Lie algebroids: a comprehensive geometric notion

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Lie algebras

A Lie algebra is vector space L over a field k with a bilinear map

$$L \times L \to L, \qquad (x, y) \mapsto [x, y]$$

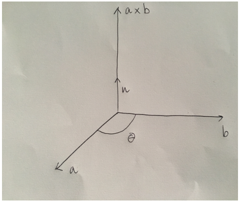
which

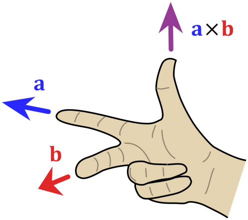
- is skew, $[x, y] = -[y, x]$
- satisfies the Jacobi identity,
  $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$

Examples

- $\mathbb{R}^3$ with the cross product,

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta\ \mathbf{u}$$





If we identify $\mathbb{R}^3$ with the space of skew-symmetric 3×3 matrices

$$(a, b, c) \mapsto A = \begin{pmatrix} 0 & b & -a \\ -b & 0 & c \\ a & -c & 0 \end{pmatrix}$$

the cross product becomes the commutator of matrices,

$$(a, b, c) \times (a', b', c') \mapsto [A, A'] = AA' - A'A$$

We denote this Lie algebra $o(3)$.

Note that if R is an orthogonal 3×3 matrix $R^T R = I$ (i.e. an element of the group $O(3)$), and we write

$$R = I + tA$$

for small t we have

$$I = R^T R = (I+tA^T)(I+tA) = I+t(A^T+A)+O(t^2) \quad \Rightarrow \quad A^T = -A$$

So the elements of the Lie algebra $o(3)$ are "infinitesimal rotations"

More generally, $n \times n$ matrices with entries in a field k make up a Lie algebra (denoted $gl_n(k)$), and the corresponding group is $GL_n(k)$ — the group of invertible $n \times n$ matrices

Lie algebra homomorphisms

Given two Lie algebras L, L', a Lie algebra homomophism $\phi\colon L \to L'$ is a linear map compatible with the brackets:

$$\phi([x,y]) = [\phi(x), \phi(y)]$$

Example:

$L = gl_n(k)$, $L' = k$ with the trivial Lie algebra structure (zero bracket)

$$\phi(A) = \operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}$$

$$\phi([A,B]) = \operatorname{tr}(AB - BA) = 0$$

$$[\phi(A), \phi(B)] = \operatorname{tr}(A)\operatorname{tr}(B) - \operatorname{tr}(B)\operatorname{tr}(A) = 0$$

Vector bundles

Given a differentiable manifold X and a vector space V, we can take its cartesian product $X \times V$, but we can also consider "twisted" products: so B is a space with a projection $\pi\colon B \to X$, and X has an open cover $\{U_i\}$ such that

