Colóquio do Departamento de Matemática - UFSC

Existence results for a one-equation turbulent model with feedback forces field[1]

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Florianópolis, October 7th, 2016

[1] Joint work with Ana Paiva, Univ. Algarve

Outline

Historical remarks: Ancient World

- Philosophical speculations and not only ....

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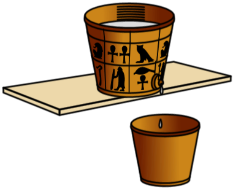

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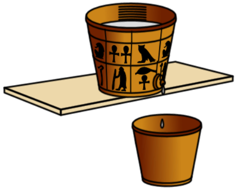

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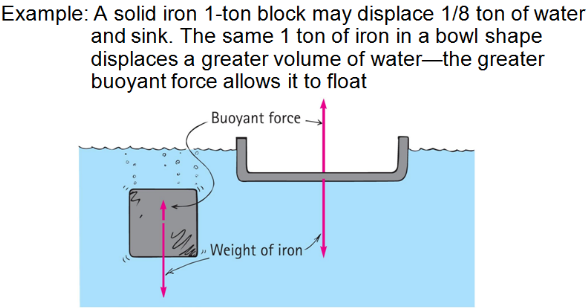
- Archimedes' Principle (c. 250 BC): the buoyant force is equal to the weight of the displaced water – applied e.g. in shipbuilding



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Historical remarks: Late Renaissance

- Torricelli's Law (1644): $v = \sqrt{2gh}$
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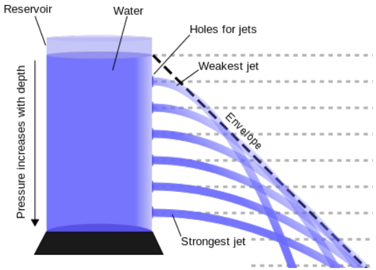

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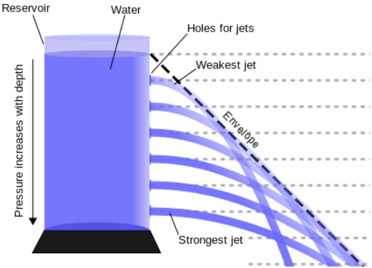

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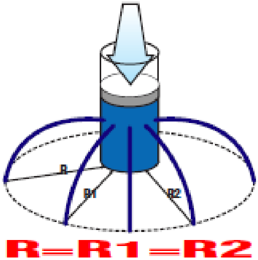

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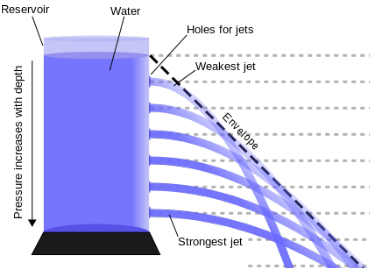

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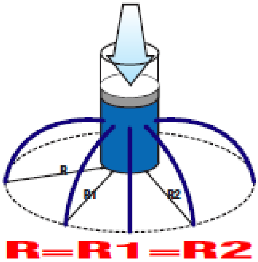

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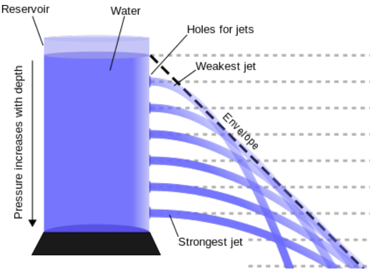

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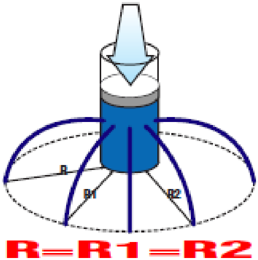

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- Hooke (1678): "deformations suffered by bodies are proportional to the forces that are applied on them"
- Newton (1687): "The resistance which arises from the lack of slipperiness of the parts of the liquid, other things being equal, is proportional to the velocity with which the parts of the liquid are separated from one another"

The rise of Differential Calculus

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- Due to the efforts of Poisson (1831), St. Venant (1843), but, mainly, of Stokes (1845):

$$\operatorname{div}\mathbf{u} = 0, \tag{4}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \mathbf{f} - \frac{1}{\rho}\nabla p + \nu\,\mathbf{div}\,(\mathbf{D}(\mathbf{u}))\,, \quad \mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}^T\right) \tag{5}$$

Derivation from the Principles of Mechanics

- Conservation of mass: incompressible and homogeneous

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\,\mathbf{u}) = 0 \quad \Rightarrow \quad \mathrm{div}\,\mathbf{u} = 0; \tag{6}$$

- Conservation of linear momentum

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = \rho\,\mathbf{f} + \mathrm{div}\mathbf{T}, \quad \mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} = 2\mu\,\mathbf{D}(\mathbf{u}) + \lambda\,\mathrm{div}\mathbf{u}\,\mathbf{I} \tag{7}$$

$$\Downarrow \qquad \text{(incompressible and homogeneous:} \quad \rho = Const., \quad \mathrm{div}\,\mathbf{u} = 0) \qquad \Downarrow$$

$$\frac{\partial\,\mathbf{u}}{\partial\,t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f} - \frac{1}{\rho}\nabla p + 2\nu\,\mathbf{div}(\mathbf{D}(\mathbf{u})), \quad \nu = \frac{\mu}{\rho}, \quad \mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) \tag{8}$$

- Conservation of angular momentum: $\mathbf{T} \in \mathbb{M}^{N\times N}_{\mathrm{sym}}$ – $\mathbf{T}$ is symmetric;
- Notation: $\mathbf{u}$ – velocity field, p – pressure, ρ – density, $\mathbf{u}$ – external forces field, $\mathbf{T}$ – Cauchy stress tensor, $\mathbf{S}$ – extra stress tensor, μ, λ - cinematic viscosities, ν – kinematic viscosity.
- Dimensions of physical interest: $N = 2$, $N = 3$; but mathematically is interesting to study a generalized dimension.

The beginning of the turbulence study

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- Reynolds (1883): into a flow through a glass tube, was injected a dye to observe the nature of flow. When the speeds were small the flow seemed to follow a straight line path. As the flow speed was increased the dye fluctuates and one observes intermittent bursts. As the flow speed is further increased the dye is blurred and seems to fill the entire pipe.

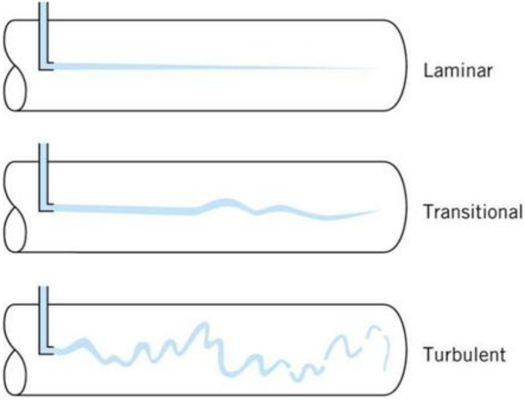

Figure: Reynolds Experiment (`www.learncax.com`)

Flow regimes

Reynolds (1883) has succeeded to prove the importance of a threshold value separating the laminar flow regime from the turbulent one within a similar fluid,

$$Re = \frac{u(l)\,l}{\nu} \equiv \frac{(\mathbf{u}\cdot\nabla)\mathbf{u}}{\nu\triangle\mathbf{u}} \equiv \frac{\text{inertia forces}}{\text{viscosity forces}}\,, \tag{9}$$

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The transition usually begins with $1000 \leq Re \leq 2000$ and extends upward to $3000 \leq Re \leq 5000$.

Transitional flow can refer to transition in either direction, that is laminar-turbulent transitional or turbulent-laminar transitional flow. The transition between laminar and turbulent flow occurs not at a specific value of the Reynolds number but in a range.

The Reynolds-averaged Navier-Stokes equations

- Reynolds hypothesis (1895): the flow has two different scales which makes possible to decompose the flow quantities

$$\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}', \quad \dots, \quad \mathbf{u}' = \textit{fluctuating velocity}, \ \overline{\mathbf{u}} = \textit{average velocity}.$$

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$$\mathcal{R} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \mathcal{R}(\mathbf{u}) = \overline{\mathbf{u}}, \quad e.g. \ \mathcal{R}(\mathbf{u}) = \frac{1}{\Delta t}\int_t^{t+\Delta t} \mathbf{u}\, dt\,, \tag{11}$$

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Properties of the Reynolds operator

1. $\mathcal{R}(\mathbf{u}+\lambda\mathbf{v}) = \mathcal{R}(\mathbf{u}) + \lambda\mathcal{R}(\mathbf{v})$ for any $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3$ and any $\lambda \in \mathbb{R}$;
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3. $\mathcal{R}(\partial\mathbf{u}) = \partial\left(\mathcal{R}(\mathbf{u})\right)$ for any $\mathbf{u} \in \mathbb{R}^3$;
4. $\mathcal{R}(\mathbf{u}\otimes\mathbf{v}) = \mathcal{R}(\mathbf{u})\otimes\mathcal{R}(\mathbf{v}) + \mathcal{R}((\mathbf{u}-\mathcal{R}(\mathbf{u}))\otimes(\mathbf{v}-\mathcal{R}(\mathbf{v})))$ for any $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3$ implies $\mathcal{R}(\mathbf{u}\otimes\mathcal{R}(\mathbf{v})) = \mathcal{R}(\mathbf{u})\otimes\mathcal{R}(\mathbf{v}) = \mathcal{R}(\mathcal{R}(\mathbf{u})\otimes\mathbf{v})$.

