Comportamento assintótico das soluções de uma família de sistemas de Boussinesq

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Outline

- Description of the model: a family of Boussinesq systems
- Setting of the problem: stabilization of a coupled system of two Benjamin-Bona-Mahony **(BBM)** equations
- Main results
- Main Idea of the proofs
- Open problems

The Benjamin-Bona-Mahony (BBM) equation

The BBM equation

$$u_t + u_x - u_{xxt} + uu_x = 0, \tag{1}$$

was proposed as an alternative model for the Korteweg-de Vries equation (KdV)

$$u_t + u_x + u_{xxx} + uu_x = 0, \tag{2}$$

to describe the propagation of one-dimensional, unidirectional small amplitude long waves in nonlinear dispersive media.

- $u(x,t)$ is a real-valued functions of the real variables x and t.

In the context of shallow-water waves, $u(x,t)$ represents the displacement of the water surface at location x and time t.

The Boussinesq system

J. L. **Bona**, M. **Chen**, J.-C. **Saut** - J. Nonlinear Sci. 12 (2002).

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b \eta_{xxt} = 0 \\ w_t + \eta_x + w w_x + c \eta_{xxx} - d w_{xxt} = 0, \end{cases} \tag{3}$$

The model describes the motion of small-amplitude long waves on the surface of an ideal fluid under the gravity force and in situations where the motion is sensibly two dimensional.

η is the elevation of the fluid surface from the equilibrium position;

$w = w_\theta$ is the horizontal velocity in the flow at height θh, where h is the undisturbed depth of the liquid;

a, b, c, d, are parameters required to fulfill the relations

$$a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \qquad c + d = \frac{1}{2}(1 - \theta^2) \geq 0,$$

where $\theta \in [0, 1]$ specifies which velocity the variable w represents.

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where $\theta \in [0, 1]$ specifies which velocity the variable w represents.

Stabilization Results: $E(t) \leq cE(0)e^{-\omega t}, \ \omega > 0, \, c > 0$

The Boussinesq system posed on a bounded interval:

- A. Pazoto and L. Rosier, Stabilization of a Boussinesq system of KdV-KdV type, System and Control Letters 57 (2008), 595-601.
- R. Capistrano Filho, A. Pazoto and L. Rosier, Control of Boussinesq system of KdV-KdV type on a bounded domain, Preprint.

The Boussinesq system posed on the whole real axis: $(-\eta_{xx}, -w_{xx})$

- M. Chen and O. Goubet, Long-time asymptotic behavior of dissipative Boussinesq systems, Discrete Contin. Dyn. Syst. Ser. 17 (2007), 509-528.

The Boussinesq system posed on a periodic domain:

- S. Micu, J. H. Ortega, L. Rosier and B.-Y. Zhang, Control and stabilization of a family of Boussinesq systems, Discrete Contin. Dyn. Syst. 24 (2009), 273-313.

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Controllability and Stabilization

• S. Micu, J. H. Ortega, L. Rosier, B.-Y. Zhang - Discrete Contin. Dyn. Syst. 24 (2009).

$$b, d \geq 0, a \leq 0, c \leq 0 \quad \text{or} \quad b, d \geq 0, a = c > 0.$$

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{xxt} = f(x,t) \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = g(x,t) \end{cases}$$

where $0 < x < 2\pi$ and $t > 0$, with boundary conditions

$$\frac{\partial^r \eta}{\partial x^r}(0,t) = \frac{\partial^r \eta}{\partial x^r}(2\pi,t), \quad \frac{\partial^r w}{\partial x^r}(0,t) = \frac{\partial^r w}{\partial x^r}(2\pi,t)$$

and initial conditions

$$\eta(x,0) = \eta^0(x), \quad w(x,0) = w^0(x).$$

• f and g are locally supported forces.

