Automorphism groups of spaces with many symmetries

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Ultrahomogeneous structures

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How to construct ultrahomogeneous structures?

Setup

- Let $\mathcal{F}$ be a family of finite structures (a structure is a set A equipped with relations $R_1^A, R_2^A, \ldots$ and functions $f_1^A, f_2^A, \ldots$).
- Maps between structures in $\mathcal{F}$ are structure preserving monomorphisms.

Examples

Example

1. $\mathcal{F}$=finite linear orders
2. $\mathcal{F}$=finite graphs
3. $\mathcal{F}$=finite Boolean algebras
4. $\mathcal{F}$=finite metric spaces with rational distances

Fraïssé family-definition

A countable family $\mathcal{F}$ of finite structures is a Fraïssé family if:

1. (F1) (joint embedding property: JEP) for any $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ and monomorphisms from A into C and from B onto C;
2. (F2) (amalgamation property: AP) for $A, B_1, B_2 \in \mathcal{F}$ and any monomorphisms $\phi_1 \colon A \to B_1$ and $\phi_2 \colon A \to B_2$, there exist C, $\phi_3 \colon B_1 \to C$ and $\phi_4 \colon B_2 \to C$ such that $\phi_3 \circ \phi_1 = \phi_4 \circ \phi_2$;
3. (F3) (hereditary property: HP) if $A \in \mathcal{F}$ and $B \subseteq A$, then $B \in \mathcal{F}$.

Fraïssé limit-definition

A countable structure $\mathbb{L}$ is a Fraïssé limit of $\mathcal{F}$ if the following two conditions hold:

1. (L1) (universality) for any $A \in \mathcal{F}$ there is an monomorphism from A into $\mathbb{L}$;
2. (L2) (ultrahomogeneity) for any $A \in \mathcal{F}$ and any monomorphisms $\phi_1 \colon A \to \mathbb{L}$ and $\phi_2 \colon A \to \mathbb{L}$ there exists an isomorphism $h \colon \mathbb{L} \to \mathbb{L}$ such that $\phi_2 = h \circ \phi_1$;

Fraïssé limit-existence and uniqueness

Theorem (Fraïssé)

Let $\mathcal{F}$ be a countable Fraïssé family of finite structures. Then:

1. *there exists a Fraïssé limit of $\mathcal{F}$;*
2. *any two Fraïssé limits are isomorphic.*

Examples

Example

1. If $\mathcal{F}$=finite linear orders, then $\mathbb{L}$=rational numbers with the order
2. If $\mathcal{F}$=finite graphs, then $\mathbb{L}$=random graph
3. If $\mathcal{F}$=finite Boolean algebras, then $\mathbb{L}$=countable atomless Boolean algebra
4. $\mathcal{F}$=finite metric spaces with rational distances, then $\mathbb{L}$=rational Urysohn metric space

Lelek fan

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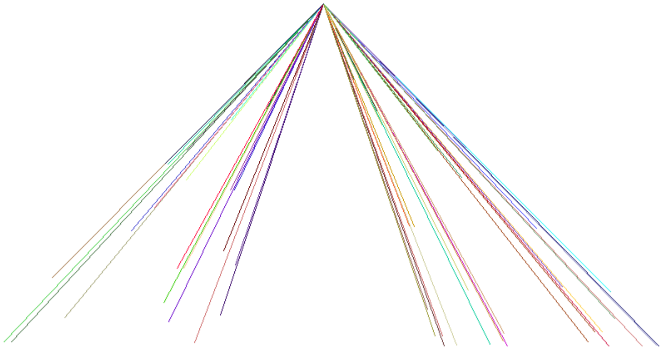
- subfan of the Cantor fan - subcontinuum of the Cantor fan that contains the top point, and is not homeomorphic to [0,1] or to a point
- Lelek fan L is a subfan of the Cantor fan with a dense set of endpoints in L

Lelek fan



About the Lelek fan

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About the Lelek fan

- Lelek fan was constructed by Lelek in 1960
- Lelek fan is unique: Any two subfans of the Cantor fan with dense set of endpoints are homeomorphic (Bula-Oversteegen 1990 and Charatonik 1989)

Endpoints of the Lelek fan

- The set of endpoints of the Lelek fan L is a dense G_δ set in L, it is a 1-dimensional space.