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- The Reynolds operator is used to filter the NS eqs (4) and (5), leading to the RANS:

$$\operatorname{div}\overline{\mathbf{u}} = 0, \tag{12}$$

$$\frac{\partial\overline{\mathbf{u}}}{\partial t} + \mathbf{div}(\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}) = \overline{\mathbf{f}} - \frac{1}{\rho}\nabla\overline{p} + \nu\,\mathbf{div}\,\mathbf{D}(\overline{\mathbf{u}}) - \mathbf{div}(\overline{\mathbf{u}'\otimes\mathbf{u}'}). \tag{13}$$

The turbulent kinetic energy

- Reynolds stress tensor: the additional term in (13),

$$\mathbf{R} := -\overline{\mathbf{u}' \otimes \mathbf{u}'}\,, \tag{14}$$

  and expresses the average of changes in $\mathbf{u}'$ due to the particle transport with the fluid movement.
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- Boussinesq hypothesis: Boussinesq (1877) has proposed (in analogy with the Stokes law for laminar flows),

$$\mathbf{R} = \nu_T\,\mathbf{D}\left(\overline{\mathbf{u}}\right)\,, \tag{15}$$

where ν_T is the eddy or turbulent viscosity, a concept introduced earlier by Saint-Venant (1843)

- By comparing the traces of the two expressions in (14) and in (15), the Boussinesq hypothesis needs to be corrected to

$$\mathbf{R} = -\frac{2}{3}k\mathbf{I} + \nu_T\,\mathbf{D}(\overline{\mathbf{u}})\,, \qquad k := \frac{1}{2}\overline{|\mathbf{u}'|^2} \quad \text{is the turbulent kinetic energy.} \tag{16}$$

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- The issue of finding an expression for the Reynolds tensor implies the search for an expression for the eddy viscosity.

Modelling the turbulent viscosity

- Prandtl's mixing-length model (1925): inspired in the kinetic theory of perfect gases,

$$\nu_T = \rho\, l_m^2 |\nabla \overline{\mathbf{u}}| \quad \Rightarrow \quad \nu_T = \rho\, l_m^2 |\mathbf{D}(\overline{\mathbf{u}})|\,, \quad \text{with } l_m \text{ algebraic prescribed} \tag{17}$$

  where l_m is the mixing length, a distance that a fluid parcel keeps its original characteristics before dispersing them into the surrounding fluid.
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- Kolmogorov's model (18) suggested that k and f should be determined by transport equations.
- Prandtl's model (1945): ν_T is determined from an equation for the transport of k,

$$\nu_T = \rho\, k^{\frac{1}{2}}\, l\,, \quad \text{but } l \text{ is still algebraically prescribed.} \tag{19}$$

The equation for the turbulent kinetic energy

- We start by considering (4) with the velocity field decomposed in the form (10) and then we subtract (13) to this equation,

$$\operatorname{div} \mathbf{u}' = 0. \tag{20}$$

- The RANS equations (13) are subtracted to the momentum equation (5), where all the quantities are decomposed as in (10). Then, we multiply the resulting equation by k and we apply the filter produced by the Reynolds operator. Using some vectorial calculus together with (14), $(16)_2$ and (20), we obtain

$$\frac{\partial k}{\partial t} + \overline{(\overline{\mathbf{u}} + \mathbf{u}') \cdot \nabla \frac{|\mathbf{u}'|^2}{2}} = \mathbf{R} : \mathbf{D}(\overline{\mathbf{u}}) - \frac{1}{\rho} \operatorname{div} (\overline{p'\mathbf{u}'}) + \nu\, \overline{\mathbf{u}' \cdot \mathbf{div}\, \mathbf{D}(\mathbf{u}')}. \tag{21}$$

- Using hypotheses of the Ergodic Theory, $\overline{(\overline{\mathbf{u}} + \mathbf{u}') \cdot \nabla \frac{|\mathbf{u}'|^2}{2}} \simeq \overline{\mathbf{u}} \cdot \nabla k - \operatorname{div} (\nu_D \nabla k)$, where

$$\nu_D := \frac{\nu_T}{\sigma_k} \quad \text{is the turbulent diffusivity and } \sigma_k \text{ is the Schmidt-Prandtl number,} \tag{22}$$

and $\operatorname{div} (\overline{p'\mathbf{u}'}) \simeq 0$ and $\overline{\mathbf{u}' \cdot \mathbf{div} \mathbf{D}(\mathbf{u}')} \simeq \overline{|\mathbf{D}(\mathbf{u}')|^2}$.

- Using these simplifications, (21) can be written as

$$\frac{\partial k}{\partial t} + \overline{\mathbf{u}} \cdot \nabla k = \operatorname{div} (\nu_D \nabla k) + \nu_T |\mathbf{D}(\overline{\mathbf{u}})|^2 - \overline{|\mathbf{D}(\mathbf{u}')|^2}. \tag{23}$$

The equation for the turbulent dissipation

- Launder and Spalding (1970) observed the importance of the rate of dissipation of the turbulent kinetic energy in the turbulent flow process,

$$\varepsilon := \nu\overline{|\nabla \mathbf{u}'|^2} \quad \Rightarrow \quad \varepsilon := \overline{|\nabla \mathbf{u}' + \nabla \mathbf{u}'^{T}|^2} \simeq \overline{|\mathbf{D}(\mathbf{u}')|^2}. \tag{24}$$

- Since ε scales as u_0^3/l_0, and independently of ν, where u_0 and l_0 are characteristic eddy's velocity and length scales, ε is modelled as

$$\varepsilon = C_D \frac{k^{\frac{3}{2}}}{l}, \qquad C_D \text{ is a closure constant.} \tag{25}$$

- As a consequence of (19), (22) and (25),

$$\nu_T = C_\mu \frac{k^2}{\varepsilon} \quad \text{and} \quad \nu_D = \frac{C_\mu}{\sigma_k}\frac{k^2}{\varepsilon}, \tag{26}$$

  where C_μ is a constant related with the kinematic viscosity.

- The derivation of the equation for the evolution of ε is much more involved: taking the curl of the RANS equation (13), some calculus tools and Ergodic hypotheses,

$$\frac{\partial \varepsilon}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \varepsilon = \operatorname{div}(\nu_D \nabla \varepsilon) + C_1 k |\mathbf{D}(\overline{\mathbf{u}})|^2 + C_2 \frac{\varepsilon^2}{k}\,. \tag{27}$$

  where C_1 and C_2 are positive constants that can be determined from the experiments.

Since experiments show that two more equations are more then enough to obtain reliable results, the turbulent k-epsilon model is the most common model used in CFD to simulate mean flow characteristics for turbulent flow conditions,

$$\operatorname{div} \overline{\mathbf{u}} = 0, \tag{28}$$

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \mathbf{div}(\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}) = \overline{\mathbf{f}} - \frac{1}{\rho} \nabla \overline{p} + \mathbf{div}(\nu + \nu_T(k, \varepsilon)\mathbf{D}(\overline{\mathbf{u}}), \tag{29}$$

$$\frac{\partial\, k}{\partial\, t} + \overline{\mathbf{u}} \cdot \nabla k = \operatorname{div}\left(\nu_D(k, \varepsilon)\nabla k\right) + \nu_T(k, \varepsilon)|\mathbf{D}(\overline{\mathbf{u}})|^2 - \varepsilon, \tag{30}$$

$$\frac{\partial \varepsilon}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \varepsilon = \operatorname{div}\left(\nu_D(k, \varepsilon)\nabla \varepsilon\right) + C_1 k|\mathbf{D}(\overline{\mathbf{u}})|^2 + C_2 \frac{\varepsilon^2}{k}\,. \tag{31}$$

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The consideration of one-equation models is acceptable in the sense that the equation for ε may be discarded by prescribing an appropriate length scale

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$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \mathbf{div}(\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}) = \overline{\mathbf{f}} - \frac{1}{\rho}\nabla \overline{p} + \mathbf{div}(\nu + \nu_T(k)\mathbf{D}(\overline{\mathbf{u}}), \tag{33}$$

$$\frac{\partial\, k}{\partial\, t} + \overline{\mathbf{u}} \cdot \nabla k = \operatorname{div}\left(\nu_D(k)\nabla k\right) + \nu_T(k)|\mathbf{D}(\overline{\mathbf{u}})|^2 - \varepsilon(k), \qquad \varepsilon(k) = \frac{C_D}{l} k^{\frac{3}{2}}. \tag{34}$$

The problem under consideration

A boundary-value problem for a general turbulent k-epsilon model

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{35}$$

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{g} - \mathbf{f}(\mathbf{u}) - \nabla p + \mathbf{div}\left((\nu + \nu_T(k))\,\mathbf{D}(\mathbf{u})\right) \quad \text{in } \Omega, \tag{36}$$

$$\mathbf{u} \cdot \nabla k = \operatorname{div}\left(\nu_D(k)\nabla k\right) + \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 + P(\mathbf{u}, k) - \varepsilon(k) \quad \text{in } \Omega, \tag{37}$$

$$\mathbf{u} = \mathbf{0} \quad \text{and} \quad k = 0 \quad \text{on } \partial\Omega\,, \tag{38}$$

where Ω is a bounded domain of $\mathbb{R}^d$, $d = 2,\ 3$.

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where:

- the velocity vector field $\mathbf{u}$, the pressure p and the external forces field $\mathbf{g}$ are, in fact, averages that result by the application of one or two average concepts;
- the averaged tensor $\mathbf{D}(\mathbf{u})$ is the symmetric part of the averaged gradient $\nabla\,\mathbf{u}$;
- the positive constant $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity, μ is the dynamic viscosity, and ρ is the mass density;
- the scalar function k characterizes the energy of the turbulence in the flow (turbulent kinetic energy, TKE);
- the rate of dissipation of k is described by the function ε (turbulent dissipation);
- the scalar function ν_T is the turbulent, or eddy, viscosity, whereas ν_D is the turbulent diffusion;

Simulation of a porous media flow

Figure: Video by Kerstin Kantiem, YouTube.