Dirichlet boundary conditions

$$\begin{aligned} &\eta_t + w_x - b\eta_{txx} = -\varepsilon a(x)\eta, && x \in (0,2\pi),\ t>0,\\ &w_t + \eta_x - d w_{txx} = 0, && x \in (0,2\pi),\ t>0, \end{aligned}$$

with boundary conditions

$$\eta(t,0) = \eta(t,2\pi) = w(t,0) = w(t,2\pi) = 0, \quad t>0,$$

and initial conditions

$$\eta(0,x) = \eta^0(x), \quad w(0,x) = w^0(x), \qquad x \in (0,2\pi).$$

We assume that

- $b, d > 0$ and $\varepsilon > 0$ are parameters.
- $a = a(x)$ is a nonnegative real-valued function satisfying

$$\begin{aligned} a(x) \geq a_0 > 0, \quad \text{in } \Omega \subset (0,2\pi),\\ a \in W^{2,\infty}(0,2\pi),\ \text{with } a(0) = a'(0) = 0. \end{aligned}$$

Dirichlet boundary conditions

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The energy associated to the model is given by

$$E(t) = \frac{1}{2}\int_0^{2\pi} (\eta^2 + b\eta_x^2 + w^2 + dw_x^2)dx \tag{4}$$

and we can (formally) deduce that

$$\frac{d}{dt}E(t) = -\varepsilon \int_0^{2\pi} a(x)\eta^2(t,x)dx. \tag{5}$$

Theorem (S.Micu, A. Pazoto - Journal d'Analyse Mathématique)

Assume that $a \in W^{2,\infty}(0,2\pi)$ and $a(0) = a'(0) = 0$. Then, there exits ε_0, such that, for any $\varepsilon \in (0,\varepsilon_0)$ and (η^0, w^0) in $(H_0^1(0,2\pi))^2$, the solution (η, w) of the system verifies

$$\lim_{t\to\infty} \|(\eta(t), w(t))\|_{(H_0^1(0,2\pi))^2} = 0.$$

Moreover, the decay of the energy is not exponential, i. e., there exists no positive constants M and ω, such that

$$\|(\eta(t), w(t))\|_{(H_0^1(0,2\pi))^2} \le Me^{-\omega t}, \qquad t \ge 0.$$

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Spectral analysis and eigenvectors expansion of solutions

Since

$$
\begin{array}{ll}
(I - b\partial_x^2)\eta_t + w_x + \varepsilon a(x)\eta = 0, & x \in (0, 2\pi),\ t > 0, \\
(I - d\partial_x^2)w_t + \eta_x = 0, & x \in (0, 2\pi),\ t > 0,
\end{array}
$$

the system can be written as

$$
\begin{array}{l}
U_t + \mathcal{A}_\varepsilon U = 0, \\
U(0) = U_0,
\end{array}
$$

where $\mathcal{A}_\varepsilon : (H_0^1(0, 2\pi))^2 \to (H_0^1(0, 2\pi))^2$ is given by

$$
\mathcal{A}_\varepsilon = \begin{pmatrix} \varepsilon \left(I - b\partial_x^2\right)^{-1} a(\cdot)\, I & \left(I - b\partial_x^2\right)^{-1} \partial_x \\ \left(I - d\partial_x^2\right)^{-1} \partial_x & 0 \end{pmatrix}. \tag{6}
$$

We have that

$\mathcal{A}_\varepsilon \in \mathcal{L}((H_0^1(0, 2\pi))^2)$ and $\mathcal{A}_\varepsilon$ is a compact operator.

The operator $\mathcal{A}_\varepsilon$ has a family of eigenvalues $(\lambda_n)_{n\geq 1}$, such that:

1. $|\Re(\lambda_n)| \leq \dfrac{c}{|n|^2}, \ \forall\, n \geq n_0,$ and $\Re(\lambda_n) < 0, \quad \forall\, n.$
2. The corresponding eigenfunctions $(\Phi_n)_{n\geq 1}$ form a Riesz basis in $(H_0^1(0, 2\pi))^2$.

Then,

$$(\eta(t), w(t)) = \sum_{n\geq 1} a_n e^{\lambda_n t} \Phi_n$$

and

$$c_1 \sum_{n\geq n_0} |a_n|^2 e^{2\Re(\lambda_n)t} \leq \|(\eta(t), w(t))\|^2_{(H_0^1(0,2\pi))^2} \leq c_2 \sum_{n\geq 1} |a_n|^2 e^{2\Re(\lambda_n)t}.$$

$E(t)$ **converges to zero as** $t \to \infty$**.**

The decay is not exponential.

