Dual Quaternions and the Study Quadric

J.M. Selig,

School of Engineering, London South Bank University, London SE1 0AA, U.K.

4 August 2017
Colóquio do Departamento de Matemática,
Universidade Federal de Santa Catarina

Introduction

- Review quaternions and rotations.
- Extend to dual quaternions and rigid displacements.
- Study quadric is the group manifold.
- Some geometry of the Study quadric.

Quaternions

Quaternions invented by Hamilton on Monday 16th October 1843 in Dublin.

$$ijk = -1$$

General quaternion has the form,

$$q = a_0 + a_1 i + a_2 j + a_3 k$$

where $a_i \in \mathbb{R}$ are just numbers and i, j, k are the unit quaterions which satisfy the relations,

$$i^2 = j^2 = k^2 = -1$$

and

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ki = j$$

Associative algebra generated by i, j, k.

Gibbs Relation

Given a quaternion $q = a_0 + a_1 i + a_2 j + a_3 k$ the scalar part of the quaternion is a_0 and the vector part is $a_1 i + a_2 j + a_3 k$. (This is where the terms *scalar* and *vector* come from). Often write quaternion as,

$$q = a_0 + \mathbf{a}$$

with $\mathbf{a} = a_1 i + a_2 j + a_3 k$. Now multiplication of quaternions can be written,

$$(a_0 + \mathbf{a})(b_0 + \mathbf{b}) = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) + (a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b})$$

Origin of dot and cross product.

Quaternion Conjugate

Similar to the complex conjugate,
with $q = a_0 + a_1 i + a_2 j + a_3 k$, its conjugate is,

$$q^- = a_0 - a_1 i - a_2 j - a_3 k$$

Note that,

$$qq^- = a_0^2 + a_1^2 + a_2^2 + a_3^2,$$

a real number.

Rotations

Rotations about vector $\mathbf{v}$ by angle θ is given by a quaternion,

$$r = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{v}$$

The action of this of this rotation on a point, written as a vector,

$$\mathbf{p} = xi + yj + zk$$

is given by

$$\mathbf{p}' = r\mathbf{p}r^{-} = (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{v})(xi + yj + zk)(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\mathbf{v})$$

Rotations — Continued

Using the Gibbs relation to expand gives,

$$\begin{aligned}\mathbf{p}' = r\mathbf{p}r^{-} &= \cos^2\frac{\theta}{2}\mathbf{p} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}(\mathbf{v}\times\mathbf{p})\\ &\quad + \sin^2\frac{\theta}{2}(\mathbf{v}\cdot\mathbf{p})\mathbf{p} + \sin^2\frac{\theta}{2}\big(\mathbf{v}\times(\mathbf{v}\times\mathbf{p})\big)\end{aligned}$$

Note: have assumed that $\mathbf{v}$ is a unit vector, $\mathbf{v}\cdot\mathbf{v} = 1$.
Using trigonometric relations and rearranging gives,

$$\mathbf{p}' = r\mathbf{p}r^{-} = \mathbf{p} + \sin\theta(\mathbf{v}\times\mathbf{p}) + (1-\cos\theta)\big(\mathbf{v}\times(\mathbf{v}\times\mathbf{p})\big)$$

The well known Rodrigues formula for rotations.

Unit Quaternions

Above identifies rotations with unit quaternions, that is quaternions satisfying $rr^{-} = 1$.

$$rr^{-} = (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{v})(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\mathbf{v}) = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\mathbf{v}\cdot\mathbf{v} = 1$$

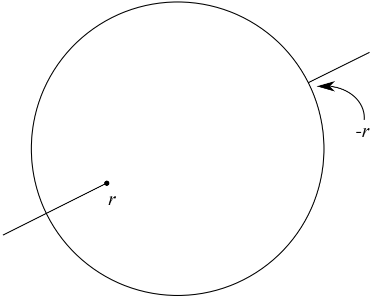
Writing $r = a_0 + a_1 i + a_2 j + a_3 k$ this means,

$$rr^{-} = a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$$

the elements of the group lie on a unit 3-dimensional sphere. Unit quaternions for a 2-to-1 cover of the group of rotations.