Darcy's law

- Fluid flows through porous media are usually described by Darcy's law (1856), an empirical flow model that represents a simple linear relationship between flow rate and the pressure drop in a porous media
- Unidirectional flow:

$$u = -\frac{K}{\mu}\frac{\partial p}{\partial x} \quad \Leftrightarrow \quad 0 = -\frac{\partial p}{\partial x} - \frac{\mu}{K}u\,, \tag{39}$$

  where the coefficient K, called permeability, is independent of the nature of the fluid but it depends on the geometry of the medium.
- In three dimensions, (39) generalizes to

$$\mathbf{u} = -\frac{\mathbf{K}}{\mu}\boldsymbol{\nabla} p \quad \Leftrightarrow \quad \mathbf{0} = -\mathbf{K}\boldsymbol{\nabla} p - \mu\mathbf{u}\,, \tag{40}$$

  where $\mathbf{K}$ is the permeability tensor (scalar if the medium is isotropic).
- If the gravity forces field is taken into account, then (40) comes as

$$\mathbf{u} = -\frac{\mathbf{K}}{\mu}\left(\boldsymbol{\nabla} p - \rho\mathbf{g}\right) \quad \Leftrightarrow \quad \mathbf{0} = -\mathbf{K}\left(\boldsymbol{\nabla} p - \rho\mathbf{g}\right) - \mu\mathbf{u}. \tag{41}$$

- For an isotropic medium, (41) reads as

$$\mathbf{u} = -\frac{K}{\mu}\left(\boldsymbol{\nabla} p - \rho\mathbf{g}\right) \quad \Leftrightarrow \quad \mathbf{0} = \rho\mathbf{g} - \boldsymbol{\nabla} p - \frac{\mu}{K}\mathbf{u}. \tag{42}$$

Brinkman's equation

- The Darcy law assumes no effect of boundaries and the fluid velocity in Darcy's equation is determined by the permeability of the matrix.
- If the boundary is impermeable, then the usual assumption is that the normal component of the velocity must vanish,

$$\mathbf{u}\cdot\mathbf{n}=\mathbf{0}\quad\text{on the solid-fluid interface.}\tag{43}$$

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- At a solid wall boundary, the fluid velocity will not reduce to the no-slip condition when the Darcy law is enforced.
- In this situation, the Brinkman equation may be employed, which is an extension of the Darcy law and facilitates the matching of boundary conditions.
- By using the theory of flow past an individual sphere, Brinkman (1947) has suggested to add the diffusion term simply to meet the boundary specifications,

$$\nabla p=\rho\mathbf{g}-\frac{\mu}{K}\mathbf{u}+\widetilde{\mu}\Delta\mathbf{u} \quad \Leftrightarrow \quad \mathbf{0}=\rho\mathbf{g}-\nabla p-\frac{\mu}{K}\mathbf{u}+\widetilde{\mu}\Delta\mathbf{u}\,, \tag{44}$$

where the coefficient $\widetilde{\mu}$ (effective viscosity) is a quantity having the dimension of viscosity.

Flow at high flow rates

The literature recognizes distinct flow regimes based on the so-called pore Reynolds number,

$$Re_p := \frac{\rho\, qd}{\nu}, \qquad q := \frac{Q}{A},$$

where q is the specific discharge (volume of fluid flowing per unit time, Q, through a unit cross-sectional area, A, normal to the direction of the flow) and d is some representative (microscopic) length characterizing the void space.

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$Re_p > 10$: breakdown in the linearity of $\mathbf{u}$

- Is due to the fact that the form drag due to solid obstacles is now comparable with the surface drag due to friction.
- Forchheimer's law (1901) remedies this situation by stating that the relationship between the flow rate and pressure gradient is nonlinear at sufficiently high velocity and that this nonlinearity increases with flow rate.

Forchheimer equations

- According to many authors (e.g. Joseph, Nield and Papanicolaou (1982)), the appropriate modification to Darcy's equation, to take into account high flow rates, is to replace (42) by the following Forchheimer Darcy model

$$\nabla p = \rho \mathbf{g} - \frac{\mu}{K}\mathbf{u} - \frac{c_F \rho}{\sqrt{K}}|\mathbf{u}|\mathbf{u} \quad \Leftrightarrow \quad \mathbf{0} = \rho \mathbf{g} - \nabla p - \frac{\mu}{K}\mathbf{u} - \frac{c_F \rho}{\sqrt{K}}|\mathbf{u}|\mathbf{u}\,, \qquad (45)$$

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- Several authors (Nakayama (1992) and Kuznetsov (1998)) have added, in their studies, a diffusion term to (45) in order to form a Brinkman-Forchheimer Darcy model,

$$\nabla p = \rho \mathbf{g} - \frac{\mu}{K}\mathbf{u} - \frac{c_F \rho}{\sqrt{K}}|\mathbf{u}|\mathbf{u} + \widetilde{\mu}\Delta \mathbf{u} \quad \Leftrightarrow \quad \mathbf{0} = \rho \mathbf{g} - \nabla p - \frac{\mu}{K}\mathbf{u} - \frac{c_F \rho}{\sqrt{K}}|\mathbf{u}|\mathbf{u} + \widetilde{\mu}\Delta \mathbf{u}. \quad (46)$$

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- Other authors (e.g. Vafai & Kim (1991)) have added, too, the advective inertia terms of the Navier-Stokes equations to form what is now commonly known as the Brinkman-Forchheimer-extended Darcy model (or generalized model)

$$\rho(\mathbf{u}\cdot\nabla)\mathbf{u} = \rho\mathbf{g} - \nabla p - \frac{\mu}{K}\mathbf{u} - \frac{c_F\rho}{\sqrt{K}}|\mathbf{u}|\mathbf{u} + \widetilde{\mu}\Delta\mathbf{u}\,, \tag{47}$$

  model that was derived based on local volume averaging.

Post-Forchheimer flows

The literature (cf. Dybbs & Edwards (1984),) recognizes distinct flow regimes based on the pore Reynolds number, Re_p:

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Post-Forchheimer flow regime (unsteady laminar flow) if $150 < Re_p < 300$

In this case, the time inertia terms need to be considered:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\right) = \rho\mathbf{g} - \boldsymbol{\nabla} p - \frac{\mu}{K}\mathbf{u} - \frac{c_F\,\rho}{\sqrt{K}}|\mathbf{u}|\mathbf{u} + \widetilde{\mu}\Delta\mathbf{u} \qquad (48)$$

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Fully turbulent flow if $Re_p > 300$: turbulence modelling is required

- The turbulence k-epsilon model is considered;
- The microscopic equations are volume-averaged and then the macroscopic equations are Reynolds-averaged; or
- The Reynolds averaging procedure is applied first and then the resultant (microscopic) turbulent equations are volume-averaged.

The governing microscopic equations

- For simplicity, we consider the isothermal case.
- The properties of the fluid (density, viscosity) are assumed to be constant and thus the fluid under consideration is incompressible and Newtonian.
- On the pore scale (microscopic scale) the flow quantities (velocity, pressure) are determined by the incompressible Navier-Stokes equations

$$\operatorname{div} \mathbf{u} = 0; \tag{49}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{g} - \frac{1}{\rho}\nabla p + \nu \mathbf{D}(\mathbf{u}), \quad \mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right), \tag{50}$$

where $\mathbf{u} \in \mathbb{R}^3$ is the velocity field, $p \in \mathbb{R}$ is the pressure, ρ is the density, ν is the kinematic viscosity and $\mathbf{g} \in \mathbb{R}^3$ is the gravity forces field.
- If the boundary is impermeable, then the usual assumption is that the normal component of the velocity must vanish,

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{0} \quad \text{on the solid-fluid interface.} \tag{51}$$

- Contrary to the the Darcy flow model (the maximum velocity occurs at the impermeable surface), the no-slip boundary condition can be used,

$$\mathbf{u} \cdot \boldsymbol{\tau} = \mathbf{0} \quad \text{on the solid-fluid interface.} \tag{52}$$

REV: Representative elementary volume

- However in typical experiments the quantities of interest are measured over a sufficiently large representative elementary volume (REV).

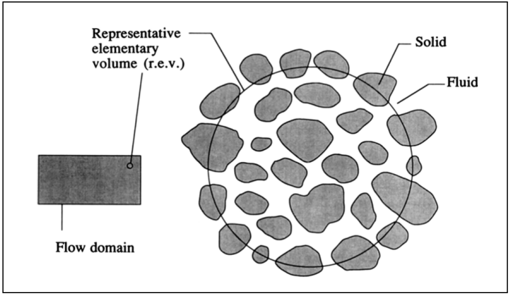

Figure: from Nield & Bejan (2006).

- REV is the smallest volume over which a measurement can be made that will yield a value representative of the whole porous medium.
- The length scale of the REV is larger than the pore scale, but much smaller than the size of the entire flow domain.

From continuum to porous-continuum

- The evolution from continuum to porous-continuum level involves an averaging process, similar to the one from molecular to continuum level.

