# Taxas de Decaimento para um Modelo Abstrato de Segunda Ordem

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22 de junho de 2018

# Apresentação

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- Problema associado a Equação da Onda

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- Método de Estabilização

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- Método de Estabilização
- Aplicação: Equação de Placas

# Parte I

# Problema associado a Equação da Onda

# Equação da Onda

Considere o seguinte problema: Encontrar uma função $u = u(t, x)$ tal que

$$\begin{cases} u_{tt}(t,x) - u_{xx}(t,x) + u_t(t,x) = 0, & t > 0 \text{ e } x \in [0, L] \\ u(t,0) = u(t,L) = 0, & t > 0 \\ u(0,x) = f(x), & x \in [0, L] \\ u_t(0,x) = g(x), & x \in [0, L]. \end{cases}$$

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E se as funções $f$ e $g$ não forem regulares?

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E se as funções $f$ e $g$ não forem regulares?

Nesse caso usamos uma generalização de funções, conhecida como distribuições.

# Distribuições

Seja $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ uma função integrável em qualquer conjunto compacto contido em $\Omega$. Então a função $f$ gera uma distribuição definida por:

$$\langle f, \phi \rangle = \int_{\Omega} f(x)\phi(x)\, dx$$

para toda função $\phi \in C_0^{\infty}(\Omega)$.

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para toda função $\phi \in C_0^\infty(\Omega)$.

A derivada distribucional da função $f$ é dada por:

$$\langle f', \phi \rangle = -\int_\Omega f(x)\phi'(x)\, dx$$

para toda função $\phi \in C_0^\infty(\Omega)$.

# Solução Generalizada

Assim, procuramos uma função $u = u(t, x)$ tal que

$$u_{tt}(t,x) - u_{xx}(t,x) + u_t(t,x) = 0$$

no sentido distribucional,

# Solução Generalizada

Assim, procuramos uma função $u = u(t,x)$ tal que

$$u_{tt}(t,x) - u_{xx}(t,x) + u_t(t,x) = 0$$

no sentido distribucional, ou seja, devemos encontrar $u = u(t,x)$ tal que

$$\int_0^L u_{tt}(x)\phi(x)\,dx - \int_0^L u(x)\phi_{xx}(x)\,dx + \int_0^L u_t(x)\phi(x)\,dx = 0$$

para toda função $\phi \in C_0^\infty[0,L]$.

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para toda função $\phi \in C_0^\infty[0,L]$. Além disso, devemos ter

$$u(0,x) = f(x) \quad \text{e} \quad u_t(0,x) = g(x) \quad \text{em } [0,L].$$

# Energia Total

Substituindo $\phi$ por $u_t$ na equação anterior tem-se que

$$\int_0^L u_{tt}(x)\,u_t(x)\;dx - \int_0^L u(x)\,u_{txx}(x)\;dx + \int_0^L u_t(x)\,u_t(x)\;dx = 0.$$

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Integrando por partes

$$\int_0^L u_{tt}(x)\, u_t(x)\; dx + \int_0^L u_x(x)\, u_{tx}(x)\; dx + \int_0^L |u_t(x)|^2\; dx = 0,$$

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Integrando por partes

$$\int_0^L u_{tt}(x)\,u_t(x)\,dx + \int_0^L u_x(x)\,u_{tx}(x)\,dx + \int_0^L |u_t(x)|^2\,dx = 0,$$

logo,

$$\frac{1}{2}\frac{d}{dt}\left\{\int_0^L |u_t(x)|^2\,dx + \int_0^L |u_x(x)|^2\,dx\right\} + \int_0^L |u_t(x)|^2\,dx = 0.$$

# Sistema dissipativo

Seja

$$E(t) = \frac{1}{2}\left\{\int_0^L |u_t(x)|^2\,dx + \int_0^L |u_x(x)|^2\,dx\right\}.$$

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Dessa forma, a energia total do sistema é dissipativa. Um dos problemas que surgem nesse caso é:

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Dessa forma, a energia total do sistema é dissipativa. Um dos problemas que surgem nesse caso é:

1) A energia total do sistema decai para zero?

2) Com que velocidade a energia total decai para zero?

