Rigidity of geodesic completeness in Lorentzian geometry

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Motivation

MOTIVATION:

- A key aspect of Lorentzian geometry is the emphasis on geodesic (in)completeness of manifolds with physically motivated geometric conditions. A primary application of Lorentzian geometry is to General Relativity, where incompleteness of the so-called causal geodesics are related to the geometric description of black holes and the "big bang singularities" in cosmological models.
- The question of geodesic completeness is much better understood in Riemannian geometry. For example, it is well known that every compact Riemannian manifold is geodesically complete (a consequence of the Hopf-Rinow theorem), and that the set of complete Riemannian metrics is dense in the space of all Riemannian metrics (with the compact-open topology) on a given manifold (Morrow '70).

MOTIVATION:

- In Lorentzian geometry, by contrast, some of the physically most important examples are geodesically incomplete. Consequently, geodesically complete Lorentzian manifolds of relevance to physics seem to be fairly special. This gives the rise to *rigidity* questions: giving a geometric description of such geodesically complete manifolds.
- In this talk, we wish to review some old and new such rigidity results which underscore this general philosophy.

Basic definitions

Lorentz vector spaces

Definition

A *Lorentz vector space* is a real vector space V of finite dimension $n \geq 2$, endowed with a bilinear symmetric form $\langle . . \rangle : V \times V \to \mathbb{R}$ with the following property: there exists a basis with respect to which

$$\langle v, w \rangle = -v_1 w_1 + \cdots + v_n w_n.$$

Such a bilinear form is called a *Lorentz scalar product*.

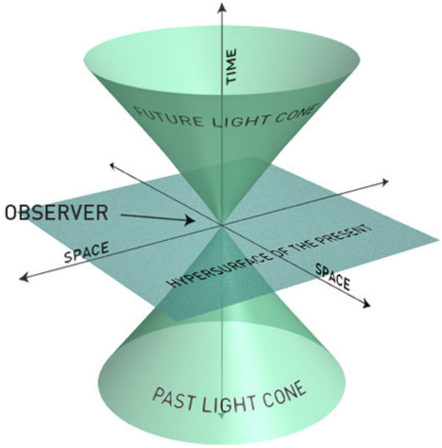
Causal character of vectors

Definition

If V is a Lorentz vector space, a nonzero vector $v \in V$ is said to be

- Timelike, if $\langle v, v \rangle < 0$;
- Spacelike, if $\langle v, v \rangle > 0$;
- Lightlike or null, if $\langle v, v \rangle = 0$;

Causal character of vectors



Spacetime: definition

Definition

A *Lorentzian metric* on a smooth manifold M of dimension $n \geq 2$ is a smooth mapping g which assigns to each $p \in M$ a Lorentz scalar product $g_p(\,.\,,\,.\,)$ on the tangent space T_pM at p. The pair (M, g) is then said to be a *Lorentzian manifold*. If in addition M is connected and (M, g) is *time-oriented*, then (M, g) is said to be a *spacetime*.

Definitions of (Levi-Civita) connection, curvature, Ricci tensor and curvature scalar are exactly as in Riemannian geometry.

Causal character extended

- A smooth curve $\alpha : I \subseteq \mathbb{R} \to M$ [resp. a vector field $X : M \to TM$] on a Lorentzian manifold (M, g) is *timelike* (resp. *spacelike, null*) if $\alpha'(t)$ [resp. $X(p) \in T_pM$] has the corresponding causal character for every $t \in I$ [resp. $p \in M$]. If $\alpha'(t)$ is everywhere nonzero and nonspacelike, then α is said to be *nonspacelike* (or *causal*).
- A submanifold $N \subset M$ is *spacelike* if the induced metric on N is Riemannian.
- A subset $A \subset M$ is *achronal* if no two points of A can be connected by a timelike curve. A *Cauchy hypersurface* is a subset $S \subset M$ which is met exactly once by every inextendible timelike curve in M. If a Cauchy hyperurface exists, then (M, g) is said to be *globally hyperbolic.*

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Geodesics and geodesic completeness defined

- A smooth curve $\alpha : I \subseteq \mathbb{R} \to M$ is a *geodesic* if $\nabla_{\alpha'}\alpha' = 0$.
- α is *complete* if we can extend its domain to $\mathbb{R}$. Otherwise it is *incomplete*.
- The notion of geodesic completeness for null, timelike and spacelike geodesics are logically independent (Geroch).
- (M, g) is *timelike* [resp. *null*, *spacelike*] *geodesically incomplete* if there exists at least one timelike [resp. null, spacelike] geodesic which is incomplete.