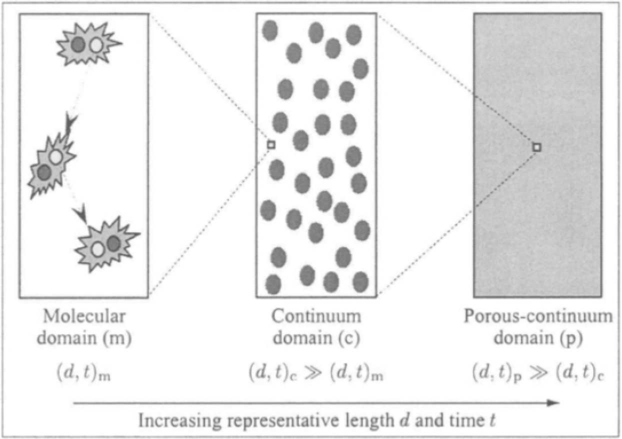

Figure: from Merrikh & Lage (2005).

Volume averaging

- The volumetric average[2] of a flow quantity, say φ, taken over a REV:

$$\text{Volumetric average: } \langle\varphi\rangle := \frac{1}{V}\int_{V_\gamma} \varphi_\gamma \, dV. \tag{53}$$

[2] Aka: seepage velocity, filtration velocity, superficial velocity, Darcy velocity, volumetric flux density.

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- These averages are related by the Dupuit-Forchheimer relationship:

$$\langle\varphi\rangle = \phi\langle\varphi\rangle_i, \quad \phi = \frac{V_\gamma}{V} \text{ is the local medium porosity.} \tag{55}$$

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- The property φ can then be defined as

$$\varphi = \langle\varphi\rangle_i + {}^i\varphi\,, \tag{56}$$

where ${}^i\varphi$ is the spatial deviation of φ w.r.t. $\langle\varphi\rangle_i$, i.e. is the difference between the real value (microscopic) and its intrinsic (fluid based average) value

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- From (54) and (56) it follows that $\langle {}^i\varphi\rangle_i = 0$.

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The governing microscopic equations

- Authors: Lee & Howell (1987), Wang & Takle (1995), Antohe & Lage (1997), Getachewa et al. (2000).
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- Assumptions: rigid, isotropic and fixed porous matrix; Newtonian fluid with constant properties; and isothermal flow (here, for simplicity).
- The volumetric average of Eqs. (49)-(50) results (cf. Hsu & Cheng (1990)):

$$\operatorname{div} \langle \mathbf{u} \rangle_i = 0; \tag{57}$$

$$\frac{\partial}{\partial t} \langle \mathbf{u} \rangle_i + \mathbf{div}(\langle \mathbf{u} \rangle_i \otimes \langle \mathbf{u} \rangle_i) = \langle \mathbf{g} \rangle_i - \frac{1}{\rho} \boldsymbol{\nabla} \langle p \rangle_i + \frac{\mu_e}{\rho} \mathbf{D}(\langle \mathbf{u} \rangle_i) - \mathbf{div}\left(\left\langle {}^i\mathbf{u} \otimes {}^i\,\mathbf{u} \right\rangle_i \right) + \mathbf{R}, \tag{58}$$

The governing microscopic equations

- Authors: Lee & Howell (1987), Wang & Takle (1995), Antohe & Lage (1997), Getachewa et al. (2000).
- Assumptions: rigid, isotropic and fixed porous matrix; Newtonian fluid with constant properties; and isothermal flow (here, for simplicity).
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- where $\mathbf{R}$ represents the total drag force per unit volume due to the presence of the porous matrix:

$$\mathbf{R} = -\frac{\mu}{K}\phi\langle\mathbf{u}\rangle_i - \frac{c_F}{\sqrt{K}}\phi^2|\langle\mathbf{u}\rangle_i|\langle\mathbf{u}\rangle_i\,. \tag{59}$$

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- Eqs. (57)-(58) model typical porous media flow for $150 < Re_p < 300$.
- When extending the analysis to turbulent flow, time-varying quantities have to be considered.

Turbulence in porous media

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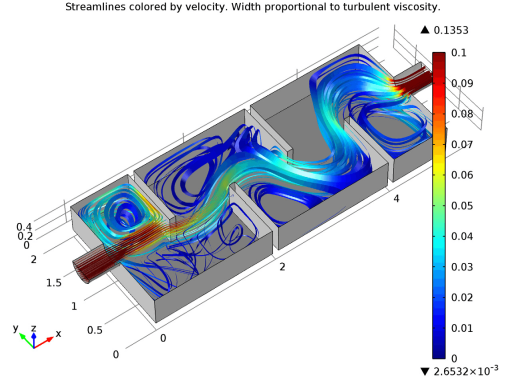
Two main differences exist between turbulent flow through porous media and turbulent flow in the absence of a porous matrix

- the size of the turbulent eddies within the pores is limited by the pore size;
- the presence of porous matrix induces additional drag while preventing motion of larger size eddies.

Figure: To illustrate this flow regime, let us look at the flow in a much larger geometry than the pore scale flow: a typical ozone purification reactor (from https://www.comsol.pt). The results show the flow patterns, flow velocity, and turbulent viscosity.



Modelling turbulence in porous media

- From a broad perspective, for high pore Reynolds number ($Re_p > 300$), turbulent models presented in the literature follow two different approaches.
- The first method (Lee & Howell (1987), Wang & Takle (1995), Antohe & Lage (1997), Getachewa et al. (2000)), starts with the volume-average of the microscopic equations and then the macroscopic equations are averaged in time to produce the turbulence equations.
- The second (Masuoka & Takatsu (1996), Kuwahara et al. (1996), Takatsu & Masuoka (1998), Kuwahara & Nakayama (1998), Nakayama & Kuwahara (1999)), makes use, first, of the time-averaged equations, and then proceeds with volume-averaging for deriving the turbulence equations.
- These two methodologies lead, in general, to distinct sets of turbulence equations because of the different averaging order, i.e., space-time and time-space, respectively.
- A third, and probably the most consistent method (Pedras-de Lemos (2000-2001)) is based on a double decomposition approach. In this method, the momentum equation is closed by using the Hazen-Dupuit-Darcy model for the total drag effect only after the space-time averaging (or time-space averaging) is performed.

- The governing equations are obtained by volume-averaging the microscopic Reynolds-averaged equations:

$$\operatorname{div} \langle \overline{\mathbf{u}} \rangle_i = 0; \tag{60}$$

$$\begin{aligned} \frac{\partial}{\partial t} \langle \overline{\mathbf{u}} \rangle_i + \mathbf{div}(\langle \overline{\mathbf{u}} \rangle_i \otimes \langle \overline{\mathbf{u}} \rangle_i) = \overline{\langle \mathbf{g} \rangle_i} - \frac{1}{\rho} \boldsymbol{\nabla} \left(\overline{\langle p \rangle_i} + \frac{2}{3} \rho \langle k \rangle_i \right) \\ + \mathbf{div} \left[\left(\nu_T + \frac{\mu_e}{\rho} \right) \mathbf{D} \left(\langle \overline{\mathbf{u}} \rangle_i \right) \right] - \frac{\mu}{K} \phi \langle \overline{\mathbf{u}} \rangle_i - \frac{c_F}{\sqrt{K}} \phi^2 | \langle \overline{\mathbf{u}} \rangle_i | \langle \overline{\mathbf{u}} \rangle_i \ . \end{aligned} \tag{61}$$

Time-averaging then volume-averaging: Nakayama & Kuwahara (1999)

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$$\begin{aligned} &\frac{\partial \langle k \rangle_i}{\partial t} + \langle \overline{\mathbf{u}} \rangle_i \cdot \boldsymbol{\nabla} \langle k \rangle_i = \mathbf{div} \left[\left(\nu + \nu_D \right) \boldsymbol{\nabla} \langle k \rangle_i \right] + 2 \nu_T \left| \mathbf{D} \left(\langle \overline{\mathbf{u}} \rangle_i \right) \right|^2 \\ &- \langle \varepsilon \rangle_i + 39 \phi^2 \sqrt[5]{(1-\phi)^2} \frac{| \langle \overline{\mathbf{u}} \rangle_i |^3}{d} \ , \end{aligned} \tag{62}$$

where d is the hydraulic diameter.

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  where d is the hydraulic diameter.
- Features of the model:
  - the hydrodynamic dispersion was incorporated in the drag forces:

$$-\mathbf{div} \left(\left\langle {}^i\overline{\mathbf{u}} \otimes {}^i \overline{\mathbf{u}} \right\rangle_i \right) + \mathbf{R} = -\frac{\mu}{K} \phi \langle \overline{\mathbf{u}} \rangle_i - \frac{c_F}{\sqrt{K}} \phi^2 | \langle \overline{\mathbf{u}} \rangle_i | \langle \overline{\mathbf{u}} \rangle_i \ ;$$

  - Additional terms, due to the presence of the porous medium, appearing in the governing equations for $\langle k \rangle_i$ and for $\langle \varepsilon \rangle_i$, were determined by using two unknown model constants.

- Volume-averaging the microscopic Reynolds-averaged equations,

$$\mathrm{div}\,\langle\overline{\mathbf{u}}\rangle_i = 0; \tag{63}$$

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- Features of the model:
  - To model the Reynolds stresses it is proposed a macroscopic Boussinesq assumption:

$$\left\langle\overline{\mathbf{u}'\otimes\mathbf{u}'}\right\rangle_i = \frac{2}{3}\langle k\rangle_i\mathbf{I} - \nu_{T_\phi}\langle\mathbf{D}(\overline{\mathbf{u}})\rangle_i, \quad \nu_{T_\phi} := C_\mu\frac{\langle k\rangle_i^2}{\langle\varepsilon\rangle_i},$$

  - where ν_{T_ϕ} is a macroscopic turbulent viscosity satisfying to: $\nu_{T_\phi}\mathbf{D}\left(\langle\overline{\mathbf{u}}\rangle_i\right) = \langle\nu_T\mathbf{D}(\overline{\mathbf{u}})\rangle_i$;
  - and ν_{D_ϕ} is a macroscopic turbulent dissipation defined by $\nu_{D_\phi} := \frac{\nu_{T_\phi}}{\sigma_k}$.
  - The total drag term $\overline{\mathbf{R}}$ is only closed after all the equations are obtained;
  - An additional term is included in the equation for k to account for the porous structure.