Parte II

# Problema Abstrato de Segunda Ordem

In this work we study decay estimates for the total energy and the $L^2$-norm of solutions for several models associated to the following abstract second order evolution equation

$$A_1 u_{tt}(t,x) + A_2 u_t(t,x) + A_3 u(t,x) = 0 \qquad (1)$$

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$$A_1 u_{tt}(t,x) + A_2 u_t(t,x) + A_3 u(t,x) = 0 \tag{1}$$

with initial conditions

$$u(0,x) = u_0(x) \quad \text{and} \quad u_t(0,x) = u_1(x)$$

where $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$ and $A_i$ are positive self adjoint differential operators with symbols given by functions $P_i(\xi)$.

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- Wave equation

$$u_{tt}(t,x) - \Delta u(t,x) + u_t(t,x) = 0.$$

- Plate equation under rotational inertia effects

$$u_{tt}(t,x) - \Delta u_{tt}(t,x) + \Delta^2 u(t,x) + u_t(t,x) = 0.$$

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- Plate equation under rotational inertia effects

$$u_{tt}(t,x) - \Delta u_{tt}(t,x) + \Delta^2 u(t,x) + u_t(t,x) = 0.$$

- Equations with fractional operators, for example

$$u_{tt}(t,x) + (-\Delta)^{\sigma} u(t,x) + (-\Delta)^{\theta} u_t(t,x) = 0.$$

- Beam equation

$$u_{tt}(t,x) + \Delta^2 u(t,x) - \Delta u(t,x) + u_t(t,x) = 0.$$

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$$u_{tt}(t,x)+\Delta^2 u(t,x)-\Delta u(t,x)+u_t(t,x)=0.$$

- Klein-Gordon equation

$$u_{tt}(t,x)-\Delta u(t,x)+k^2 u(t,x)+u_t(t,x)=0.$$

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- IBq equation

$$u_{tt}(t,x)-\Delta u_{tt}(t,x)-\Delta u(t,x)+\nu(-\Delta)^{\theta} u_t(t,x)=0.$$

To obtain decay estimates to the above problem the idea is to work with the associated problem in the Fourier space.

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To this end we take the Fourier transform of the above problem to obtain the following initial problem

$$P_1(\xi)v_{tt}(t,\xi)+P_2(\xi)v_t(t,\xi)+P_3(\xi)v(t,\xi)=0$$

$$v(0,\xi)=v_0(\xi) \quad \text{and} \quad v_t(0,\xi)=v_1(\xi)$$

where $v$ is the Fourier transform of $u$.

The above problem becomes an initial value one for a linear ordinary differential equation of second order with coefficients depending on a frequency parameter $\xi \in \mathbb{R}^n_\xi$.

# Regularity-loss

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This decay structure still remains true for the case $\theta > 0$, but it becomes weaker when $\theta$ increases.

In fact, for $\theta > 0$ the eigenvalues are given by

$$\lambda(\xi) = -\frac{|\xi|^{2\theta}}{2(1+|\xi|^2)}\left[1 \pm \sqrt{1 - 4\left(1+|\xi|^2\right)|\xi|^{4-4\theta}}\right].$$

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In particular, the regularity-loss property ceases to occur in the case when $\theta = 1$ .

# Regularity-loss

For the wave equation the eigenvalues in the Fourer space are:

$$\lambda(\xi) = -|\xi|^{2\theta}\left[1 \pm \sqrt{1 - 4\,|\xi|^{2-4\theta}}\right].$$

The property of regularity-loss does not hold for the wave equation.

Parte III

# Método de Estabilização

In the sequel we study the following general linear second order ordinary differential equations with coefficients and initial data depending on a parameter $\xi$ for a function $v(t)$:

$$P_1(\xi)\, v_{tt}(t,\xi) + P_2(\xi)\, v_t(t,\xi) + P_3(\xi)\, v(t,\xi) = 0 \qquad (2)$$

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with initial conditions

$$v(0,\xi) = v_0(\xi) \quad \text{and} \quad v_t(0,\xi) = v_1(\xi) \tag{3}$$

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where $t \in \mathbb{R}^+$, $\xi \in \Omega$ and $P_i = P_i(\xi)$ $(i = 1,2,3)$ are positive measurable functions (except possibly on a set of measure zero) defined in a regular subset $\Omega$ of $\mathbb{R}^n$.