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An example: Schwarzschild spacetime

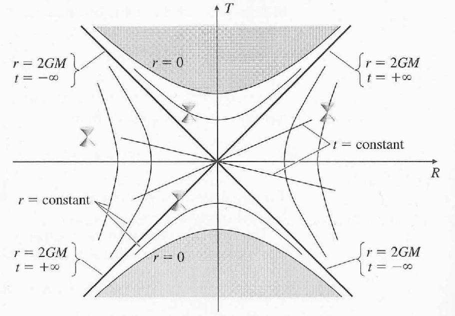

Figure: The extended Schwarzschild-Kruskal spacetime

Conformaly compactfied Schwarzschild-Kruskal spacetime

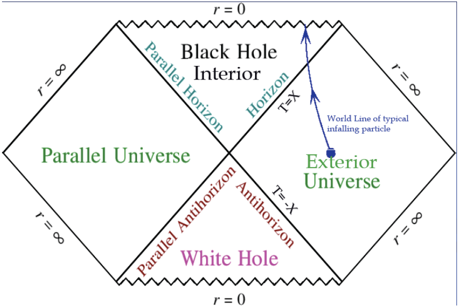

Figure: Schwarzschild-Kruskal spacetime: Penrose diagram

The Lorentzian Splitting Theorem

A motivating example: Robertson-Walker spacetimes

- A *Robertson-Walker spacetime* is of the form $((a,b)\times S, -dt^2 + f(t)^2 h)$, where $-\infty \le a < b \le +\infty$, $f : (a,b) \to (0,+\infty)$ a positive smooth function and (S,h) is a Riemannian space form, i.e., a connected simply connected geodesically complete Riemannian manifold of constant curvature.
- Such spacetimes are timelike geodesically incomplete if $a > -\infty$ and/or $b < +\infty$, and null geodesically incomplete if $\lim_{t\to a^+}\int_t^c f(s)ds$ and/or $\lim_{t\to b^-}\int_c^t f(s)ds$ are finite.
- In General Relativity, these spacetimes are used as idealized models for the universe as a whole. Singularities are interpreted as the *big bang* and/or the *big crunch*.

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Space forms

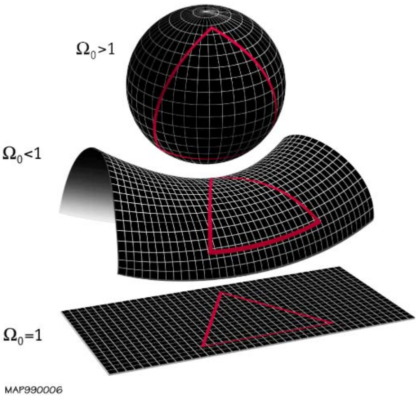

Figure: Space forms: sphere, euclidean or hyperbolic space.

The Big Bang

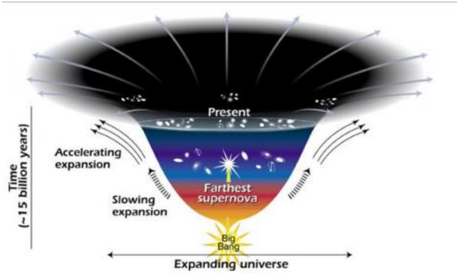

Figure: The Big bang

The Hawking-Penrose singularity theorem

Theorem

[Hawking & Penrose (1970)] Let (M^n, g) be a spacetime, with $n \geq 3$. Assume that:

i) (M, g) is *chronological*, i.e., has no closed timelike curves.
ii) (M, g) satisfies the nonspacelike generic condition.
iii) $Ric(v, v) \geq 0$, $\forall v \in TM$ timelike.
iv) There exists an achronal, spacelike, connected, compact hypersurface without boundary $S \subset M$.

Then some causal geodesic in (M, g) is incomplete.

The Lorentzian splitting theorem

Theorem

[Galloway, Horta, Beem, Erhlich, Markovsen, ...] Let (M^n, g) be a spacetime, with $n \geq 3$. Assume that:

i) (M, g) is either globally hyperbolic or timelike geodesically complete.
ii) (M, g) has a timelike geodesic line.
iii) $Ric(v, v) \geq 0$, $\forall v \in TM$ timelike.

Then (M, g) splits isometrically as $(\mathbb{R} \times S, -dt^2 + h)$, where (S, h) is a complete Riemannian manifold.

Open problem: the Bartnik conjecture

Conjecture

[R. Bartnik, 1988] Let (M^n, g) be a spacetime, with $n \geq 2$. Assume that:

i) (M, g) is globally hyperbolic with a compact Cauchy hypesurface.

ii) (M, g) is timelike geodesically complete.

iii) $Ric(v, v) \geq 0$, $\forall v \in TM$ timelike.