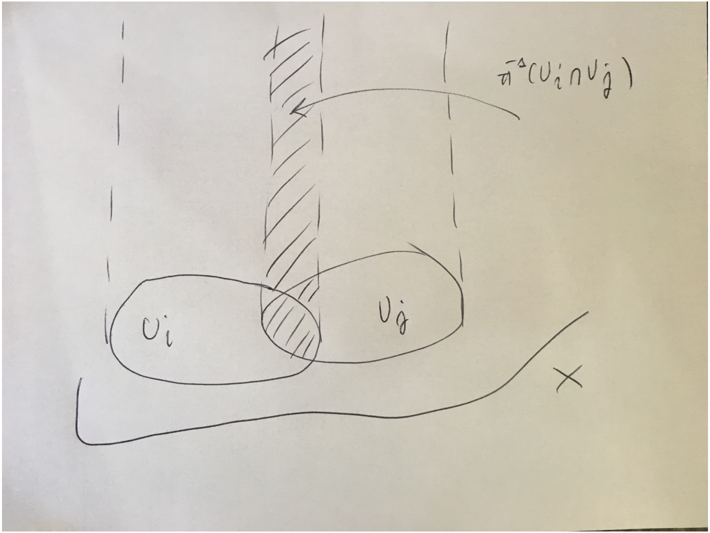
$$\pi^{-1}(U_i) \simeq U_i \times V$$

If $U_i \cap U_j \neq \emptyset$, the identifications

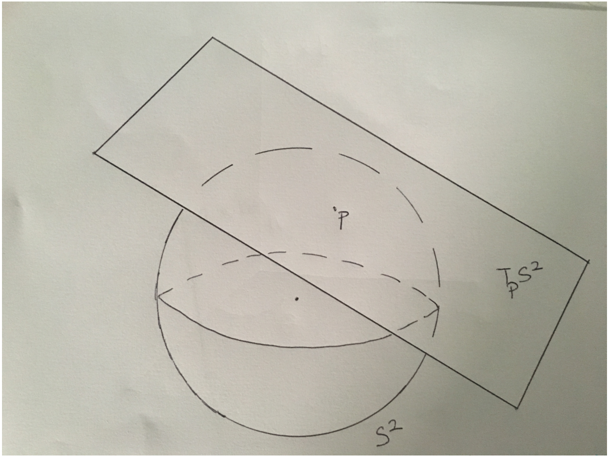
$$(U_i \cap U_j) \times V \to (U_i \cap U_j) \times V$$

are required to be linear in their second argument, thus obtaining maps (transition functions)

$$g_{ij}\colon U_i \cap U_j \to GL_n(k)$$

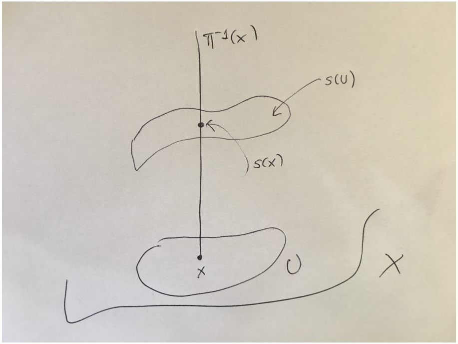

$\pi^{-1}(U_i \cap U_j)$
U_i
U_j
X

Example: the tangent bundle to the 2-sphere S^2



Not a trivial bundle (i.e., $TS^2 \not\simeq S^2 \times \mathbb{R}^2$)

Notion of section: if $U \subset X$, a section $s\colon U \to B$ is a morphism such that $\pi \circ s = id$.



Sections of a tangent bundle are vector fields.

Vector fields act on functions as derivations: if in given local coordinates $(x^1, \dots, n^n)$ a vector field has components $(v^1, \dots, v^n)$, then

$$v(f) = \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}$$

So one can consider the commutator of vector fields:

$$[u, v](f) = u(v(f)) - v(u(f))$$

This is a first-order object, as

$$[u, v](f) = \sum_{i,j=1}^{n} \left(u^i \frac{\partial v^j}{\partial x^i} - v^i \frac{\partial u^j}{\partial x^i} \right) \frac{\partial f}{\partial x^j}$$

One can also consider bundles of Lie algebras: i.e., the "standard fibre" V is a Lie algebra L, and the transition functions are Lie algebra homomorphisms.

Examples: if $B \to X$ is a vector bundle, then $\mathrm{End}(B)$ is a bundle of Lie algebras. The standard fibre is $gl_k(n)$ if $n = \dim V$.

Lie algebroids

X: a differentiable manifold, or some other kind of geometric space (complex manifold, analytic space, scheme...)

T_X the tangent bundle

Lie algebroid: a vector bundle $\mathcal{A}$ with a morphism $a\colon \mathcal{A} \to T_X$ (the anchor) and a Lie bracket on the sections of $\mathcal{A}$ satisfying

$$[s, ft] = f[s, t] + a(s)(f)\, t$$

Note that if s, t are sections of $\ker(a)$ then

$$[s, ft] = f[s, t]$$

i.e., $\ker(a)$ is a bundle of Lie algebras.

Examples

- A bundle of Lie algebras, with $a = 0$
- T_X, with $a = \mathrm{id}$
- Poisson structures, symplectic structures, etc.

Foliations

A foliation $\mathcal{F}$ in a manifold X is an involutive subbundle of T_X, i.e.:

- it is a vector bundle on X which injects into T_X
- it is closed under the bracket, i.e.

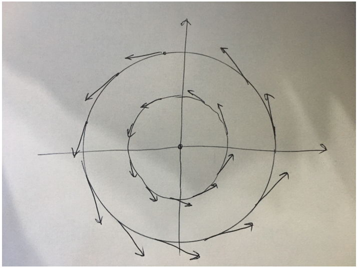
$$u, v \text{ sections of } \mathcal{F} \quad \Rightarrow \quad [u, v] \text{ is a section of } \mathcal{F}$$

By definition, a foliation is a Lie algebroid, with anchor given by the injection $\mathcal{F} \hookrightarrow T_X$

Foliations can be integrated, i.e., they are tangent to submanifolds of X; hence their name