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$E(t)$ **converges to zero as** $t \to \infty$**.**

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We obtain the asymptotic behavior of the high eigenfunctions and prove that they are quadratically close to a Riesz basis $(\Psi_m)_{m\geq 1}$ formed by eigenvectors of a well chosen dissipative differential operator with constant coefficients:

$$\sum_{m\geq N+1} ||\Phi_m - \Psi_m||^2_{(H_0^1(0,2\pi))^2} \sim \frac{1}{m^2}.$$

Theorem (S.Micu, A. Pazoto - Journal d'Analyse Mathématique)

Let (η, w) be a finite energy solution of the system with $a \equiv 0$. If there exist $T > 0$ and an open set $\Omega \subset (0, 2\pi)$, such that

$$\eta(x,t) = 0 \quad , \forall\ (x,t) \in \Omega \times (0,T), \tag{7}$$

then

$$\eta = w \equiv 0 \quad \textit{in} \quad \mathbb{R} \times (0, 2\pi).$$

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Periodic boundary conditions

For b, $d > 0$ and β_1, β_2, α_1, $\alpha_2 \geq 0$, we consider the system

$$\begin{aligned} &\eta_t + w_x - b\eta_{txx} + (\eta w)_x + \beta_1 M_{\alpha_1}\eta = 0, \\ &w_t + \eta_x - dw_{txx} + ww_x + \beta_2 M_{\alpha_2} w = 0, \end{aligned} \tag{8}$$

with periodic boundary conditions

$$\begin{aligned} &\eta(0,t) = \eta(2\pi,t);\ \eta_x(0,t) = \eta_x(2\pi,t), \\ &w(0,t) = w(2\pi,t);\ w_x(0,t) = w_x(2\pi,t), \end{aligned}$$

and initial conditions

$$\eta(x,0) = \eta^0(x), \quad w(x,0) = w^0(x).$$

In (8), M_{α_j} are Fourier multiplier operators given by

$$M_{\alpha_j}\left(\sum_{k\in\mathbb{Z}} v_k e^{ikx}\right) = \sum_{k\in\mathbb{Z}} (1+k^2)^{\frac{\alpha_j}{2}} \widehat{v}_k e^{ikx}.$$

The energy associated to the model is given by

$$E(t) = \frac{1}{2}\int_0^{2\pi} (\eta^2 + b\eta_x^2 + w^2 + dw_x^2)dx \tag{9}$$

and we can (formally) deduce that

$$\begin{aligned} \frac{d}{dt}E(t) = -\beta_1 \int_0^{2\pi} (M_{\alpha_1}\eta)\, \eta\, dx - \beta_2 \int_0^{2\pi} (M_{\alpha_2}w)\, w\, dx \\ - \int_0^{2\pi} (\eta w)_x\, \eta\, dx. \end{aligned} \tag{10}$$

Since $\beta_1, \beta_2 \geq 0$ and

$$(M_{\alpha_j}v, v)_{L^2(0,2\pi)} \geq 0, \qquad j = 1, 2,$$

the terms $M_{\alpha_1}\eta$ and $M_{\alpha_2}w$ play the role of feedback damping mechanisms, at least for the linearized system.

Assumptions on the Dissipation: $\int_{\mathbb{T}} M_{\alpha_i}\varphi(x)\varphi(x)dx \geq 0$

- Applications and study of asymptotic behavior os solutions:
  - J. L. Bona and J. Wu, M2AN Math. Model. Numer. Anal. (2000).
  - J.-P. Chehab, P. Garnier and Y. Mammeri, J. Math. Chem. (2001).
  - D. Dix, Comm. PDE (1992).
  - C. J. Amick, J. L. Bona and M. Schonbek, Jr. Diff. Eq. (1989).
  - P. Biler, Bull. Polish. Acad. Sci. Math. (1984).
  - J.-C. Saut, J. Math. Pures et Appl. (1979).