The global energy $E = E(t)$ associated with the linear ordinary differential equation (2) depending on a parameter $\xi \in \Omega \subset \mathbb{R}^n$ is defined by

$$E(t) := \frac{1}{2}\int_{\Omega}\left\{P_1(\xi)\,|v_t(t,\xi)|^2 + P_3(\xi)\,|v(t,\xi)|^2\right\}d\xi, \quad t \geq 0.$$

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To derive the estimates for the energy, we employ an improvement of the energy method in the Fourier space, which was developed in our previous works ([2], [3]).

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The idea to obtain the decay estimates em time $t$ for an associated functional of energy $E(t)$ is to construct another functional of energy type $F(t)$ such that

**(A)** $\displaystyle\int_S^T F(s)\,ds \leq C_1 E(S), \qquad \forall\, 0 \leq S < T;$

**(B)** $\big[E(t)\big]^{1+\beta} \leq C_2 F(t), \qquad \forall\, t \geq 0,$

for a suitable $\beta > 0$ and $C_2$, $C_1$ positive constants.

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for a suitable $\beta > 0$ and $C_2$, $C_1$ positive constants.

Therefore, if the estimates i) and ii) hold then we can conclude that

$$\int_S^\infty \left[E(s)\right]^{1+\beta} ds \leq CE(S), \qquad \forall\, S \geq 0$$

with $C$ a positive constant depending on the initial data.

# Haraux-Komornik inequality

## Lemma

*Let* $E : [0,+\infty) \to [0,+\infty)$ *be a non-increasing function and assume that there are two constants* $\gamma \geq 0$ *and* $T_0 > 0$ *such that:*

$$\int_S^{\infty} [E(s)]^{1+\gamma} ds \leq T_0 [E(0)]^{\gamma} E(S), \qquad \forall\, S \geq 0.$$

*i) If* $\gamma > 0$ *then* $E(t) \leq E(0)\, T_0^{1/\gamma} \left(1 + \frac{1}{\gamma}\right)^{1/\gamma} t^{-1/\gamma}, \qquad \forall\, t \geq T_0.$

*ii) If* $\gamma = 0$ *then* $E(t) \leq E(0)\, e^{1-t/T_0}, \qquad \forall\, t \geq T_0.$

We multiply both sides of (2) by $\overline{v}_t$ and we take the real part of the resulting identity to have

$$\begin{aligned} &\frac{1}{2}P_1(\xi)\,|v_t(T)|^2+\frac{1}{2}P_3(\xi)\,|v(T)|^2+\int_S^T P_2(\xi)\,|v_t(s)|^2\,ds \\ &=\frac{1}{2}P_1(\xi)\,|v_t(S)|^2+\frac{1}{2}P_3(\xi)\,|v(S)|^2. \end{aligned} \tag{4}$$

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$$= \frac{1}{2}P_1(\xi)\,|v_t(S)|^2 + \frac{1}{2}P_3(\xi)\,|v(S)|^2. \tag{4}$$

Let $\rho : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be the function of $\xi$ defined by

$$\rho(\xi) = \min\left\{P_3(\xi)\,P_2(\xi)^{-1},\, P_2(\xi)\,P_1(\xi)^{-1}\right\}. \tag{5}$$

Multiplying the equation (2) by $\rho\overline{v}$ and taking the real part we obtain

$$\begin{aligned}
&\frac{1}{2}\rho(\xi)P_2(\xi)\,|v(T)|^2+\int_S^T\rho(\xi)P_3(\xi)\,|v(s)|^2\,ds\\
&=\frac{1}{2}\rho(\xi)P_2(\xi)\,|v(S)|^2+\int_S^T\rho(\xi)P_1(\xi)\,|v_t(s)|^2\,ds\\
&\quad-\rho(\xi)P_1(\xi)\,Re(v_t(T)\,\overline{v}(T))+\rho(\xi)P_1(\xi)\,Re(v_t(S)\,\overline{v}(S)) \qquad (6)\\
&\leq\frac{1}{2}P_3(\xi)\,|v(S)|^2+\int_S^T P_2(\xi)\,|v_t(s)|^2\,ds+\frac{1}{2}P_1(\xi)\,|v_t(T)|^2\\
&\quad+\frac{1}{2}\rho(\xi)^2P_1(\xi)\,|v(T)|^2+\frac{1}{2}P_1(\xi)\,|v_t(S)|^2+\frac{1}{2}\rho(\xi)^2P_1(\xi)\,|v(S)|^2,
\end{aligned}$$