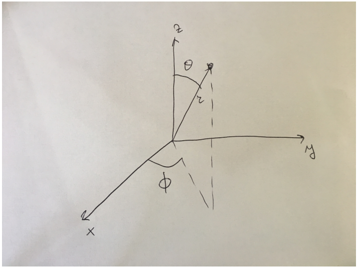
$X = \mathbb{R}^2$

$v = -y\,\mathbf{i} + x\,\mathbf{j}$



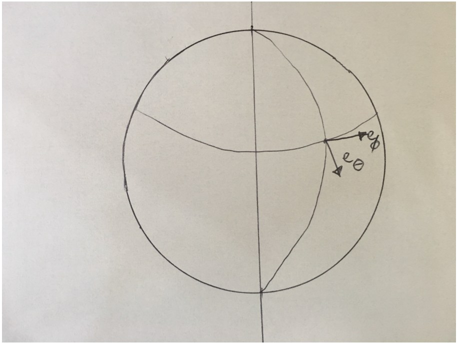
(Singular at the origin)

$$\begin{cases} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{cases}$$



$$[\mathbf{e}_\phi, \mathbf{e}_\theta] = \frac{1}{r}\cot\theta\,\mathbf{e}_\phi$$

$\mathcal{A}$ = subbundle of $T\mathbb{R}^3$ generated by $\mathbf{e}_\phi$, $\mathbf{e}_\theta$



Atiyah algebroid of a vector bundle

$\mathcal{E}$ vector bundle on a manifold X

$\mathcal{D}_{\mathcal{E}}$: bundle of 1-st order differential operators on $\mathcal{E}$ with scalar symbol

$$D(s)^\alpha = \sum_{i,\beta} A(x)^{\alpha i}_\beta \frac{\partial s^\beta}{\partial x^i} + \sum_\beta B(x)^\alpha_\beta \, s^\beta$$

D has scalar symbol if

$$A(x)^{\alpha i}_\beta = \delta^\alpha_\beta \, v^i(x)$$

Bracket: commutator of operators

Anchor: the symbol map, $\sigma \colon D \mapsto v$

Exact sequence

$$0 \to End(\mathcal{E}) \to \mathcal{D}_{\mathcal{E}} \xrightarrow{\sigma} T_X \to 0$$

Connections

Connection on $\mathcal{E}$:

$$\begin{tikzcd} 0 \arrow[r] & End(\mathcal{E}) \arrow[r] & \mathcal{D}_{\mathcal{E}} \arrow[r, "\sigma"] & T_X \arrow[r] \arrow[l, bend left, "\nabla"] & 0 \end{tikzcd}$$

$$\sigma \circ \nabla = \mathrm{id}_{T_x}$$

Curvature of a connection:

$$R(u, v) = [\nabla_u, \nabla_v] - \nabla_{[u,v]}$$

Lie algebroids as tangent objects

A grupoid is a small category $\mathfrak{G}$ where al morphisms (arrows) are isomorphisms

(a group is a groupoid with one object)

A Lie groupoid is a groupoid where

- The sets $M = \mathrm{Ob}(\mathfrak{G})$ and $\mathcal{G} = \mathrm{Mor}(\mathfrak{G})$ are manifolds
- the source and target operations are submersions
- the category operations

$$\mathrm{id}\colon M \to \mathcal{G}, \quad \mathrm{comp}\colon \mathcal{G} \;{}_t\!\times_s \mathcal{G} \to \mathcal{G}$$

  are differentiable

$$\mathrm{id}\colon M \to \mathcal{G}, \qquad t\colon \mathcal{G} \to M$$

$$T^t\mathcal{G} = \bigcup_{p\in M} T(t^{-1}(p)) \subset T\,\mathcal{G}$$

$$\mathcal{A} = \mathrm{id}^*\, T^t\mathcal{G}$$

This is a vector bundle on M

Bracket is the commutator of left-invariant vector fields in $T\,\mathcal{G}$

Anchor is the differential of the source map

Summing up:

The notion of Lie algebroid generalizes both the notion of Lie algebra, and that of tangent bundle. In a sense it is an interpolation between the two.

Main examples are

- Foliations, allowing for singularities
- Poisson and symplectic structures
- Atiyah algebroids (connections, curvatures etc.)
- Tangent objects to Lie groupoids

Of course this was just the tip of the iceberg: the basic definition and a few examples.

The theory has a lot of structure, for example Lie algebroids give rise to a rich cohomology theory (generalizing both the Chevalley-Eilenberg cohomology of Lie algebras, and the de Rham cohomology of manifolds – varieties – schemes).

This cohomology theory can be applied to study all the examples cited in the previous slide