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- The set of endpoints of the Lelek fan L is a dense G_δ set in L, it is a 1-dimensional space.
- It is homeomorphic to: the complete Erdős space, the set of endpoints of the Julia set of the exponential map, the set of endpoints of the separable universal $\mathbb{R}$-tree. (Kawamura, Oversteegen, Tymchatyn)

The pseudo-arc

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The pseudo-arc is the unique hereditarily indecomposable chainable continuum.

- continuum = compact and connected metric space;
- indecomposable = not a union of two proper subcontinua;
- chainable = each open cover is refined by an open cover $U_1, U_2, \ldots, U_n$ such that for i, j, $U_i \cap U_j \neq \emptyset$ if and only if $|j - i| \leq 1$

A few properties of the pseudo-arc

Theorem (Bing)

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In the space of all subcontinua of either $[0,1]^n$, $n > 1$, or the Hilbert space, equipped with the Hausdorff metric, homeomorphic copies of the pseudo-arc form a dense G_δ set.

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Theorem (Bing, Moise)

The pseudo-arc is homogeneous.

Projective Fraïssé theory – setup

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2. A topological L-structure is a compact zero-dimensional second-countable space A equipped with closed relations $R_i^A, i \in I$ and continuous functions $f_j^A, j \in J$.

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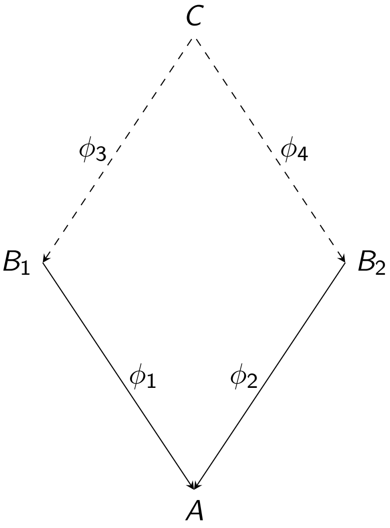
2. A topological L-structure is a compact zero-dimensional second-countable space A equipped with closed relations $R_i^A, i \in I$ and continuous functions $f_j^A, j \in J$.
3. Epimorphisms are continuous surjections preserving the structure.

Projective Fraïssé family – definition

A family $\mathcal{F}$ of finite topological L-structure is a projective Fraïssé family if:

1. (F1) (joint projection property: JPP) for any $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ and epimorphisms from C onto A and from C onto B;
2. (F2) (amalgamation property: AP) for $A, B_1, B_2 \in \mathcal{F}$ and any epimorphisms $\phi_1 \colon B_1 \to A$ and $\phi_2 \colon B_2 \to A$, there exist C, $\phi_3 \colon C \to B_1$ and $\phi_4 \colon C \to B_2$ such that $\phi_1 \circ \phi_3 = \phi_2 \circ \phi_4$.



amalgamation property

Projective Fraïssé limit – definition

A topological L-structure $\mathbb{L}$ is a projective Fraïssé limit of $\mathcal{F}$ if the following three conditions hold:

1. (L1) (projective universality) for any $A \in \mathcal{F}$ there is an epimorphism from $\mathbb{L}$ onto A;
2. (L2) (projective ultrahomogeneity) for any $A \in \mathcal{F}$ and any epimorphisms $\phi_1 \colon \mathbb{L} \to A$ and $\phi_2 \colon \mathbb{L} \to A$ there exists an isomorphism $h \colon \mathbb{L} \to \mathbb{L}$ such that $\phi_2 = \phi_1 \circ h$;
3. (L3) for any finite discrete topological space X and any continuous function $f \colon \mathbb{L} \to X$ there is an $A \in \mathcal{F}$, an epimorphism $\phi \colon \mathbb{L} \to A$, and a function $f_0 \colon A \to X$ such that $f = f_0 \circ \phi$.