Group of Rotations

But both r and $-r$ correspond to the same rotation, since $r\mathbf{p}r^{-} = (-r)\mathbf{p}(-r)^{-}$.
Line through the centre of the 3-sphere meets it in antipodal points r and $-r$. So can identify elements of the rotation group with lines through the origin in $\mathbb{R}^4$ with coordinates (a_0, a_1, a_2, a_3).



Space of lines through the origin in $\mathbb{R}^4$ know as 3-dimensional projective space, $\mathbb{RP}^3$. Group manifold of rotation group $SO(3)$.

Rigid Body Displacements

Rotations and translations—no reflections. Proper rigid-body displacements. General displacement $(r, \mathbf{t})$, where r is a rotation and $\mathbf{t}$ a translation vector. Action of rigid displacement on a point $\mathbf{p}$ given by,

$$\mathbf{p}' = r\mathbf{p}r^{-} + \mathbf{t}$$

Rotations act on translations. Effect of applying $(r_1, \mathbf{t}_1)$ then $(r_2, \mathbf{t}_2)$ is

$$\mathbf{p}'' = r_2\mathbf{p}'r_2^{-} + \mathbf{t}_2 = r_2(r_1\mathbf{p}r_1^{-} + \mathbf{t}_1)r_2^{-} + \mathbf{t}_2$$

The combined effect of the two displacements is a displacement, $(r_2 r_1, \, r_2\mathbf{t}_1 r_2^{-} + \mathbf{t}_2)$.
Group $SE(3)$ is the semi-direct product of rotation with translations, $SE(3) = SO(3) \ltimes \mathbb{R}^3$

Dual Quaternions

Clifford introduced new generator ε which squares to zero $\varepsilon^2 = 0$ and commutes with all other quaternions $i\varepsilon = \varepsilon i$ and so on. Called the dual unit.

General dual quaternion now has the form,

$$h = q_0 + \varepsilon q_1 = (a_0 + a_1 i + a_2 j + a_3 k) + \varepsilon(c_0 + c_1 i + c_2 j + c_3 k)$$

The dual quaternion conjugate is simply

$$h^- = q_0^- + \varepsilon q_1^- = (a_0 - a_1 i - a_2 j - a_3 k) + \varepsilon(c_0 - c_1 i - c_2 j - c_3 k)$$

but we will need another conjugate

$$h^\dagger = q_0^- - \varepsilon q_1^- = (a_0 - a_1 i - a_2 j - a_3 k) + \varepsilon(-c_0 + c_1 i + c_2 j + c_3 k)$$

Dual Quaternions and Rigid Body Displacements

Use dual quaternions to represent rigid displacements, displacement $(g, \mathbf{t})$ represented by the dual quaternion,

$$h = r + \frac{1}{2}\varepsilon \mathbf{t} r$$

Composing displacements represented by multiplication of dual quaternions,

$$(r_2 + \frac{1}{2}\varepsilon \mathbf{t}_2 r_2)(r_1 + \frac{1}{2}\varepsilon \mathbf{t}_1 r_1) = r_2 r_1 + \frac{1}{2}\varepsilon(\mathbf{t}_2 + r_2 \mathbf{t}_1 r_2^-) r_2 r_1$$

relys on fact that r_2 is a unit quaternion (rotation) $r_2 r_2^- = 1$.

Action on Points

Now represent a point $\mathbf{p} = xi + yj + zk$ as the dual quaternion,

$$\tilde{\mathbf{p}} = 1 + \varepsilon \mathbf{p} = 1 + \varepsilon(xi + yj + zk)$$

Action of a rigid displacement on a point given by,

$$\tilde{\mathbf{p}}' = h\mathbf{p}h^{\dagger} = (r + \frac{1}{2}\varepsilon \mathbf{t} r)(1 + \varepsilon \mathbf{p})(r^{-} - \frac{1}{2}\varepsilon r^{-}\mathbf{t}^{-})$$

notice that since $\mathbf{t}$ is a vector $\mathbf{t}^{-} = -\mathbf{t}$, so

$$(r + \frac{1}{2}\varepsilon \mathbf{t} r)(1 + \varepsilon \mathbf{p})(r^{-} - \frac{1}{2}\varepsilon r^{-}\mathbf{t}^{-}) = 1 + \varepsilon(r\mathbf{p}r^{-} + \mathbf{t})$$

again $rr^{-} = 1$ has been use to simplify.
Notice once more that h and $-h$ give the same rigid-body displacement.