Combining (4) and (6) we obtain that

$$\int_S^T \rho(\xi) P_3(\xi) |v(s)|^2 ds \leq P_1(\xi) |v_t(S)|^2 + \frac{3}{2} P_3(\xi) |v(S)|^2 \tag{7}$$

for all $\xi \in \Omega$ and $0 \leq S < T < +\infty$, because $\rho(\xi)^2 P_1(\xi) \leq P_3(\xi)$ for all $\xi \in \Omega$.

Combining (4) and (6) we obtain that

$$\int_S^T \rho(\xi) P_3(\xi) |v(s)|^2 \, ds \leq P_1(\xi) |v_t(S)|^2 + \frac{3}{2} P_3(\xi) |v(S)|^2 \tag{7}$$

for all $\xi \in \Omega$ and $0 \leq S < T < +\infty$, because $\rho(\xi)^2 P_1(\xi) \leq P_3(\xi)$ for all $\xi \in \Omega$.

Motivated by estimates (4) and (7) we introduce the following functional

$$F(t) := \int_\Omega P_2(\xi) |v_t(t)|^2 \, d\xi + \int_\Omega \rho(\xi) P_3(\xi) |v(t)|^2 \, d\xi, \quad t \geq 0.$$

Combining (4) and (6) we obtain that

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Then, (4) and (7) imply that

$$\int_S^T F(s)\,ds \leq CE(S). \tag{8}$$

## Theorem 1

*Let* $P_1, P_2, P_3 : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ *be positive measurable functions of* $\xi \in \Omega$ *such that* $C_1 P_1(\xi) \leq P_2(\xi) \leq C_2 P_3(\xi)$ *(except possibly on a set of measure zero) where* $C_1$ *and* $C_2$ *are positive constants. Then, the global energy* $E(t)$ *in* $\Omega$ *associated with the initial value problem* (2)-(3) *satisfies*

$$\int_S^T E(s)\, ds \leq CE(S)$$

*for all* $0 \leq S < T < +\infty$*, where* $C > 0$ *is a constant.*

## Corollary 1

*Assuming the hypotheses of Theorem* 1 *we have*

$$E(t) \leq C\, e^{-\gamma t} \quad (t \to \infty)$$

*where* $C > 0$ *is a constant depending on the initial data and* $\gamma$ *is a positive constant.*

The proof of Corollary 1 follows from the Theorem 1 and Haraux-Komornik Lemma.

In the sequel, we impose suitable conditions on the functions $P_i(\xi)$ in (2) to get a fixed number $\beta > 0$ such that

$$\left[E(t)\right]^{1+\beta} \leq CF(t), \qquad \forall\, t \geq 0 \tag{9}$$

for some positive constant $C$, which may depend on $\beta$ and the initial data.

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$$\left[E(t)\right]^{1+\beta} \leq CF(t), \qquad \forall\, t \geq 0 \tag{9}$$

for some positive constant $C$, which may depend on $\beta$ and the initial data. Thus, if the estimates (8) and (9) hold, then we can conclude that

$$\int_S^T \left[E(s)\right]^{1+\beta} ds \leq C\,E(S), \qquad \forall\, 0 \leq S < T < +\infty$$

with a constant $C > 0$ depending on the initial data.

Remains to find conditions on the functions $P_i$ in order to have that the estimate (8) is true.