- Fractional derivative (Weyl fractional derivative operator):

$$h(x) = \sum_{k\in\mathbb{Z}} a_k e^{ikx} \Rightarrow W_x^{\alpha}(h)(x) = \sum_{k\in\mathbb{Z}} (ik)^{\alpha} a_k e^{ikx}, \quad \alpha \in (0,1).$$

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Main results

The energy $E(t)$ satisfies

$$\frac{dE}{dt} = -\beta_1 \int_0^{2\pi} (M_{\alpha_1}\eta)\,\eta\,dx - \beta_2 \int_0^{2\pi} (M_{\alpha_2}w)\,w\,dx - \int_0^{2\pi} (\eta w)_x\,\eta\,dx,$$

where

$$M_{\alpha_j} v = \sum_{k\in\mathbb{Z}} (1+k^2)^{\frac{\alpha_j}{2}} \widehat{v}_k e^{ikx}.$$

Firstly, we analyze the linearized system:

- $\alpha_1 = \alpha_2 = 2$ and $\beta_1, \beta_2 > 0 \Longrightarrow$ the exponential decay of solutions in the H^s-setting, for any $s \in \mathbb{R}$.
- $\max\{\alpha_1, \alpha_2\} \in (0,2)$, $\beta_1, \beta_2 \geq 0$ and $\beta_1^2 + \beta_2^2 > 0 \Longrightarrow$ polynomial decay rate of solutions in the H^s-setting, by considering more regular initial data.

Exponential decay estimate and contraction mapping argument $\Longrightarrow$ global well-posedness and the exponential stability property of the nonlinear system.

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Exponential decay estimate and contraction mapping argument $\Longrightarrow$ global well-posedness and the exponential stability property of the nonlinear system.

The Linearized System

Since

$$(I - b\partial_x^2)\eta_t + w_x + \beta_1 M_1 \eta = 0,$$
$$(I - d\partial_x^2)w_t + \eta_x + \beta_2 M_2 \eta = 0,$$

the linear system can be written as

$$U_t + AU = 0,$$
$$U(0) = U_0,$$

where A is given by

$$A = \begin{pmatrix} \beta_1 \left(I - b\partial_x^2\right)^{-1} M_{\alpha_1} & \left(I - b\partial_x^2\right)^{-1} \partial_x \\ \\ \left(I - d\partial_x^2\right)^{-1} \partial_x & \beta_2 \left(I - b\partial_x^2\right)^{-1} M_{\alpha_2} \end{pmatrix}. \qquad (11)$$

For $\alpha > 0$, the operator $(I - \alpha\partial_x^2)^{-1}$ is defined in the following way:

$$(I - \alpha\partial_x^2)^{-1}\varphi = v \Leftrightarrow \begin{cases} v - \alpha v_{xx} = \varphi & \text{in } (0, 2\pi), \\ \\ v(0) = v(2\pi), & v_x(0) = v_x(2\pi). \end{cases}$$

Spectral Analysis

If we assume that

$$(\eta^0, w^0) = \sum_{k\in\mathbb{Z}} (\widehat{\eta}_k^0, \widehat{w}_k^0) e^{ikx},$$

the solution can be written as

$$(\eta, \omega)(x,t) = \sum_{k\in\mathbb{Z}} (\widehat{\eta}_k(t), \widehat{\omega}_k(t)) e^{ikx},$$

where the pair $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ fulfills

$$\begin{aligned}
&(1+bk^2)(\widehat{\eta}_k)_t + ik\widehat{w}_k + \beta_1(1+k^2)^{\frac{\alpha_1}{2}}\widehat{\eta}_k = 0,\\
&(1+dk^2)(\widehat{w}_k)_t + ik\widehat{\eta}_k + \beta_2(1+k^2)^{\frac{\alpha_2}{2}}\widehat{w}_k = 0,\\
&\widehat{\eta}_k(0) = \widehat{\eta}_k^0, \qquad \widehat{w}_k(0) = \widehat{w}_k^0,
\end{aligned} \tag{12}$$

where $t \in (0, T)$.