A boundary-value problem for a general turbulent k-epsilon model

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{66}$$

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{g} - \mathbf{f}(\mathbf{u}) - \nabla p + \mathbf{div}\left((\nu + \nu_T(k))\, \mathbf{D}(\mathbf{u})\right) \quad \text{in } \Omega, \tag{67}$$

$$\mathbf{u} \cdot \nabla k = \operatorname{div}\left(\nu_D(k)\nabla k\right) + \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 + P(\mathbf{u}, k) - \varepsilon(k) \quad \text{in } \Omega, \tag{68}$$

$$\mathbf{u} = \mathbf{0} \quad \text{and} \quad k = 0 \quad \text{on } \partial\Omega\,, \tag{69}$$

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$$\mathbf{f}(\mathbf{u}) = -\frac{\mu}{K}\phi\mathbf{u} - \frac{c_F}{\sqrt{K}}\phi^2|\mathbf{u}|\mathbf{u}, \quad P(\mathbf{u}, k) \equiv P(\mathbf{u}) = 39\phi^2\sqrt[5]{(1-\phi)^2}\frac{|\mathbf{u}|^3}{d},$$

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The feedback terms in other fields:

A boundary-value problem for a general turbulent k-epsilon model

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Turbulent flows in a rotating frame

$\mathbf{f}(\mathbf{u}) = 2\boldsymbol{\Omega} \times \mathbf{u}\,, \quad P(\mathbf{u}, k) = 0\,, \quad \boldsymbol{\Omega}$ is the vector of angular velocity;

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Turbulent flows controlled by a magnetic field $\mathbf{B}$

$$\mathbf{f}(\mathbf{u}, \mathbf{B}) = -\mathbf{J} \times \mathbf{B}\,, \quad \mathbf{J} = \sigma(\nabla\Phi - \mathbf{u} \times \mathbf{B})\,, \quad \triangle\Phi = \operatorname{div}(\mathbf{u} \times \mathbf{B})\,,$$

where $\mathbf{J}$ is the total electric current intensity, Φ is the electric potential, and σ is the conductivity, a material dependent parameter.

Assumptions physically admissible

$$\mathbf{f} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d \quad \text{is a Carathéodory function,} \tag{74}$$

$$\varepsilon,\ \nu_T,\ \nu_D : \Omega \times \mathbb{R} \to \mathbb{R}^+ \quad \text{are Carathéodory functions,} \tag{75}$$

$$P : \Omega \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \quad \text{is a Carathéodory function.} \tag{76}$$

In particular, the assumption (75) fits with turbulent dissipation, turbulent viscosity and turbulent diffusion functions involved in realistic models when giving, for instance, by the following formula

$$\varepsilon(k) = \frac{k\sqrt{k}}{l}\,, \quad \nu_T(k) = C_1 l\sqrt{k}\,, \quad \nu_D(k) = \mu_e + C_2 l\sqrt{k}\,, \quad l \neq 0, \quad k \geq 0\,, \tag{77}$$

where μ_e is an effective (dynamic) viscosity, C_1, C_2 are dimensionless constants and $l : \Omega \to \mathbb{R}$ is the mixing length function which is usually assumed to satisfy $l(\mathbf{x}) \geq l_0$ for a.e. $\mathbf{x} \in \Omega$ and for some positive constant l_0.

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Assumptions mathematically needed

$$|\nu_T(k)| \leq C_T, \quad |\nu_D(k)| \leq C_D, \tag{78}$$

for some positive constants C_T and C_D.

Notion of weak solutions

Let us introduce the following function spaces:

$\mathcal{V} := \{\mathbf{v} \in \mathbf{C}_0^\infty(\Omega) : \mathrm{div}\mathbf{v} = 0\}$, $\mathbf{H}$:= closure of $\mathcal{V}$ in $\mathbf{L}^2(\Omega)$, $\mathbf{V}$:= closure of $\mathcal{V}$ in $\mathbf{H}^1(\Omega)$.

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Definition

Let the conditions (74)-(78) be fulfilled and assume that $\mathbf{g} \in \mathbf{V}'$. We say a pair $(\mathbf{u}, k)$ is a weak solution to the problem (70)-(73), if:

1. $\mathbf{u} \in \mathbf{V}$ and for every $\mathbf{v} \in \mathbf{V} \cap \mathbf{L}^d(\Omega)$ there hold $\mathbf{f}(\mathbf{u}) \cdot \mathbf{v} \in \mathbf{L}^1(\Omega)$ and
$$\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}\, d\mathbf{x} + \int_\Omega (\nu + \nu_T(k))\, \mathbf{D}(\mathbf{u}) : \nabla \mathbf{v}\, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u}) \cdot \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{g} \cdot \mathbf{v}\, d\mathbf{x}; \quad (79)$$
2. $k \in \mathrm{W}_0^{1,q}(\Omega)$, with $\frac{2d}{d+2} \leq q < d'$, and for every $\varphi \in \mathrm{W}_0^{1,q'}(\Omega)$ there hold $\varepsilon(k)\,\varphi$, $P(\mathbf{u}, k)\,\varphi \in \mathrm{L}^1(\Omega)$ and
$$\begin{aligned} &\int_\Omega (\mathbf{u} \cdot \nabla k)\varphi\, d\mathbf{x} + \int_\Omega \nu_D(k)\nabla k \cdot \nabla \varphi\, d\mathbf{x} + \int_\Omega \varepsilon(k)\,\varphi\, d\mathbf{x} = \\ &\int_\Omega \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 \varphi\, d\mathbf{x} + \int_\Omega P(\mathbf{u}, k)\,\varphi\, d\mathbf{x}; \end{aligned} \quad (80)$$
3. $k \geq 0$ and $\varepsilon(k) \geq 0$ *a.e.* in Ω.

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On the functions $\mathbf{f}(\mathbf{u})$ and $\varepsilon(k)$

We assume the existence of nonnegative constants C_f and C_ε such that (a.e. in Ω)

$$|\mathbf{f}(\mathbf{u})| \leq C_f |\mathbf{u}|^\alpha \quad \text{for} \quad 0 \leq \alpha \leq \frac{d+2}{d-2} \text{ if } d \neq 2, \quad \text{or for any } \alpha \geq 0 \text{ if } d = 2, \tag{81}$$

$$|\varepsilon(k)| \leq C_\varepsilon |k|^\theta \quad \text{for} \quad 0 \leq \theta \leq \frac{d+2}{d-2} \text{ if } d \neq 2, \quad \text{or for any } \theta \geq 0 \text{ if } d = 2. \tag{82}$$

Growth conditions on the feedbacks

On the functions $\mathbf{f}(\mathbf{u})$ and $\varepsilon(k)$

We assume the existence of nonnegative constants C_f and C_ε such that (a.e. in Ω)

$$|\mathbf{f}(\mathbf{u})| \leq C_f |\mathbf{u}|^\alpha \quad \text{for} \quad 0 \leq \alpha \leq \frac{d+2}{d-2} \text{ if } d \neq 2, \quad \text{or for any } \alpha \geq 0 \text{ if } d = 2, \tag{81}$$

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On the term $P(\mathbf{u}, k)$, we consider $P(\mathbf{u}, k) \equiv \pi(\mathbf{u})$ and $P(\mathbf{u}, k) \equiv \varpi(\mathbf{u})k$, where

$$\pi, \ \varpi : \Omega \times \mathbb{R}^d \to \mathbb{R}_0^+ \text{ are Carathéodory.}$$

- If $P(\mathbf{u}, k) \equiv \pi(\mathbf{u})$, we assume the existence of a constant $C_\pi \geq 0$ such that

$$|\pi(\mathbf{u})| \leq C_\pi |\mathbf{u}|^\beta \quad \text{for} \quad 0 \leq \beta \leq \frac{d+2}{d-2} \text{ if } d \neq 2, \quad \text{or for} \quad \text{any } \beta \geq 0 \text{ if } d = 2\,. \tag{83}$$

- If $P(\mathbf{u}, k) \equiv \varpi(\mathbf{u})k$, we assume the existence of a constant $C_\varpi > 0$ such that

$$|\varpi(\mathbf{u})| \leq C_\varpi |\mathbf{u}|^\beta \quad \text{for} \quad 0 \leq \beta \leq \frac{4}{d-2} \text{ if } d \neq 2\,, \quad \text{or for} \quad \text{any } \beta \geq 0 \text{ if } d = 2\,. \tag{84}$$

More assumptions

Sign conditions

$$\mathbf{f}(\mathbf{u}) \cdot \mathbf{u} \geq 0 \quad \text{for all } \mathbf{u} \in \mathbb{R}^d \text{ and a.e. in } \Omega\,, \tag{85}$$

$$\varepsilon(k)\, k \geq 0 \quad \text{for all } k \in \mathbb{R} \text{ and a.e. in } \Omega\,. \tag{86}$$

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We are particularly interested in the case of

$$\varepsilon(k) := e(k)k\,, \quad e : \Omega \times \mathbb{R} \to \mathbb{R}_0^+ \text{ is Carathéodory} \quad \left[\varepsilon = C_D \frac{k^{\frac{3}{2}}}{l}\right]. \tag{87}$$

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Remark

Condition (85) is satisfied by any feedback's forces field that we have considered above: Coriolis, Darcy or Darcy-Forchheimer. With respect to (86), this condition is always verified due to the definition of k and $\varepsilon(k)$ in the turbulent k–epsilon model.