Projective Fraïssé limit – existence and uniqueness

Theorem (Irwin-Solecki)

Let $\mathcal{F}$ be a countable projective Fraïssé family of finite structures. Then:

1. *there exists a projective Fraïssé limit of $\mathcal{F}$;*
2. *any two projective Fraïssé limits are isomorphic.*

Example

Let $\mathcal{F}$ be the family of all finite sets.

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The projective Fraïssé limit is the Cantor set.

Pseudo-arc from a projective Fraïssé limit, part 1

Let r be a binary relation symbol. Let $\mathcal{G}$ be the family of all finite linear reflexive graphs.

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Theorem (Irwin-Solecki)

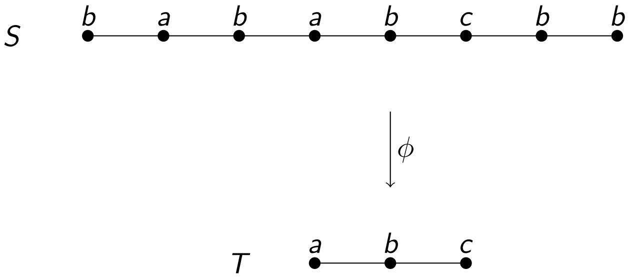
$\mathcal{G}$ is a projective Fraïssé family.

Epimorphisms

A continuous surjection $\phi\colon S \to T$ is an epimorphism iff

$$r^T(a, b) \iff \exists c, d \in S\left(\phi(c) = a, \phi(d) = b, \text{ and } r^S(c, d)\right).$$

An example of an epimorphism



Pseudo-arc from a projective Fraïssé limit, part 2

Lemma (Irwin-Solecki)

Let $\mathbb{P}$ be the projective Fraïssé limit of $\mathcal{G}$. Then $r^{\mathbb{P}}$ is an equivalence relation such that each equivalence class has at most two elements.

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Theorem (Irwin-Solecki)

$\mathbb{P}/r^{\mathbb{P}}$ is the pseudo-arc.

Lelek fan from a projective Fraïssé limit, part 1

Let R be a binary relation symbol. Let $\mathcal{F}$ be the family of all finite reflexive fans.

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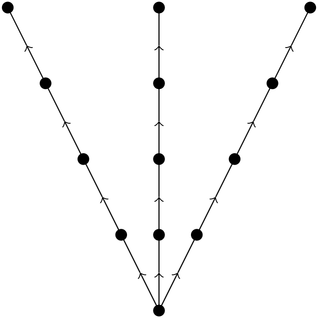
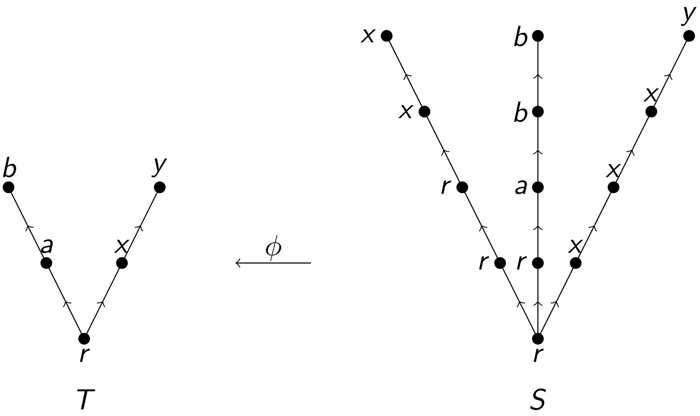


Theorem

$\mathcal{F}$ is a projective Fraïssé family.

An example of an epimorphism



Lelek fan from a projective Fraïssé limit, part 2

Lemma

Let $\mathbb{L}$ be the projective Fraïssé limit of $\mathcal{F}$. Then $R_S^{\mathbb{L}}$, where $R_S^{\mathbb{L}}(x,y)$ iff $R^{\mathbb{L}}(x,y)$ or $R^{\mathbb{L}}(y,x)$, is an equivalence relation such that each equivalence class has at most two elements.

Lelek fan from a projective Fraïssé limit, part 2

Lemma

Let $\mathbb{L}$ be the projective Fraïssé limit of $\mathcal{F}$. Then $R_S^{\mathbb{L}}$, where $R_S^{\mathbb{L}}(x, y)$ iff $R^{\mathbb{L}}(x, y)$ or $R^{\mathbb{L}}(y, x)$, is an equivalence relation such that each equivalence class has at most two elements.