We set

$$A(k) = \begin{pmatrix} \frac{\beta_1 (1+k^2)^{\frac{\alpha_1}{2}}}{1+bk^2} & \frac{ik}{1+bk^2} \\ \\ \frac{ik}{1+dk^2} & \frac{\beta_2 (1+k^2)^{\frac{\alpha_2}{2}}}{1+dk^2} \end{pmatrix}.$$

Then system (12) is equivalent to

$$\begin{pmatrix} \widehat{\eta}_k \\ \\ \widehat{w}_k \end{pmatrix}_t (t) + A(k) \begin{pmatrix} \widehat{\eta}_k \\ \\ \widehat{w}_k \end{pmatrix}(t) = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \widehat{\eta}_k \\ \\ \widehat{w}_k \end{pmatrix}(0) = \begin{pmatrix} \widehat{\eta}_k^0 \\ \\ \widehat{w}_k^0 \end{pmatrix}.$$

Hence, the solution of (12) is given by

$$\begin{pmatrix} \widehat{\eta}_k \\ \\ \widehat{w}_k \end{pmatrix}(t) = e^{-A(k)\, t} \begin{pmatrix} \widehat{\eta}_k^0 \\ \\ \widehat{w}_k^0 \end{pmatrix}. \qquad (13)$$

Lemma

The eigenvalues of the matrix A are given by

$$\lambda_k^{\pm} = \frac{1}{2}\left(\frac{\beta_1(1+k^2)^{\frac{\alpha_1}{2}}}{1+bk^2} + \frac{\beta_2(1+k^2)^{\frac{\alpha_2}{2}}}{1+dk^2} \pm \frac{2|k|\sqrt{e_k^2-1}}{\sqrt{(1+bk^2)(1+dk^2)}}\right),$$

where

$$e_k = \frac{1}{2k}\left(\beta_1(1+k^2)^{\frac{\alpha_1}{2}}\sqrt{\frac{1+dk^2}{1+bk^2}} - \beta_2(1+k^2)^{\frac{\alpha_2}{2}}\sqrt{\frac{1+bk^2}{1+dk^2}}\right),$$

and $k \in \mathbb{Z}^*$.

Observe that

- $\lambda_k^{\pm} = \lambda_{-k}^{\pm}$.
- If $e_k < 1$, the eigenvalues $\lambda_k^{\pm} \in \mathbb{C}$.
- If $e_k \geq 1$, the eigenvalues $\lambda_k^{\pm} \in \mathbb{R}$.

Lemma

The solution $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ *of* (12) *is given by*

$$\widehat{\eta}_k(t) = \frac{1}{1-\zeta_k^2}\left[\left(\widehat{\eta}_k^0 + i\alpha_k\zeta_k\widehat{w}_k^0\right)e^{-\lambda_k^+ t} - \left(\zeta_k^2\widehat{\eta}_k^0 + i\alpha_k\zeta_k\widehat{w}_k^0\right)e^{-\lambda_k^- t}\right],$$

$$\widehat{w}_k(t) = \frac{1}{1-\zeta_k^2}\left[\left(i\theta_k\zeta_k\widehat{\eta}_k^0 - \zeta_k^2\widehat{w}_k^0\right)e^{-\lambda_k^+ t} - \left(i\theta_k\zeta_k\widehat{\eta}_k^0 - \widehat{w}_k^0\right)e^{-\lambda_k^- t}\right],$$

if $|e_k| \neq 1$ *and* $k \neq 0$*,*

$$\widehat{\eta}_k(t) = \left[\left(1 - \frac{k\zeta_k}{\sqrt{(1+bk^2)(1+dk^2)}}t\right)\widehat{\eta}_k^0 - \frac{ikt}{1+bk^2}\widehat{w}_k^0\right]e^{-\lambda_k^+ t},$$

$$\widehat{w}_k(t) = \left[-\frac{ikt}{1+dk^2}\widehat{\eta}_k^0 + \left(1 + \frac{k\zeta_k}{\sqrt{(1+bk^2)(1+dk^2)}}t\right)\widehat{w}_k^0\right]e^{-\lambda_k^+ t},$$

if $|e_k| = 1$ *and* $k \neq 0$*, and finally,*

$$\widehat{\eta}_0(t) = \widehat{\eta}_0^0 e^{-\beta_1 t}, \qquad \widehat{w}_0(t) = \widehat{w}_0^0 e^{-\beta_2 t}.$$

Here, $\alpha_k = \sqrt{\frac{1+dk^2}{1+bk^2}}$, $\theta_k = \sqrt{\frac{1+bk^2}{1+dk^2}}$ *and* $\zeta_k = e_k - \sqrt{e_k^2 - 1}$.