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Assumptions, already touched on at (78), mathematically needed

$$0 \leq \nu_T(k) \leq C_T \quad \text{for all } k \in \mathbb{R} \text{ and a.e. in } \Omega, \quad C_T \in \mathbb{R}^+, \tag{88}$$

$$0 < c_D \leq \nu_D(k) \leq C_D \quad \text{for all } k \in \mathbb{R} \text{ and a.e. in } \Omega, \quad c_D, C_D \in \mathbb{R}^+. \tag{89}$$

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Existence under growth conditions

Theorem

Let Ω be a bounded domain of $\mathbb{R}^d$, $d = 2, 3$, with a Lipschitz-continuous compact boundary $\partial\Omega$. Assume $\mathbf{g} \in \mathbf{L}^2(\Omega)$ and all the conditions (74), (81)-(82), (85)-(86), (134)-(90) hold. In addition assume that one, but only one, of the following conditions is satisfied:

1. $P(\mathbf{u}, k) = \pi(\mathbf{u})$ *a.e. in* Ω *and (129) holds;*
2. $P(\mathbf{u}, k) = \varpi(\mathbf{u})k$ *a.e. in* Ω*, (130) holds, and*

$$c_D > C \left(\frac{\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}}{\nu} \right)^{\beta} \tag{91}$$

for the positive constant $C := C_{\varpi}\, \lambda(2, d)^2 \Lambda(2, d)^{\beta} C_K^{2\beta} \Lambda_P(d)^{\beta}$.

Then there exists, at least, a weak solution to the problem (70)-(73).

Remark

Observe that the possibility of $P(\mathbf{u}, k) = 0$ *a.e. in* Ω *is covered by condition (1), in particular when we take* $C_\pi = 0$ *in (129). On the other hand, the case of* $P(\mathbf{u}, k) = k$ *a.e. in* Ω *is contained in the condition (2) as a limit situation.*

Proof: STEP 1 - The regularized problem

We start by considering, for each $n \in \mathbb{N}$, the following regularized problem

$$\mathrm{div}\,\mathbf{u} = 0 \quad \text{in} \quad \Omega, \tag{92}$$

$$(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = \mathbf{g} - \mathbf{f}(\mathbf{u}) - \nabla p + \mathbf{div}\left((\nu + \nu_T(k))\,\mathbf{D}(\mathbf{u})\right) \quad \text{in } \Omega, \tag{93}$$

$$\mathbf{u}\cdot\boldsymbol{\nabla} k = \mathrm{div}\left(\nu_D(k)\nabla k\right) + \nu_T(k)\,\mathcal{R}_n\left(|\mathbf{D}(\mathbf{u})|^2\right) + P(\mathbf{u},k) - \varepsilon(k) \quad \text{in } \Omega, \tag{94}$$

$$\mathbf{u} = 0 \quad \text{and} \quad k = 0 \quad \text{on } \partial\Omega, \tag{95}$$

where $\mathcal{R}_n(a)$ denotes the following regularization of the nonnegative term a

$$\mathcal{R}_n(a) := \frac{a}{1 + \frac{1}{n}a}\,. \tag{96}$$

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$$\mathbf{u} \cdot \nabla k = \operatorname{div}(\nu_D(k)\nabla k) + \nu_T(k)\,\mathcal{R}_n\left(|\mathbf{D}(\mathbf{u})|^2\right) + P(\mathbf{u}, k) - \varepsilon(k) \quad \text{in } \Omega, \tag{94}$$

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Definition

Under the assumptions of Definition 1, we say a pair $(\mathbf{u}, k)$ is a weak solution to the regularized problem (92)-(95) if, for each $n \in \mathbb{N}$, (1) and (3) of Definition 1 hold, and

(2') $k \in H^1_0(\Omega)$ and for every $\varphi \in H^1_0(\Omega) \cap \mathrm{L}^d(\Omega)$ there holds

$$\begin{aligned}\int_\Omega (\mathbf{u} \cdot \nabla k)\varphi\, d\mathbf{x} + \int_\Omega \nu_D(k)\nabla k \cdot \nabla\varphi\, d\mathbf{x} + \int_\Omega \varepsilon(k)\,\varphi\, d\mathbf{x} = \\ \int_\Omega \nu_T(k)\mathcal{R}_n\left(|\mathbf{D}(\mathbf{u})|^2\right)\varphi\, d\mathbf{x} + \int_\Omega P(\mathbf{u}, k)\,\varphi\, d\mathbf{x}.\end{aligned} \tag{97}$$

Proof: STEP 2 - Proof of the existence for the regularized problem

Proposition

Let the conditions of Theorem 2 be fulfilled. Then (for each $n \in \mathbb{N}$) there exists, at least, a weak solution to the problem (92)-(95).

Proof.

Using the Galerkin approximations together with compactness arguments, we prove that, for each $n \in \mathbb{N}$, there exists a weak solution $(\mathbf{u}_n, k_n) \in \mathbf{V} \times H^1_0(\Omega)$ to the problem (92)-(95) and such that

$$\int_\Omega (\mathbf{u}_n \cdot \boldsymbol{\nabla})\mathbf{u}_n \cdot \mathbf{v}\, d\mathbf{x} + \int_\Omega (\nu + \nu_T(k_n))\mathbf{D}(\mathbf{u}_n) : \boldsymbol{\nabla}\mathbf{v}\, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u}_n) \cdot \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{g} \cdot \mathbf{v}\, d\mathbf{x} \tag{98}$$

and

$$\int_\Omega (\mathbf{u}_n \cdot \boldsymbol{\nabla} k_n) v\, d\mathbf{x} + \int_\Omega \nu_D(k_n) \boldsymbol{\nabla} k_n \cdot \boldsymbol{\nabla} v\, d\mathbf{x} + \int_\Omega \varepsilon(k_n) v\, d\mathbf{x} = \int_\Omega \nu_T(k_n)\, \mathcal{R}_n\left(|\mathbf{D}(\mathbf{u}_n)|^2\right) v\, d\mathbf{x} + \int_\Omega P(\mathbf{u}_n, k_n)\, v\, d\mathbf{x} \tag{99}$$

hold for all $(\mathbf{v}, v) \in (\mathbf{V} \cap \mathbf{L}^d(\Omega)) \times (H^1_0(\Omega) \cap L^d(\Omega))$. $\square$

Proof: STEP 3 - A priori estimates

We start by obtaining an estimate for $\mathbf{u}_n$, Since the sought solutions and the test functions are in the same function space, we can take $\mathbf{v} = \mathbf{u}_n$ in (98) and we obtain,

$$\|\nabla \mathbf{u}_n\|_{\mathbf{L}^2(\Omega)} \leq \frac{C_K^2}{\nu} \Lambda_P(d) \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \tag{100}$$

In view of (100) and of the assumption that $\mathbf{g} \in \mathbf{L}^2(\Omega)$, we have

$$\mathbf{u}_n \to \mathbf{u} \quad \text{weakly in } \mathbf{H}_0^1(\Omega), \quad \text{as } n \to \infty, \tag{101}$$

$$\mathbf{u}_n \to \mathbf{u} \quad \text{strongly in } \mathbf{L}^s(\Omega), \quad \text{as } n \to \infty, \quad \text{for any } s : 1 \leq s < 2^* \tag{102}$$

$$\mathbf{u}_n \to \mathbf{u} \quad \text{a.e. in } \Omega, \quad \text{as } n \to \infty \tag{103}$$

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$$\mathbf{u}_n \to \mathbf{u} \quad \text{a.e. in } \Omega, \quad \text{as } n \to \infty \tag{103}$$

We take

$$v = \varphi(k_n), \qquad \text{where} \quad \varphi(k_n) := 1 - \frac{1}{(1+k_n)^\delta}, \tag{104}$$

where δ is a positive constant, in (99) to get

$$\int_\Omega |\nabla k_n|^q \, d\mathbf{x} \leq C\,, \quad C = C(\nu, \beta, c_D, C_T, C_\pi, d, q, \Omega, \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}), \quad \text{if} \ \ P(\mathbf{u}, k) = \pi(\mathbf{u}), \tag{105}$$

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In view of (100) and of the assumption that $\mathbf{g} \in \mathbf{L}^2(\Omega)$, we have

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Consequently,

$$k_n \to k \quad \text{weakly in } \mathrm{W}_0^{1,q}(\Omega), \quad \text{as } n \to \infty, \quad \text{for } q < d', \tag{107}$$

$$k_n \to k \quad \text{strongly in } \mathrm{L}^s(\Omega), \quad \text{as } n \to \infty, \quad \text{for all } s\text{: } 1 \leq s < q^*, \tag{108}$$

$$k_n \to k \quad \text{a.e. in } \Omega, \quad \text{as } n \to \infty. \tag{109}$$

STEP 4A - Passing $\mathcal{R}_n(|\mathbf{D}(\mathbf{u}_n)|^2)$ to the limit $n \to \infty$

To prove this, first we need to show that

$$\mathbf{D}(\mathbf{u}_n) \to \mathbf{D}(\mathbf{u}) \quad \text{strongly in } \mathbf{L}^2(\Omega), \quad \text{as } n \to \infty, \tag{110}$$

$$\mathbf{D}(\mathbf{u}_n) \to \mathbf{D}(\mathbf{u}) \quad \text{a.e. in } \Omega, \quad \text{as } n \to \infty. \tag{111}$$