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For this purpose, we first note that for $\beta > 0$ it holds the following estimate

$$
\begin{aligned}
&[E(t)]^{1+\beta} \\
&\leq \frac{1}{2}\left[\int_\Omega P_1(\xi)\,|v_t(t)|^2\,d\xi\right]^{1+\beta} + \frac{1}{2}\left[\int_\Omega P_3(\xi)\,|v(t)|^2\,d\xi\right]^{1+\beta} \\
&\leq \frac{1}{2}\left\{\left[\int_\Omega P_2(\xi)^{-\frac{1}{\beta}}\,P_1(\xi)^{\frac{1+\beta}{\beta}}\,|v_t(t)|^2\,d\xi\right]^{\beta}\right. \\
&\qquad \left. + \left[\int_\Omega \rho(\xi)^{-\frac{1}{\beta}}\,P_3(\xi)\,|v(t)|^2\,d\xi\right]^{\beta}\right\} F(t)
\end{aligned}
$$

for all $t \geq 0$.

To prove our result we need to estimate the two integrals in the right hand side of the above equation. For example, to estimate the term

$$\int_{\Omega} \rho(\xi)^{-\frac{1}{\beta}} P_3(\xi)\,|v(t)|^2\, d\xi$$

we can use the following estimates:

$$\frac{1}{2} P_3(\xi)\,|v(t)|^2 \leq \frac{1}{2} P_1(\xi)\,|v_1|^2 + \frac{1}{2} P_3(\xi)\,|v_0|^2;$$

$$\int_{\Omega} \rho(\xi)^{-\frac{1}{\beta}} P_3(\xi)\,|v(t)|^2\, d\xi \leq \int_{\Omega} \rho(\xi)^{-\frac{1}{\beta}} P_1(\xi)\,|v_1|^2\, d\xi + \int_{\Omega} \rho(\xi)^{-\frac{1}{\beta}} P_3(\xi)\,|v_0|^2\, d\xi.$$

Under different hypotheses on the functions $P_i$ we can estimate the last two integrals in terms of the initial data.

Under different hypotheses on the functions $P_i$ we can estimate the last two integrals in terms of the initial data.

**Hypothesis 1:** There exists a number $\beta > 0$ such that at least one of the following two conditions for $P_i$ holds:

i) $C_\beta^3 = \int_\Omega \rho(\xi)^{-\frac{1}{\beta}} P_1(\xi) d\xi < \infty, \quad C_\beta^4 = \int_\Omega \rho(\xi)^{-\frac{1}{\beta}} P_3(\xi) d\xi < \infty.$

ii) $C_\beta^1 = \int_\Omega P_2(\xi)^{-\frac{1}{\beta}} P_1(\xi)^{\frac{1+\beta}{\beta}} d\xi < \infty, \; C_\beta^4 = \int_\Omega \rho(\xi)^{-\frac{1}{\beta}} P_3(\xi)\, d\xi < \infty,$

$C_\beta^5 = \int_\Omega \rho(\xi)^{-\frac{1}{\beta}} P_3(\xi)\, P_2(\xi)^{-2}\, P_1(\xi)^2\, d\xi < \infty$ and

$C_\beta^6 = \int_\Omega \rho(\xi)^{-\frac{1}{\beta}} P_2(\xi)^{-2}\, P_1(\xi)\, P_3(\xi)^2\, d\xi < \infty.$

## Theorem 2

*Let* $[v_0, v_1] \in L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ *and let* $P_i(\xi)$ $(i = 1, 2, 3)$ *and* $\beta > 0$ *satisfying the Hypothesis* 1. *Then, the global energy* $E(t)$ *in* $\Omega$ *associated with the problem* (2)-(3) *satisfies*

$$\int_S^T \left[E(t)\right]^{1+\beta} dt \leq C_\beta \left\{ \|v_0\|^2_{L^\infty} + \|v_1\|^2_{L^\infty} \right\}^\beta E(S)$$

*for all* $0 \leq S < T < \infty$, *where* $C_\beta$ *is a positive constant depending on* $\beta$