The case $s = 0$

For any $t \geq 0$ and $k \in \mathbb{Z}$, we have that

$$b|\widehat{\eta}_k(t)|^2 + d|\widehat{w}_k(t)|^2 \leq M\left(b|\widehat{\eta}_k^0|^2 + d|\widehat{w}_k^0|^2\right) e^{-2t\min\left\{|\Re(\lambda_k^+)|,\, |\Re(\lambda_k^-)|\right\}},$$

where

$$\min\{|\Re(\lambda_k^+)|,\, |\Re(\lambda_k^-)|\} \geq D > 0,$$

and D is a positive number, depending on the parameters β_1, β_2, α_1, α_2, b and d.

Moreover,

- If $\beta_1\beta_2 = 0$, then $\Re(\lambda_k^\pm) \to 0$, as $|k| \to \infty$, and we cannot expect uniform exponential decay of the solutions.
- The fact that the decay of the solutions is not exponential is equivalent to the non uniform decay rate: given any non increasing positive function Θ, there is an initial data $\left(\eta^0, w^0\right)$ such that the $H_p^s \times H_p^s-$norm of the corresponding solution decays slower that Θ.

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Let us introduce the space

$$V^s = H_p^s(0, 2\pi) \times H_p^s(0, 2\pi).$$

Then, the following holds:

Theorem

The family of linear operators $\{S(t)\}_{t\geq 0}$ *defined by*

$$S(t)(\eta^0, w^0) = \sum_{k\in\mathbb{Z}} (\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx}, \qquad (\eta^0, w^0) \in V^s, \tag{14}$$

is an analytic semigroup in V^s *and verifies the following estimate*

$$\|S(t)(\eta^0, w^0)\|_{V^s} \leq C\|(\eta^0, w^0)\|_{V^s}, \tag{15}$$

where C *is a positive constant. Moreover, its infinitesimal generator is the compact operator* $(D(A), A)$, *where* $D(A) = V^s$ *and* A *is given by*

$$A = \begin{pmatrix} \beta_1 \left(I - b\partial_x^2\right)^{-1} M_{\alpha_1} & \left(I - b\partial_x^2\right)^{-1} \partial_x \\ \left(I - d\partial_x^2\right)^{-1} \partial_x & \beta_2 \left(I - b\partial_x^2\right)^{-1} M_{\alpha_2} \end{pmatrix}. \tag{16}$$

Definition

The solutions *decay exponentially in V^s* if there exist two positive constants M and μ, such that

$$\|S(t)(\eta^0, w^0)\|_{V^s} \leq M e^{-\mu t} \|(\eta^0, w^0)\|_{V^s}, \tag{17}$$

$\forall t \geq 0$ and $(\eta^0, w^0) \in V^s$.

We have the following result:

Theorem

The solutions decay exponentially in V^s if and only if $\alpha_1 = \alpha_2 = 2$ and $\beta_1,\ \beta_2 > 0$. Moreover, μ from (17) *is given by*

$$\mu = \inf_{k \in \mathbb{Z}} \left\{ \left|\Re(\lambda_k^+)\right|,\ \left|\Re(\lambda_k^-)\right| \right\}, \tag{18}$$

where $\lambda_k^\pm$ are the eigenvalues of the operator A.

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Theorem

Suppose that $\beta_1, \beta_2 \geq 0$, $\beta_1^2 + \beta_2^2 > 0$ *and* $\min\{\alpha_1, \alpha_2\} \in [0, 2)$. *Then, there exists* δ *and* $M > 0$, *such that*

$$\|S(t)(\eta^0, w^0)\|_{V^s} \leq \frac{M}{(1+t)^{\frac{1}{\delta}(q-\frac{1}{2})}} ||(\eta^0, w^0)||_{V^{s+q}}, \ \forall t > 0,$$

where $s \in \mathbb{R}$ *and* $q > \frac{1}{2}$. *Moreover,* $\delta > 0$ *is defined by*

$$\delta = \begin{cases} 2 - \max\{\alpha_1, \alpha_2\} & \textit{if } \alpha_1 + \alpha_2 \leq 2, \quad \max\{\alpha_1, \alpha_2\} \leq 1, \\ \max\{\alpha_1, \alpha_2\} & \textit{if } \alpha_1 + \alpha_2 \leq 2, \quad \max\{\alpha_1, \alpha_2\} > 1, \\ 2 - \min\{\alpha_1, \alpha_2\} & \textit{if } \alpha_1 + \alpha_2 > 2. \end{cases}$$

