeld equation as follows. If we perform the change of variables

$$\begin{aligned} a &= U_x \\ b &= U_t \end{aligned} \tag{5.2}$$

the Monge-Ampère equation can be written as a first order system

$$\begin{aligned} a_t &= b_x \\ b_t &= \frac{b_x^2 - 1}{a_x} \end{aligned} \tag{5.3}$$

and this equation can be related to the Chaplygin gas equation (4.37) through the following change of variables

$$
\begin{aligned}
\widetilde{u} &= -\frac{b_x}{a_x} \\
\widetilde{v} &= a_x
\end{aligned}
\tag{5.4}
$$

Thus, we can give a Lax description for the hyperbolic Monge-Ampère equation through the Lax representation derived in Section 4.3. Finally, the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations (3.9), for $n = 3$, with

$$
F(t^1, t^2, t^3) = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 + f(t^2, t^3) \tag{5.5}
$$

where $t^2 \equiv x$ and $t^3 \equiv t$, yields the third order Monge-Ampère equation

$$
f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt} \tag{5.6}
$$

This equation is a bi-Hamiltonian system and has a matrix Lax representation. It is then possible to generate a whole set of nonlocal charges much like the nonlinear sigma model (details are given in [7]). It is likely that a dispersionless sort of Lax representation for (5.6) may exist.

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