**Hypothesis 2:** We assume the existence of real numbers $\beta > 0$ and $r_i \geq 0$ $(i = 1, 2, 3, 4)$ such that the measurable and positive (except possibly on a set of measure zero)
functions $P_i(\xi)$ $(i = 1, 2, 3)$ and the initial data $v_0, v_1$ satisfy at least one of the following two conditions:

i) $$\int_\Omega \rho(\xi)^{-\frac{1}{\beta}} P_3(\xi) \,|v_0|^2 \, d\xi \leq C_1 \int_\Omega (1 + |\xi|^2)^{r_1} |v_0|^2 \, d\xi < +\infty,$$

$$\int_\Omega \rho(\xi)^{-\frac{1}{\beta}} P_1(\xi) \,|v_1|^2 \, d\xi \leq C_2 \int_\Omega (1 + |\xi|^2)^{r_2} \,|v_1|^2 \, d\xi < +\infty.$$

ii)

$$\int_\Omega \rho(\xi)^{-\frac{1}{\beta}} P_2(\xi)^{-2} P_1(\xi) P_3(\xi)^2 |v_0|^2 d\xi \leq C_3 \int_\Omega (1+|\xi|^2)^{r_3} |v_0|^2 d\xi < \infty,$$

$$\int_\Omega \rho(\xi)^{-\frac{1}{\beta}} P_3(\xi) \, |v_0|^2 \, d\xi \leq C_3 \int_\Omega (1+|\xi|^2)^{r_3} \, |v_0|^2 \, d\xi < +\infty,$$

$$\int_\Omega P_2(\xi)^{-\frac{1}{\beta}} P_1(\xi)^{\frac{1+\beta}{\beta}} \, |v_1|^2 \, d\xi \leq C_4 \int_\Omega (1+|\xi|^2)^{r_4} \, |v_1|^2 \, d\xi < +\infty,$$

$$\int_\Omega \rho(\xi)^{-\frac{1}{\beta}} P_3(\xi) P_2(\xi)^{-2} P_1(\xi)^2 |v_1|^2 d\xi \leq C_4 \int_\Omega (1+|\xi|^2)^{r_4} |v_1|^2 d\xi < \infty.$$

## Theorem 3

*For global energy* $E(t)$ *of the problem* (2)–(3) *we have the following two possibilities:*

**I)** *If the item* (i) *of the Hypothesis* 2 *holds then*

$$\int_S^T \left[E(t)\right]^{1+\beta} dt \leq C \left\{ \int_\Omega \langle\xi\rangle^{2r_1} |v_0|^2 d\xi + \int_\Omega \langle\xi\rangle^{2r_2} |v_1|^2 d\xi \right\}^\beta E(S)$$

**II)** *If the item* (ii) *of the Hypothesis* 2 *holds then*

$$\int_S^T \left[E(t)\right]^{1+\beta} dt \leq C \left\{ \int_\Omega \langle\xi\rangle^{2r_3} |v_0|^2 d\xi + \int_\Omega \langle\xi\rangle^{2r_4} |v_1|^2 d\xi \right\}^\beta E(S)$$

*for* $0 \leq S < T < \infty$*, where* $\langle\xi\rangle^2 = \left(1 + |\xi|^2\right)$. *The positive constant* $C$ *is independent of the initial data.*

As consequence of Theorem 2 or Theorem 3 we get the following result.

## Corollary 2

*Under the hypotheses of Theorem 2 or Theorem 3, the global energy to problem (2)–(3) has the following asymptotic behavior*

$$E(t) \leq C\, t^{-1/\beta} \quad (t \to \infty)$$

*with $\beta > 0$ given in the Hypotheses 1 or 2 and $C$ a positive constant depending on the initial data.*

# Parte IV

## Aplicação

In the sequel we apply the previous results to obtain decay rates for the total energy associated with a plate equation under rotational inertia effects in $\mathbb{R}^n$ is given by

$$u_{tt}(t,x) - \Delta u_{tt}(t,x) + \Delta^2 u(t,x) + u_t(t,x) = 0. \tag{10}$$

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decay rates for the total energy associated with a plate equation under rotational inertia effects in $\mathbb{R}^n$ is given by

$$u_{tt}(t,x) - \Delta u_{tt}(t,x) + \Delta^2 u(t,x) + u_t(t,x) = 0. \tag{10}$$

The total energy $E_u(t)$ associated with the solution $u(t,x)$ of the equation (10) is defined by

$$E_u(t) = \frac{1}{2}\left(\|u_t(t)\|^2 + \|\nabla u_t(t)\|^2 + \|\Delta u(t)\|^2\right), \qquad t \geq 0.$$

To get those decay estimates to the initial value problem
associated with equation (10) we take the Fourier transform in both sides of the equation (10) for obtain

$$v_{tt}(t,\xi)+|\xi|^2 v_{tt}(t,\xi)+|\xi|^4 v(t,\xi)+v_t(t,\xi)=0$$

with $v=\hat{u}$, where $\hat{u}$ is the partial Fourier transform of $u$.