Remark: If $\alpha_1 = \alpha_2 = 2$ and $\beta_1 = 0$ or $\beta_2 = 0$, then $\delta = 2$.

The nonlinear problem

Theorem

Let $s \geq 0$ and suppose that $\beta_1, \beta_2 > 0$ and $\alpha_1 = \alpha_2 = 2$. There exist $r > 0$, $C > 0$ and $\mu > 0$, such that, for any $(\eta^0, w^0) \in V^s$, satisfying

$$||(\eta^0, w^0)||_{V^s} \leq r,$$

the system admits a unique solution $(\eta, w) \in C([0,\infty); V^s)$ which verifies

$$\|(\eta(t), w(t))\|_{V^s} \leq Ce^{-\mu t}\|(\eta^0, w^0)\|_{V^s}, \quad t \geq 0.$$

Moreover, μ may be taken as in the linearized problem.

The energy $E(t)$ satisfies

$$\frac{dE}{dt} = -\beta_1 \int_0^{2\pi} (M_{\alpha_1}\eta)\,\eta\,dx - \beta_2 \int_0^{2\pi} (M_{\alpha_2}w)\,w\,dx - \int_0^{2\pi} (\eta w)_x\,\eta\,dx.$$

We define the space

$$Y_{s,\mu} = \{(\eta, w) \in C_b(\mathbb{R}^+; V^s) : e^{\mu t}(\eta, w) \in C_b(\mathbb{R}^+; V^s)\},$$

with the norm

$$||(\eta, w)||_{Y_{s,\mu}} := \sup_{0 \le t < \infty} ||e^{\mu t}(\eta, w)(t)||_{V^s},$$

and the function $\Gamma : Y_{s,\mu} \to Y_{s,\mu}$ by

$$\Gamma(\eta, w)(t) = S(t)(\eta^0, w^0) - \int_0^t S(t-\tau) N(\eta, w)(\tau)\, \mathrm{d}\tau,$$

where $N(\eta, w) = (-(I - b\partial_x^2)^{-1}(\eta w)_x, -(I - d\partial_x^2)^{-1} w w_x)$ and $\{S(t)\}_{t \ge 0}$ is the semigroup associated to the linearized system.

Combining the estimates obtained for the linearized system we have

$$||\Gamma(\eta, w)(t)||_{V^s} \leq Me^{-\mu t}||(\eta^0, w^0)||_{V^s} + MCe^{-\mu t} \sup_{0 \leq \tau \leq t} ||e^{\mu\tau}(\eta, w)||_{V^s},$$

for any $t \geq 0$ and some positive constants M and C.

- If we take $(\eta, w) \in B_R(0) \subset Y_{s,\mu}$, the following estimate holds

$$||\Gamma(\eta, w)||_{Y_{s,\mu}} \leq M||(\eta^0, w^0)||_{V^s} + MC||(\eta, w)||^2_{Y_{s,\mu}} \leq Mr + MCR^2.$$

- A similar calculations shows that,

$$||\Gamma(\eta_1, w_1) - \Gamma(\eta_2, w_2)||_{Y_{s,\mu}} \leq 2RMC||(\eta_1, w_1) - (\eta_2, w_2)||_{Y_{s,\mu}},$$

  for any $(\eta_1, w_1), (\eta_2, w_2) \in B_R(0)$.

A suitable choice of R guarantees that Γ is a contraction.

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- **Dirichlet boundary conditions:**
- Less regularity for the potential a.
- Stabilization results for the nonlinear problem.
- Dissipative mechanisms, like $-[a(x)\varphi_x]_x$, ensures the uniform decay?
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