Maximal Monotone Operators in General Banach Spaces

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September 16, 2016

Motivation

- Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \| := \sqrt{\langle \cdot, \cdot \rangle}$.
- Dual of $\mathcal{H}$:

$$\mathcal{H}^* := \{\varphi : \mathcal{H} \to \mathbb{R} \mid \varphi \text{ is linear and continuous}\}.$$

- $\mathcal{H}^*$ is a vector space. (Dual) norm in $\mathcal{H}^*$:

$$\|\varphi\| := \sup_{\|x\| \leq 1} |\varphi(x)|.$$

Motivation

- For each $z \in \mathcal{H}$, define

$$\varphi_z : \mathcal{H} \to \mathbb{R}, \quad \varphi_z(x) = \langle x, z \rangle \quad \forall x \in \mathcal{H}.$$

  Then, $\varphi_z \in \mathcal{H}^*$ and $\|\varphi_z\| = \|z\|$.

- **Theorem (Riesz)**

  For each $\varphi \in \mathcal{H}^*$, there is a unique $z \in \mathcal{H}$ such that

$$\varphi(x) = \langle x, z \rangle \qquad \forall x \in \mathcal{H}.$$

  In addition, $\|\varphi\| = \|z\|$.

Motivation

Theorem (Lax-Milgram)

Assume that $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a bilinear mapping, for which there exist constants $\alpha, \beta > 0$ such that

$$|B(x,y)| \leq \alpha\|x\|\|y\|, \qquad B(x,x) \geq \beta\|x\|^2 \qquad \forall x, y \in \mathcal{H}.$$

Then, for each $\varphi \in \mathcal{H}^*$, there exist a unique $z \in \mathcal{H}$ such that

$$B(x,z) = \varphi(x) \qquad \forall x \in \mathcal{H}.$$

Motivation

- **Definition**

  A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ is

  1. *positive* (monotone) if
  $$\langle Ax, x \rangle \geq 0 \qquad \forall x \in \mathcal{H}.$$
  2. *strictly positive* (strongly monotone) if there exists $\beta > 0$ such that
  $$\langle Ax, x \rangle \geq \beta \|x\|^2 \qquad \forall x \in \mathcal{H}.$$

Motivation

Theorem

If $A : \mathcal{H} \to \mathcal{H}$ is strictly positive then

$$R(A) = \mathcal{H}.$$

In particular, for each $u \in \mathcal{H}$ there exists a unique $z \in \mathcal{H}$ solving the linear equation

$$Az = u.$$

Motivation

Corollary

If $A : \mathcal{H} \to \mathcal{H}$ is positive then

$$R(A+I) = \mathcal{H}.$$

In particular, for each $u \in \mathcal{H}$ there exists a unique $z \in \mathcal{H}$ solving the regularized equation

$$(A+I)z = u.$$

Proof.

Use the previous theorem for $A+I$. □

Motivation

Theorem

If $f : \mathcal{H} \to \mathbb{R}$ is differentiable and convex then

$$R(\nabla f + I) = \mathcal{H}.$$

In particular, for each $u \in \mathcal{H}$ there exists a unique $z \in \mathcal{H}$ solving the nonlinear equation

$$(\nabla f + I)z = u.$$

Proof.

Let $u \in \mathcal{H}$ and define

$$z = \arg\min_{x \in \mathcal{H}} f(x) + \frac{1}{2}\|x - u\|^2.$$

Motivation

Then,

$$0 = \nabla f(z) + z - u$$

and so

$$u = \nabla f(z) + z = (\nabla f + I)z,$$

i.e.,

$$z = (\nabla f + I)^{-1}u.$$

Motivation

Proposition

Let $f : \mathcal{H} \to \mathbb{R}$ be differentiable. Then, f is convex if and only if

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0 \qquad \forall x, y \in \mathcal{H}.$$

In other words, f is convex if and only if ∇f is monotone.