The Group Manifold

Not all dual quaternions represent rigid body displacements only those $h = q_0 + \varepsilon q_1$ where,

$$q_0 = r \qquad \text{and} \qquad q_1 = \frac{1}{2}\mathbf{t}r$$

As quaternion equations we have,

$$q_0 q_0^- = rr^- = 1$$

and

$$q_0 q_1^- + q_1 q_0^- = -\frac{1}{2}rr^-\mathbf{t} + \frac{1}{2}\mathbf{t}rr^- = 0$$

Writing $q_0 = a_0 + a_1 i + a_2 j + a_3 k$ and $q_1 = c_0 + c_1 i + c_2 j + c_3 k$ these give two algebraic equations,

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1 \qquad \text{and} \qquad a_0 c_0 + a_1 c_1 + a_2 c_2 + a_3 c_3 = 0$$

The Study Quadric

Can do the same as we did with rotation to remove the double valued nature of the representation. Take $(a_0 : a_1 : a_2 : a_3 : c_0 : c_1 : c_2 : c_3)$ as homogeneous coordinates in a $\mathbb{RP}^7$. Now h and $-h$ are the same point. The lines through the origin in $\mathbb{R}^8$ each meet the space $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$ in just two antipodal points. So just have to satisfy the second equation,

$$a_0 c_0 + a_1 c_1 + a_2 c_2 + a_3 c_3 = 0$$

This is a homogeneous, degree 2 equation so defines a quadric in $\mathbb{RP}^7$ — the Study quadric.

The Study Quadric — continued

Each rigid-body displacement corresponds to a point in this quadric, but not all points in the quadric correspond to displacements. The points with $a_0 = a_1 = a_2 = a_3 = 0$ form a 3-dimensional plane ($\mathbb{RP}^3$), lying entirely inside the quadric. This is because points on this 3-plane are given by $(0 : 0 : 0 : 0 : c_0 : c_1 : c_2 : c_3)$ and these satisfy the equation above for the Study quadric. Call this 3-plane A_∞.

Clearly, such points do not correspond to rigid displacements. Set of ideal points, like points at infinity in projective space. The group manifold for $SE(3)$ is the Study quadric with this 3-plane deleted.

Displacement Subvarieties

Look at subvarieties of the Study quadric. That is subspaces determined by algebraic equations in the homogeneous coordinates $a_0, \ldots, c_3$.

- Curves correspond to rigid-body motions, a subject in its own right with many applications to robot path-planning and computer animation.
- Subvarieties determined by the possible motion of a body attached to the end of a serial linkage. Intersect these to get information about closed loop mechanisms and parallel robots.
- Subvarieties determined by geometric constraints. For example the set of displacements which move a particular point so that it stays on a given plane.

Begin with subvarieties determined by linear equations.

Lines in the Study Quadric I

Consider a line in $\mathbb{P}^7$ joining the points $1 = 1 + \varepsilon 0$ and $\ell = q_0 + \varepsilon q_1$. The point 1 is the identity in the group and it is clear that any line in the Study quadric can be moved to a line of this form by transforming any point on the line to the identity.

A general line through the identity will have the form,

$$g(c, s) = c + s\ell$$

where c and s are homogeneous parameters.

For such a line to lie entirely in the Study quadric ℓ must lie in the quadric and every other point on the line must too. This can be expressed as

$$q_0 q_1^- + q_1 q_0^- = 0, \quad \text{and} \quad q_1 + q_1^- = 0$$

Lines in the Study Quadric II

If $q_0 = a_0 + a_1 i + a_2 j + a_3 k$ and $q_1 = c_0 + c_1 i + c_2 j + c_3 k$ then these equations give,

$$\ell = (a_1 i + a_2 j + a_3 k) + \varepsilon(c_1 i + c_2 j + c_3 k)$$

with $a_1 c_1 + a_2 c_2 + a_3 c_3 = 0$. These elements ℓ correspond to lines in space. If we assume that the parameters $c = \cos(\theta/2)$ and $s = \sin(\theta/2)$ then the line in the Study quadric $c + s\ell$ corresponds to the one parameter sub-group of rotations about the line in space represented by ℓ.
Notice this set of displacements are the same as those that can be generated by a revolute (hinge) joint.