Then (M, g) splits isometrically as $(\mathbb{R} \times S, -dt^2 + h)$, where (S, h) is a compact Riemannian manifold.

Recent progress towards proving this conjecture has been made by G. Galloway and C. Vega.

Rigidity of stationary and Brinkmann spacetimes

Definition

A spacetime (M, g) is said to be

i) *stationary* if there exists a complete timelike Killing vector field $X \in \Gamma(TM)$,

ii) *Brinkmann* if there exists a complete null parallel vector field $X \in \Gamma(TM)$, i.e., $\nabla X = 0$.

Stationary spacetimes have important application in black hole physics, while certain Brinkmann spacetimes model an idealized class of gravitational wave solutions in GR.

Anderson's rigidity of stationary spacetimes

Theorem

[M. Anderson (2000)] Any geodesically complete **chronological** Ricci-flat 4-d stationary spacetime is isometric to (a quotient of) Minkowski spacetime.

- **Query**: is there an analogue of this result for Brinkmann spacetimes?
- **Answer**: There exists a *conjectural*, partially proven analogue!

Plane waves

Definition

A spacetime (M^{n+2}, g) is said to be a *(standard) pp-wave* if $M = \mathbb{R}^{n+2}$ with metric

$$g = 2du(dv + H(u,x)du) + \sum_{i,j=1}^{n} dx_i^2,$$

where $\mathcal{X} = \partial_v$ is a null parallel vector field. If H (the *potential*) does not depend of u, the pp-wave is said to be *autonomous*. Moreover, if

$$H(u,x) = \sum_{i,j=1}^{n} a_{ij}(u)x^i x^j,$$

the corresponding pp-wave is called a *plane wave.*

The Ehlers-Kundt conjecture

Conjecture

[Ehlers and Kundt (1962)] Any geodesically complete Ricci-flat 4-d pp-wave is a plane wave.

We will instead consider an alternative version with stronger assumptions:

Brinkmann Rigidity Theorem

[IPCS, J.L. Flores, J. Herrera'16] Let (M, g) be a 4-d **strongly causal**, geodesically complete, Ricci-flat **transversally Killing** Brinkmann spacetime. Then, the universal covering spacetime $(\overline{M}, \overline{g})$ of (M, g) is isometric to a plane wave.

RESULTS II:

- A concrete situation where this result applies is when the Brinkmann spacetime is an *autonomous* pp-wave. In this case the pp-wave is indeed transversally Killing. Therefore, we deduce the following version of the Ehlers-Kundt conjecture.

Corollary: Autonomous Ehlers-Kundt

[IPCS, J.L. Flores, J. Herrera '16] Every geodesically complete, **strongly causal, autonomous,** Ricci-flat, 4-dimensional pp-wave is a Cahen-Wallach space.

- A *Cahen-Wallach space* is an indecomposable, solvable geodesically complete symmetric Lorentzian manifold. These were classified by M. Cahen and N. Wallach in 1970.

Outline of Proof

- Geodesic completeness implies that $\mathcal{X}$ is complete, so it has a global flow $\phi : \mathbb{R} \times M \to M$.
- The flow ϕ is an (isometric) $\mathbb{R}$-action, which by virtue of strong causality is free and proper.
- Since $\mathcal{Y}$ is complete and commutes with $\mathcal{X}$, its flow φ together with ϕ defines an $\mathbb{R}^2$-action Φ which is also free and proper.
- M is then a (trivial) principal $\mathbb{R}^2$-bundle over the (smooth) quotient $Q := M/\Phi$. In particular $M \simeq \mathbb{R}^2 \times Q$.
- The metric splits accordingly. Passing to the covering and using Ricci-flatness we show that Q is Euclidean and the distribution $\mathcal{X}^\perp \cap \mathcal{Y}^\perp$ is integrable.

Outline of Proof

Definition:

A function $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}$ is *at most quadratic* if there exist numbers $a, b > 0$ such that

$$\mathcal{F}(x) \leq a\|x\|^2 + b, \forall x \in \mathbb{R}^n.$$

- Note that if a function $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}$ is *not* at most quadratic, then there exists a sequence $\{x_k\}_k$ in $\mathbb{R}^n$ for which

$$\mathcal{F}(x_k) > k\|x_k\|^2 + k, \forall k \in \mathbb{N},$$

  and, in particular, $\|x_k\| \to +\infty$ as $k \to +\infty$.
- Clearly, if $\mathcal{F}$ remains bounded above by a polynomial of degree at most 2 outside a compact subset of $\mathbb{R}^n$ then $\mathcal{F}$ is at most quadratic.

The following Lemma is key.