Maximal Monotone Operators

- Let X be a Banach space with dual X^*.
- Let $\langle \cdot, \cdot \rangle$ denote the duality product in $X \times X^*$:

$$\langle x, x^* \rangle = \langle x^*, x \rangle := x^*(x) \qquad \forall (x, x^*) \in X \times X^*.$$

- Norm (dual) in X^*:

$$\|x^*\| := \sup_{\|x\| \leq 1} |\langle x, x^* \rangle|.$$

Maximal Monotone Operators

Definition

Let $T : X \rightrightarrows X^*$ be a set-valued mapping.

- *domain* of T:

$$D(T) := \{x \in X \mid T(x) \neq \emptyset\}.$$

- *range* of T:

$$R(T) := \bigcup_{x \in X} T(x).$$

- *graph* of T:

$$G(T) := \{(x, x^*) \in X \times X^* \mid x^* \in T(x)\}.$$

Maximal Monotone Operators

Definition

A set-valued mapping $T : X \rightrightarrows X^*$ is

- *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0 \qquad \forall\, x^* \in T(x), \quad y^* \in T(y).$$

- *maximal monotone* if it is monotone and maximal in the family of all monotone operators from X to X^* (with the order of inclusion).

Important contributions to monotone operator theory: Kato, Minty, Brezis, Browder, Rockafellar, Gossez, Borwein, Simons, Phelps, Attouch, Thera, Burachik, Iusem, Svaiter, Zalinescu, Haraux, and many others ...

Maximal Monotone Operators

Theorem (G. Minty, 1962)

Let $T : \mathcal{H} \rightrightarrows \mathcal{H}$ be monotone. Then, T is maximal monotone if and only if

$$R(T + I) = \mathcal{H}.$$

Moreover, if T is maximal monotone then $(T + I)^{-1} : \mathcal{H} \to \mathcal{H}$ is everywhere defined and firmly nonexpansive.

Maximal Monotone Operators

Resolvent operator of T:

$$R_\lambda := (\lambda T + I)^{-1}.$$

Moreau-Yosida regularization of T:

$$T_\lambda := \frac{I - R_\lambda}{\lambda}.$$

Example of Maximal Monotone Operators

Definition

An extended-real valued function $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is *convex* if

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y) \qquad \forall x, y \in X, \quad \forall \lambda \in [0,1].$$

Moreover, f is *proper* is $f \neq \infty$.

Example of Maximal Monotone Operators

Definition

The *epigraph* of f is:

$$\text{epi}(f) := \{(x, \mu) \in X \times \mathbb{R} \mid \mu \geq f(x)\}.$$

f is said to be *lower semicontinuous* (lsc) if $\text{epi}(f)$ is closed in $X \times \mathbb{R}$.

Example of Maximal Monotone Operators

Definition

Let $f : X \to \overline{\mathbb{R}}$ be proper convex and lower semicontinuous (cplsc). The *subdifferential* of f at x is defined by

$$\partial f(x) := \{x^* \in X^* \mid f(y) \geq f(x) + \langle y - x, x^* \rangle \quad \forall y \in X\}.$$

Maximality of the subdifferential: Rockafellar (1976), Simons (1991), M.A. and Svaiter (2007).

Many other examples of maximal monotone operators in Convex Programming, Saddle-Point Problems, Equilibrium Theory, PDEs, etc.

The Fenchel-Legendre Conjugation

Definition

Let $f : X \to \overline{\mathbb{R}}$ be cplsc. The *Fenchel conjugate* of f is defined by

$$f^* : X^* \to \overline{\mathbb{R}}, \quad f^*(x^*) = \sup_{x \in X} \langle x, x^* \rangle - f(x).$$

Fenchel-Young inequality:

$$f(x) + f^*(x^*) \geq \langle x, x^* \rangle \quad \forall (x, x^*) \in X \times X^*.$$

Moreover,

$$f(x) + f^*(x^*) = \langle x, x^* \rangle \iff x^* \in \partial f(x).$$

Fitzpatrick Functions

Definition

Let $T : X \rightrightarrows X^*$ be maximal monotone. The *Fitzpatrick function* of T is defined by

$$\varphi_T : X \times X^* \to \overline{\mathbb{R}}, \quad \varphi_T(x, x^*) := \sup_{(y,y^*) \in G(T)} \langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle.$$