Theorem

$\mathbb{L}/R_S^{\mathbb{L}}$ is the Lelek fan.

Non-triviality of $H(L)$

Remark

The group $H(L)$ is non-trivial, that is, there is $f \in H(L)$ such that $f \neq Id$.

$\mathrm{Aut}(\mathbb{L})$ as a subgroup of $H(L)$

- Each automorphism $h \in \mathrm{Aut}(\mathbb{L})$ can be identified in a natural way with a homeomorphism $h^* \in H(L)$.

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- Each automorphism $h \in \mathrm{Aut}(\mathbb{L})$ can be identified in a natural way with a homeomorphism $h^* \in H(L)$.
- $\mathrm{Aut}(\mathbb{L})$ is equipped with the compact-open topology.
- The topology on $\mathrm{Aut}(\mathbb{L})$ is finer than the compact-open topology on $H(L)$.

Projective universality and Projective Ultrahomogeneity

smooth fan = subfan of the Cantor fan

Projective universality and Projective Ultrahomogeneity

smooth fan = subfan of the Cantor fan

Theorem

1. *Each smooth fan is a continuous image of the Lelek fan L via a map that takes the root to the root and is monotone on segments.*
2. *Let X be a smooth fan with a metric d. If $f_1, f_2 : L \to X$ are two continuous surjections that take the root to the root and are monotone on segments, then for any $\epsilon > 0$ there exists $h \in \mathrm{Aut}(\mathbb{L})$ such that for all $x \in L$, $d(f_1(x), f_2 \circ h^*(x)) < \epsilon$.*

Corollary

Corollary

The group $\mathrm{Aut}(\mathbb{L})$ *is dense in* $H(L)$.

Homeomorphism group of the Lelek fan–totally disconnected

A topological space X is totally disconnected if for any $x, y \in X$ there is a clopen set $C \subseteq X$ such that $x \in C$ and $y \in (X \setminus C)$.

Homeomorphism group of the Lelek fan–totally disconnected

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Proposition

The group $H(L)$ is totally disconnected.

Homeomorphism group of the Lelek fan–'locally generated'

A homeomorphism $h \in H(L)$ is called an ϵ-homeomorphism if $d_{\sup}(h, \mathrm{Id}) < \epsilon$.

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Theorem

For every $\epsilon > 0$ and $h \in H(L)$ there are ϵ-homeomorphisms $h_1, \ldots, h_n \in H(L)$ such that $h = h_1 \circ \ldots \circ h_n$.

Homeomorphism group of the Lelek fan–‘locally generated’

A homeomorphism $h \in H(L)$ is called an ϵ-homeomorphism if $d_{\text{sup}}(h, \text{Id}) < \epsilon$.

Theorem

For every $\epsilon > 0$ and $h \in H(L)$ there are ϵ-homeomorphisms $h_1, \ldots, h_n \in H(L)$ such that $h = h_1 \circ \ldots \circ h_n$.
Moreover, if $h \in \mathrm{Aut}(\mathbb{L})$, then we can choose required $h_1, \ldots, h_n$ in $\mathrm{Aut}(\mathbb{L})$.

$H(L)$ is a 'large' group

Corollary

The group $H(L)$ is not locally compact.

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To show the corollary above we needed:

Theorem (van Dantzig)

A totally disconnected locally compact group admits a basis at the identity that consists of compact open subgroups.

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A totally disconnected locally compact group admits a basis at the identity that consists of compact open subgroups.

Corollary

The group $H(L)$ is not a non-archimedean group.

Conjugacy classes of $H(L)$

Theorem

The group of all homeomorphisms of the Lelek fan, $H(L)$, has a dense conjugacy class, i.e. there is $g \in H(L)$ such that $\{hgh^{-1} \colon h \in H(L)\}$ is dense.

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Theorem

The group of all automorphisms of $\mathbb{L}$, $\mathrm{Aut}(\mathbb{L})$, has a dense conjugacy class.

$H(L)$ is simple

Recall that a group is simple if it has no proper normal subgroups.

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Theorem

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