To do it, we take $\mathbf{v}_n = \mathbf{u}_n - \mathbf{u}$ in (98) and after some algebraic manipulations, we get

$$\begin{aligned}
&\int_\Omega (\nu + \nu_T(k_n))\, |\mathbf{D}(\mathbf{u}_n) - \mathbf{D}(\mathbf{u})|^2 \, dx \\
= &\int_\Omega (\mathbf{u}_n \cdot \nabla)\mathbf{u}_n \cdot (\mathbf{u} - \mathbf{u}_n)\, dx + \int_\Omega [\nu_T(k_n)\mathbf{D}(\mathbf{u}) - \nu_T(k)\mathbf{D}(\mathbf{u})] : (\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}_n))\, dx \\
&+ \int_\Omega (\nu + \nu_T(k))\, \mathbf{D}(\mathbf{u}) : (\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}_n))\, dx + \int_\Omega \mathbf{g} \cdot (\mathbf{u}_n - \mathbf{u})\, dx + \int_\Omega \mathbf{f}(\mathbf{u}_n) \cdot (\mathbf{u} - \mathbf{u}_n)\, dx
\end{aligned}$$

Using Hölder's and Cauchy's inequalities together with (81) and (134), we get

$$\begin{aligned}
\frac{\nu}{2}\int_\Omega |\mathbf{D}(\mathbf{u}_n) - \mathbf{D}(\mathbf{u})|^2 \, dx \leq{}& \|\mathbf{u}_n\|_{\mathbf{L}^4(\Omega)} \|\nabla \mathbf{u}_n\|_{\mathbf{L}^2(\Omega)} \|\mathbf{u}_n - \mathbf{u}\|_{\mathbf{L}^4(\Omega)} + \frac{1}{2\nu}\|\nu_T(k_n)\mathbf{D}(\mathbf{u}) - \nu_T(k)\mathbf{D}(\mathbf{u})\|^2_{\mathbf{L}^2(\Omega)} \\
&+ \int_\Omega (\nu + \nu_T(k))\, \mathbf{D}(\mathbf{u}) : (\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}_n))\, dx + \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{u}_n - \mathbf{u}\|_{\mathbf{L}^2(\Omega)} + C_f \|\mathbf{u}_n\|^\alpha_{\mathbf{L}^{2^*}(\Omega)} \|\mathbf{u} - \mathbf{u}_n\|_{\mathbf{L}^2(\Omega)}.
\end{aligned} \tag{112}$$

Then can apply Lebesgue's dominated convergence theorem, to show that

$$|\nu_T(k_n)\mathbf{D}(\mathbf{u}) - \nu_T(k)\mathbf{D}(\mathbf{u})|^2 \to 0 \quad \text{strongly in } \mathbf{L}^1(\Omega), \quad \text{as } j \to \infty. \tag{113}$$

As a consequence (110) holds and, due to Riesz-Fisher's theorem, (111) also holds.

STEP 4B - Passing $\mathcal{R}_n(|\mathbf{D}(\mathbf{u}_n)|^2)$ to the limit $n \to \infty$

Using the definition of $\mathcal{R}_n$, we have

$$\int_\Omega \left| \left(\nu_T(k_n)\, \mathcal{R}_n\left(|\mathbf{D}(\mathbf{u}_n)|^2 \right) - \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 \right) v \right| \, d\mathbf{x}$$

$$\leq \int_\Omega \left| \nu_T(k_n)|\mathbf{D}(\mathbf{u}_n)|^2 - \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 \right| \, |v| \, d\mathbf{x} + \int_\Omega \frac{1}{n} \frac{\nu_T(k)|\mathbf{D}(\mathbf{u})|^2 |\mathbf{D}(\mathbf{u}_n)|^2}{1 + \frac{1}{n}|\mathbf{D}(\mathbf{u}_n)|^2} \, |v| \, d\mathbf{x}. \tag{114}$$

Testing (98) with $\mathbf{v} = \mathbf{u}_n$, we have, due to the symmetry of $\mathbf{D}(\mathbf{u}_n)$, that

$$\int_\Omega (\nu + \nu_T(k_n))|\mathbf{D}(\mathbf{u}_n)|^2 \, d\mathbf{x} = \int_\Omega \mathbf{g} \cdot \mathbf{u}_n \, d\mathbf{x} - \int_\Omega \mathbf{f}(\mathbf{u}_n) \cdot \mathbf{u}_n \, d\mathbf{x}. \tag{115}$$

Then, we take $\mathbf{v} = \mathbf{u}$ in (79),

$$\int_\Omega (\nu + \nu_T(k))\, |\mathbf{D}(\mathbf{u})|^2 \, d\mathbf{x} = \int_\Omega \mathbf{g} \cdot \mathbf{u} \, d\mathbf{x} - \int_\Omega \mathbf{f}(\mathbf{u}) \cdot \mathbf{u} \, d\mathbf{x}\,. \tag{116}$$

Now, in view of (115)-(116), we have

$$\int_\Omega \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 \, d\mathbf{x} \leq \liminf \int_\Omega \nu_T(k_n)|\mathbf{D}(\mathbf{u}_n)|^2 \, d\mathbf{x} \leq \limsup \int_\Omega \nu_T(k_n)|\mathbf{D}(\mathbf{u}_n)|^2 \, d\mathbf{x} \leq$$

$$\limsup \left(\int_\Omega \mathbf{g} \cdot \mathbf{u}_n \, d\mathbf{x} - \int_\Omega \mathbf{f}(\mathbf{u}_n) \cdot \mathbf{u}_n \, d\mathbf{x} - \int_\Omega \nu |\mathbf{D}(\mathbf{u}_n)|^2 \, d\mathbf{x} \right) \leq$$

$$\int_\Omega \mathbf{g} \cdot \mathbf{u} \, d\mathbf{x} - \int_\Omega \mathbf{f}(\mathbf{u}) \cdot \mathbf{u} \, d\mathbf{x} - \int_\Omega \nu |\mathbf{D}(\mathbf{u})|^2 \, d\mathbf{x} = \int_\Omega \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 \, d\mathbf{x}\,.$$

Thus,

$$\nu_T(k_n)|\mathbf{D}(\mathbf{u}_n)|^2 \to \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 \quad \text{strongly in } \mathbf{L}^1(\Omega), \quad \text{as } m \to \infty$$

and, consequently, the first integral of the right-hand side of (114) also converges to zero.

Uniqueness

Additional assumptions of monotonicity

$$(\mathbf{f}(\mathbf{u}_1)-\mathbf{f}(\mathbf{u}_2))\cdot(\mathbf{u}_1-\mathbf{u}_2)\geq 0 \quad \text{for all } \mathbf{u}_1,\ \mathbf{u}_2\in\mathbb{R}^d \quad \text{and} \quad \text{a.e. in } \Omega, \tag{117}$$

$$(\varepsilon(k_1)-\varepsilon(k_2))\,(k_1-k_2)\geq 0 \quad \text{for all } k_1,\ k_2\in\mathbb{R} \quad \text{and} \quad \text{a.e. in } \Omega. \tag{118}$$

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$$(\mathbf{f}(\mathbf{u}_1) - \mathbf{f}(\mathbf{u}_2)) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \geq 0 \quad \text{for all } \mathbf{u}_1,\ \mathbf{u}_2 \in \mathbb{R}^d \quad \text{and} \quad \text{a.e. in } \Omega, \tag{117}$$

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Additional assumptions of Lipschitz-continuity

There exist positive constants L_{ν_T}, L_{ν_D} and $\pi(u)$ or $\varpi(u)$:

$$|\nu_T(k_1) - \nu_T(k_2)| \leq L_{\nu_T}|k_1 - k_2| \quad \text{for all } k_1,\ k_2 \in \mathbb{R} \quad \text{and} \quad \text{a.e. in } \Omega, \tag{119}$$

$$|\nu_D(k_1) - \nu_D(k_2)| \leq L_{\nu_D}|k_1 - k_2| \quad \text{for all } k_1,\ k_2 \in \mathbb{R} \quad \text{and} \quad \text{a.e. in } \Omega. \tag{120}$$

$$|\pi(\mathbf{u}_1) - \pi(\mathbf{u}_2)| \leq L_\pi\,|\mathbf{u}_1 - \mathbf{u}_2| \quad \text{for all } \mathbf{u}_1,\ \mathbf{u}_2 \in \mathbb{R}^d \quad \text{and} \quad \text{a.e. in } \Omega, \tag{121}$$

$$|\varpi(\mathbf{u}_1) - \pi(\mathbf{u}_2)| \leq L_\varpi\,|\mathbf{u}_1 - \mathbf{u}_2| \quad \text{for all } \mathbf{u}_1,\ \mathbf{u}_2 \in \mathbb{R}^d \quad \text{and} \quad \text{a.e. in } \Omega. \tag{122}$$

Uniqueness

Additional assumptions of monotonicity

$$(\mathbf{f}(\mathbf{u}_1)-\mathbf{f}(\mathbf{u}_2))\cdot(\mathbf{u}_1-\mathbf{u}_2)\geq 0 \quad \text{for all } \mathbf{u}_1,\ \mathbf{u}_2\in\mathbb{R}^d \quad \text{and} \quad \text{a.e. in } \Omega, \tag{117}$$

$$(\varepsilon(k_1)-\varepsilon(k_2))\,(k_1-k_2)\geq 0 \quad \text{for all } k_1,\ k_2\in\mathbb{R} \quad \text{and} \quad \text{a.e. in } \Omega. \tag{118}$$

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There exist positive constants L_{ν_T}, L_{ν_D} and $\pi(u)$ or $\varpi(u)$:

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$$|\pi(\mathbf{u}_1)-\pi(\mathbf{u}_2)|\leq L_{\pi}\,|\mathbf{u}_1-\mathbf{u}_2| \quad \text{for all } \mathbf{u}_1,\ \mathbf{u}_2\in\mathbb{R}^d \quad \text{and} \quad \text{a.e. in } \Omega, \tag{121}$$

$$|\varpi(\mathbf{u}_1)-\pi(\mathbf{u}_2)|\leq L_{\varpi}\,|\mathbf{u}_1-\mathbf{u}_2| \quad \text{for all } \mathbf{u}_1,\ \mathbf{u}_2\in\mathbb{R}^d \quad \text{and} \quad \text{a.e. in } \Omega. \tag{122}$$

Theorem

Let $(\mathbf{u}, k)$ be a weak solution to (70)-(73) in the conditions of Theorem 2. Assume (117)-(118), (119)-(120) and (121)-(122) are fulfilled. If $\exists\ C_1 = C(\nu)\,,\ C_2 = C(\nu, c_D) > 0$:

$$\|\nabla\,\mathbf{u}\|_{\mathbf{L}^{\sigma}(\Omega)} < C_1 \quad \textit{for some } \sigma>2 \textit{ if } d=2, \quad \textit{or for some } \sigma\geq d \textit{ if } d\neq 2 \tag{123}$$

$$\|\nabla k\|_{L^{\tau}(\Omega)} < C_2 \quad \textit{for some } \tau>2 \textit{ if } d=2, \quad \textit{or for some } \tau\geq d \textit{ if } d\neq 2 \tag{124}$$

Proposition

Let $(\mathbf{u}, k)$ be a weak solution to the problem (70)-(73) in the conditions of Theorem 2. Then there exist positive constants C, C_1 and C_2 such that

$$\|\nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)} \leq C\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}, \tag{125}$$

$$\|\nabla k\|_{L^q(\Omega)} \leq C_1\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} + C_2. \tag{126}$$

Existence of a unique pressure

Proposition

Let $(\mathbf{u}, k)$ be a weak solution to the problem (70)-(73) in the conditions of Theorem 2. Then there exist positive constants C, C_1 and C_2 such that

$$\|\nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)} \leq C\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}\,, \tag{125}$$

$$\|\nabla k\|_{L^q(\Omega)} \leq C_1\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} + C_2. \tag{126}$$

Theorem

Let $(\mathbf{u}, k)$ be a weak solution to the problem (70)-(73) in the conditions of Theorem 2. Then there exists a unique $p \in L^\eta(\Omega)$, with $\int_\Omega p\, d\mathbf{x} = 0$ and $\eta \geq \max\left\{2, \frac{d}{2}, \frac{2d}{2d-\alpha(d-2)+2}\right\}$ if $d \neq 2$, or $\eta \geq 2$ if $d = 2$, such that

$$\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\, dx + \int_\Omega (\nu+\nu_T(k))\mathbf{D}(\mathbf{u}) : \nabla\mathbf{v}\, dx + \int_\Omega \mathbf{f}(\mathbf{u})\cdot\mathbf{v}\, dx - \int_\Omega \mathbf{g}\cdot\mathbf{v}\, dx = \int_\Omega p\, \mathrm{div}\,\mathbf{v}\, dx \tag{127}$$

for any $\mathbf{v} \in \mathbf{W}^{1,\eta}_0(\Omega)$. Moreover, there exist positive constants C_1, C_2, C_3 and C_4 such that

$$\|p\|_{L^\eta(\Omega)} \leq C_1\|\nabla\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + C_2\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\Omega)} + C_3\|\nabla\mathbf{u}\|^\alpha_{\mathbf{L}^2(\Omega)} + C_4\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \tag{128}$$

Growth conditions only on $P(\mathbf{u}, k)$:

If $P(\mathbf{u}, k) = \pi(\mathbf{u})$, $\quad |\pi(\mathbf{u})| \leq C_\pi |\mathbf{u}|^\beta \quad 0 \leq \beta \leq \dfrac{2d}{d-2}$ if $d \neq 2$, $\quad$ any $\beta \geq 0$ if $d = 2$

If $P(\mathbf{u}, k) = \varpi(\mathbf{u})k$, $\quad |\varpi(\mathbf{u})| \leq C_\varpi |\mathbf{u}|^\beta$ for $\quad 0 \leq \beta \leq \dfrac{d}{d-2}$ if $d \neq 2$, $\quad$ or for any

Existence under no growth conditions

Growth conditions only on $P(\mathbf{u}, k)$:

If $P(\mathbf{u}, k) = \pi(\mathbf{u})$, $\quad |\pi(\mathbf{u})| \leq C_\pi |\mathbf{u}|^\beta \quad 0 \leq \beta \leq \dfrac{2d}{d-2}$ if $d \neq 2$, $\quad$ any $\beta \geq 0$ if $d = 2$

If $P(\mathbf{u}, k) = \varpi(\mathbf{u})k$, $\quad |\varpi(\mathbf{u})| \leq C_\varpi |\mathbf{u}|^\beta$ $\quad$ for $\quad 0 \leq \beta \leq \dfrac{d}{d-2}$ if $d \neq 2$, $\quad$ or for any

Theorem

Let Ω be a bounded domain of $\mathbb{R}^d$, $d = 2, 3$, with a Lipschitz-continuous compact boundary $\partial\Omega$. Assume $\mathbf{g} \in \mathbf{L}^2(\Omega)$ and all the conditions (74), (85)-(86), (134)-(90) and (129)-(130) hold. In addition, assume that

$$\exists\ \tau > 0 : |\angle(\mathbf{f}(\mathbf{u}), \mathbf{u})| \notin \left(\frac{\pi}{2} - \tau, \frac{\pi}{2} + \tau\right) \quad \forall\ \mathbf{u} : |\mathbf{u}| \geq \mathrm{L}, \quad \forall\ L > 0, \tag{131}$$

$$H_L \in \mathbf{L}^1(\Omega) \quad \forall\ \mathrm{L} > 0, \quad \textit{where} \quad H_L := \sup_{|\mathbf{u}| \leq \mathrm{L}} |\mathbf{f}(\mathbf{u})|, \tag{132}$$

$$G_M \in \mathrm{L}^1(\Omega) \quad \forall\ M > 0, \quad \textit{where} \quad G_M := \sup_{|k| \leq M} |\varepsilon(k)|. \tag{133}$$

If one of the following conditions is satisfied:

1. $P(\mathbf{u}, k) = \pi(\mathbf{u})$ *a.e. in* Ω *and (129) holds;*
2. $P(\mathbf{u}, k) = \varpi(\mathbf{u})k$ *a.e. in* Ω, *(130) holds and (91) is satisfied for the positive constant*

Concluding remarks

- Comparing the models: The Nakayama & Kuwahara (1999) model requires less assumptions then the model from Pedras & de Lemos (2000):

$$\mathbf{f}(\mathbf{u}) = \frac{\mu}{K}\phi\mathbf{u} - \frac{c_F}{\sqrt{K}}\phi^2|\mathbf{u}|\mathbf{u}\,,$$

$$P_{NK}(\mathbf{u},k) \equiv P(\mathbf{u}) = 39\phi^2\sqrt[5]{(1-\phi)^2}\frac{|\mathbf{u}|^3}{d}\,, \qquad P_{PL}(\mathbf{u},k) = \frac{c_k\phi^3}{\sqrt{K}}k|\mathbf{u}|.$$

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- Work on progress: with growth conditions on $\mathbf{f}(\mathbf{u})$ and on $\varepsilon(k)$:
  - Regularity up to boundary;

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- Work on progress: with growth conditions on $\mathbf{f}(\mathbf{u})$ and on $\varepsilon(k)$:
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- Work on progress: with no growth conditions on $\mathbf{f}(\mathbf{u})$ and on $\varepsilon(k)$:
  - Uniqueness;
  - Existence of the pressure;
  - Regularity.

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- Work on progress: with growth conditions on $\mathbf{f}(\mathbf{u})$ and on $\varepsilon(k)$:
  - Regularity up to boundary;
- Work on progress: with no growth conditions on $\mathbf{f}(\mathbf{u})$ and on $\varepsilon(k)$:
  - Uniqueness;
  - Existence of the pressure;
  - Regularity.
- Open problems:
  - Removing not admissible physics restrictions:

$$0 \le \nu_T(k) \le C_T \quad \text{for all } k \in \mathbb{R} \text{ and a.e. in } \Omega, \quad C_T \in \mathbb{R}^+,$$

$$0 < c_D \le \nu_D(k) \le C_D \quad \text{for all } k \in \mathbb{R} \text{ and a.e. in } \Omega, \quad c_D, C_D \in \mathbb{R}^+;$$

  - Generalized dimension d;
  - The parabolic version of problem.

Related works

- On a One-Equation Turbulent Model with Feedbacks. Springer Proceedings in Mathematics & Statistics 164, pp 51–61 (with A. Paiva).
  http://link.springer.com/chapter/10.1007/978-3-319-32857-7_5
- A stationary turbulent one-equation model with applications in porous media. Submitted. (with A. Paiva)
  Preprint available in http://cmafcio.campus.ciencias.ulisboa.pt.
- Anisotropically diffused and damped Navier-Stokes equations, Discrete Contin. Dyn. Syst. 2015, Suppl., 349-358 (2015).
- Existence for a one-equation turbulent model with strong nonlinearities. Submitted. (with A. Paiva)
  Preprint available in http://cmafcio.campus.ciencias.ulisboa.pt.

Muito obrigado!