Lemma 1

[H.P. Boas and R.P. Boas, '88] A harmonic function $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}$ bounded from one side by a polynomial of degree m is also a polynomial of degree at most m. In particular, if $\mathcal{F}$ is at most quadratic, then there exist numbers $a_{ij}, b_j \in \mathbb{R}$ ($i, j \in \{1, \ldots, n\}$) such that

$$\mathcal{F}(x) = \sum_{i,j=1}^{n} a_{ij} x^i x^j + \sum_{j=1}^{n} b_j x^j + \mathcal{F}(0).$$

Lemma 2

Let $\Omega \subseteq \mathbb{C} \equiv \mathbb{R}^2$ be an open set containing 0, and let $\mathcal{F} : \Omega \to \mathbb{R}$ be a harmonic function such that $\mathcal{F}(0) = 0$. Then, for each $R > 0$ such that $\overline{B_R(0)} \subset \Omega$, and for each $p \in \partial B_R(0)$, there exists a piecewise smooth curve $z : [0,1] \to \overline{B_R(0)}$ such that

i) $z(0) = z(1) = 0$ and $z(t_0) = p$ for some $t_0 \in (0,1)$,

ii) $\int_0^1 \mathcal{F}(z(t))dt \geq \frac{1}{5}\mathcal{F}(p)$, and

iii) $\int_0^1 \|\dot{z}(t)\|^2 dt \leq 50\pi^2 R^2$.

And now, for something NOT completely different....

- We can then assume without loss of generality that (M, g) is a pp-wave.
- In particular, we have global coordinates $\{u, v, x, y\}$ for which g has the expression

$$g = 2du(dv + H(u, x, y)du) + dx^2 + dy^2,$$

with H harmonic in x, y.
- We wish to show that H is quadratic in the coordinates x, y.

- Assume then, by way of contradiction, that H is *not* quadratic. Since $-H$ is spatially harmonic, due to Lemma 1 $-H$ *can not* be at most quadratic in x, y. Therefore we can pick a sequence $p_k = (x_k, y_k)$ in $\mathbb{R}^2$ for which

$$-H(u, p_k) > k\|p_k\|^2 + k, \quad \forall k \in \mathbb{N} \tag{1}$$

and $R_k := \|p_k\| \to +\infty$ as $k \to +\infty$.

- **Basic strategy:** We show the existence of some open set $\mathcal{U}_0$ containing the origin $(0,0,0,0)$ of $M \equiv \mathbb{R}^4$ and timelike curve segments with endpoints arbitrarily close to the origin, such that they are not contained in $\mathcal{U}_0$, in violation of our assumption of strong causality for (M,g).
- This contradiction then yields that H is indeed quadratic, which in turn establishes the theorem.
- For any two numbers $\Delta, E > 0$ and for each $k \in \mathbb{N}$, we can define the curve $\Gamma_k^E : [0,1] \to \mathbb{R}^4$ given, for each $t \in [0,1]$, for the form

$$\Gamma_k^E(t) := (V_k^E(t), \Delta t, Z_k(t)),$$

where $Z_k(t)$ is chosen using Lemma 2, and $V_k^E(t)$ is chosen to make Γ_k^E be timelike curve, with

$$g(\dot{\Gamma}_k^E, \dot{\Gamma}_k^E) = -E.$$

- Consider the smooth functions $h_k : E \in (0, +\infty) \mapsto V_k^E(1) \in \mathbb{R}$ $(k \in \mathbb{N})$. For each $k \in \mathbb{N}$, $h_k(E) < 0$ for large enough E. However, collecting appropriate estimates we get

$$h_k(1) \geq (\frac{k}{5}\Delta - \frac{C(\alpha, \Delta)}{2\Delta})R_k^2 + \frac{k}{5} - \frac{1}{2\Delta}.$$

- It is clear from this inequality that we can pick $k_0 \in \mathbb{N}$ for which $h_{k_0}(1) > 0$, and since h_{k_0} is continuous, there exists $E_0 > 0$ for which $h_{k_0}(E_0) = 0$.
- We then conclude that $\Gamma_{k_0}^{E_0}$ is a timelike curve such that $\Gamma_{k_0}^{E_0}(0) = (0,0,0,0)$ and $\Gamma_{k_0}^{E_0}(1) = (0, \Delta, 0, 0) \in \mathcal{U}_0$ but $\Gamma_{k_0}^{E_0}(t_{k_0}) = (V_{k_0}^{E_0}(t_{k_0}), \Delta t_{k_0}, p_{k_0}) \notin \mathcal{U}_0$, for some $t_0 \in [0,1]$ as desired, thus completing the proof.

THANK YOU!