Lines in the Study Quadric III

If the components $a_1 = a_2 = a_3 = 0$ then ℓ represents a line at infinity. Here the line in the Study quadric $c + s\ell$ represents a subgroup of translations, all in the directed along the vector $c_1 i + c_2 j + c_3 k$. These displacements can be generated by a prismatic (sliding) joint.

A line of rotations in the Study quadric doesn't meet A_∞ the ideal element of the Study quadric. However, a line representing a translation subgroup does meet A_∞ at a single point; when $c = 0$.

These properties are not changed if we take a line that doesn't pass through the identity since points on A_∞ stay on A_∞ after a left (or right) translation in the group.

Hence, all lines in the Study quadric are either translates of a one-parameter subgroup of rotations or translates of a one-parameter subgroup of translations.

A and B-planes

Two families of 3-planes lying entirely within the Study quadric. In $\mathbb{P}^7$ 4 linearly independent equations define a 3-plane, these can be written,

$$(I + M)\vec{a} + (I - M)\vec{c} = \vec{0}$$

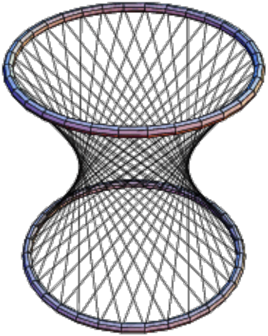
where

$$\vec{a} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad \text{and} \qquad \vec{c} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

and $M \in O(4)$. (To see how this works change coordinates to $\vec{x} = \vec{a} + \vec{c}$, $\vec{y} = \vec{a} - \vec{c}$.)

Gives A-planes if $\det(M) = 1$ and B-planes when $\det(M) = -1$.

A and B-planes — Continued



A and B-planes behave like lines on a 2-D hyperboloid.

A-plane are sets of displacements which move one point in space to another, or move one plane in space to another.

B-planes through the identity are sets of rotations with axes lying in a particular plane, or the subgroup of all translations.

Subspaces of all Rotations

Consider the set of all rotations about arbitrary lines in space. If, as before, r is a rotation about some vector $\mathbf{v}$ then $r = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{v}$ then this is a rotation about a line through the origin. A general rotation about a line through some point $\mathbf{p}$ will be given by the conjugation,

$$(1 + \frac{1}{2}\varepsilon\mathbf{p})r(1 - \frac{1}{2}\varepsilon\mathbf{p}) = r + \frac{1}{2}\varepsilon(\mathbf{p}r - \mathbf{p}r)$$

$$= \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{v} + \varepsilon\sin\frac{\theta}{2}\mathbf{p}\times\mathbf{v}$$

So the set of rotations lie on the intersection of the Study quadric with the hyperplane c_0.

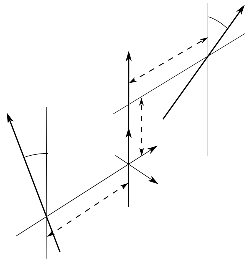
Intersection with Quadrics

Sets of displacements lie on the intersection of the Study quadric with another quadric in $\mathbb{P}^7$, only time for a quick look.

- The variety of displacements which move a given point in such a way that it meets a given plane.
- The variety of displacements which move a given points to meet a given sphere.
- The variety of displacements which move a given line in such a way that it lies in a particular linear line complex.

Some of these varieties model the capabilities of open-loop mechanical chains.

The RRR Linkage



The end-effector of a 3R linkage generates a $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ Segre variety.

Lies on the intersection of 9 linearly independent quadrics in $\mathbb{P}^7$, not a complete intersection. One of the 9 quadrics is the Study quadric here.

Variety has degree 6. Also it meets a general A-plan in 4 points and a generic B-plane in 2 points.

Conclusions

- Biquaternions, different groups according to relation for ε^2. Get $SO(4)$ with $\varepsilon^2 = 1$ and $SO^+(3,1)$ with $\varepsilon^2 = -1$.
- Can do Euclidean geometry with dual quaternions but simpler to use Clifford algebra $Cl(0,3,1)$, note that dual quaternions form the even sub-algebra of this Clifford algebra. Or $Cl(3,0,1)$.
- Means to an end. Really want to look at algebraic geometry of configurations spaces defined by mechanisms.
- Many other application of this geometry and algebra.

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THANK YOU