Fitzpatrick (1988):

$$\varphi_T(x, x^*) \geq \langle x, x^* \rangle \quad \forall (x, x^*) \in X \times X^*, \tag{1}$$

$$\varphi_T(x, x^*) = \langle x, x^* \rangle \iff x^* \in T(x). \tag{2}$$

Recent contributions: Borwein, Bauschke, Simons, Zalinescu, Thera, Burachik, M. A. and Svaiter, Ghoussoub, Visintin, $\cdots$

Type (D) and type (NI) maximal monotone operators

Definition (Gossez, 1971)

The Gossez's monotone closure of a maximal monotone operator $T : X \rightrightarrows X^*$, is $\overline{T}^g : X^{**} \rightrightarrows X^*$,

$$\overline{T}^g = \{(x^{**}, x^*) \in X^{**} \times X^* \mid \langle x^* - y^*, x^{**} - y \rangle \geq 0,\ \forall (y, y^*) \in G(T)\}.$$

A maximal monotone operator $T : X \rightrightarrows X^*$ is of (Gossez) type (D) if for any $(x^{**}, x^*) \in \overline{T}^g$, there exists a *bounded* net $\{(x_i, x_i^*)\}_{i \in I}$ in $G(T)$ which converges to (x^{**}, x^*) in the $\sigma(X^{**}, X^*) \times$strong topology of $X^{**} \times X^*$.

Tools from (classical) nonlinear functional analysis; Applications to nonlinear PDEs ...

Type (D) and type (NI) maximal monotone operators

Definition (Simons, 1996)

A maximal monotone operator $T : X \rightrightarrows X^*$ is of (Simons) type *(NI)* if

$$\inf_{(y,y^*)\in G(T)} \langle y^* - x^*, y - x^{**}\rangle \leq 0, \qquad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

Tools from minimax theory . . .
Simons (1996): Type (D)$\Rightarrow$ type (NI).

Type (D) and type (NI) maximal monotone operators

Theorem (M.A. and Svaiter, 2010)

Type (D) $\Longleftrightarrow$ *type (NI).*

Tools from Fitzpatrick theory and convex analysis . . .

Surjectivity of Perturbations

Theorem (M.A. and Svaiter, 2010)

A maximal monotone operator $T : X \rightrightarrows X^*$ *is of type (D) if and only if*

$$R(T + J_\varepsilon) = X^* \qquad \forall \varepsilon > 0,$$

where $J_\varepsilon := \partial_\varepsilon \frac{1}{2}\|\cdot\|^2$. Tools from Fitzpatrick theory and convex analysis . . .

Maximality of the Sum

Theorem (Rockafellar, 1970)

Let X be reflexive and $S, T : X \rightrightarrows X^*$ be maximal monotone operators. If

$$\mathit{int}(D(T)) \cap D(S) \neq \emptyset,$$

then $T + S$ is also maximal monotone.

Problem remains open in general Banach spaces (Rockafellar's conjecture); partial results by Borwein, Simons, Voisei, M. A. and Svaiter (2012), etc.

Maximality of the Sum

Definition (Attouch, Revalski, Thera, 1999)

Given two maximal monotone operators of type (D) $T_1, T_2 : X \rightrightarrows X^*$, their *variational sum* is defined as follows

$$T_1 \underset{v}{+} T_2 = \bigcap_{\mathcal{I}} \liminf_n (T_{1,\lambda_n} + T_{2,\mu_n}),$$

where

$$\mathcal{I} = \{(\lambda_n, \mu_n)_n \subset \mathbb{R}^2 \,:\, \lambda_n, \mu_n \geq 0,\ \lambda_n + \mu_n > 0,\ \lambda_n, \mu_n \to 0\}$$

and T_λ denotes the Moreau-Yosida regularization of T, for $\lambda \geq 0$.

The Variatinal Sum

Theorem (Bueno, Garcia and M.A., 2016)

Let X be a real Banach space and let $T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone operators of type (D). Then their variational sum $T_1 \underset{v}{+} T_2$ is representable.

Obrigado!

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