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with $v=\hat{u}$, where $\hat{u}$ is the partial Fourier transform of $u$.

As usual we denote by $E_T=E_T(t)$ the total energy of the plate equation in the Fourier space, that is,

$$E_T(t)=\frac{1}{2}\int_{\mathbf{R}^n}\left\{(1+|\xi|^2)\,|v_t(t)|^2+|\xi|^4|v(t)|^2\right\}\,d\xi,\quad t\geq 0.$$

## Theorem 4

*Let* $n \geq 1$ *and* $\beta$ *be a positive fixed number satisfying* $\beta > \dfrac{4}{n+4}$ *and let* $[u_0, u_1] \in \left(H^{2+\frac{1}{\beta}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)\right) \times \left(H^{1+\frac{1}{\beta}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)\right)$. *Then it is true that*

$$E_u(t) \leq K_\beta \left\{ \|u_0\|_{L^1}^2 + \|u_1\|_{L^1}^2 + \|u_0\|_{H^r}^2 + \|u_1\|_{H^{r-1}}^2 \right\} t^{-1/\beta}$$

*for* $t \geq T_0$ *where* $T_0$ *is a constant depending on the initial data,* $K_\beta > 0$ *a constant depending on* $\beta$ *and* $r = 2 + \dfrac{1}{\beta}$.

To the plate equation we have:

$$P_1(\xi) = 1 + |\xi|^2, \qquad P_2(\xi) = 1 \quad \text{and} \quad P_3(\xi) = |\xi|^4.$$

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Then, it is easy to see that there exists $b > 0$ such that the corresponding function $\rho(\xi)$ given in (5) satisfies

$$\rho(\xi) \geq \begin{cases} C\,|\xi|^4, & \text{for } |\xi| \leq b \\ \dfrac{1}{1 + C|\xi|^2}, & \text{for } |\xi| \geq b \end{cases}$$

where $C$ is a positive constant.

We broke the integral of the energy in two parts: on the low and the high frequency regions:

$$E_l(t) = \frac{1}{2}\int_{|\xi|\leq b} \left\{ (1+|\xi|^2)\,|v_t(t)|^2 + |\xi|^4\,|v(t)|^2 \right\} d\xi$$

$$E_h(t) = \frac{1}{2}\int_{|\xi|\geq b} \left\{ (1+|\xi|^2)\,|v_t(t)|^2 + |\xi|^4\,|v(t)|^2 \right\} d\xi.$$

Let $P_1(\xi) = 1 + |\xi|^2,\ P_2(\xi) = 1$ and $P_3(\xi) = |\xi|^4$. By choosing $\beta > \frac{4}{n+4}$ it is easy to verify that the item (ii) of the Hypotheses 1 holds for $\Omega = \{\xi \in \mathbb{R}^n \mid |\xi| \leq b\}$.

Let $P_1(\xi) = 1 + |\xi|^2$, $P_2(\xi) = 1$ and $P_3(\xi) = |\xi|^4$. By choosing $\beta > \frac{4}{n+4}$ it is easy to verify that the item (ii) of the Hypotheses 1 holds for $\Omega = \{\xi \in \mathbb{R}^n \,|\, |\xi| \leq b\}$. For example,

$$C_\beta^4 = \int_{|\xi|\leq b} \rho(\xi)^{-\frac{1}{\beta}} P_3(\xi)\, d\xi = \int_{|\xi|\leq b} \rho(\xi)^{-\frac{1}{\beta}} |\xi|^4\, d\xi$$

$$\leq K \int_{|\xi|\leq b} |\xi|^{-\frac{4}{\beta}+4}\, d\xi < +\infty$$

due to the hypothesis that $\frac{4}{\beta} - 4 < n$.

Now, choosing $\beta > \dfrac{4}{n+4}$ the item (ii) in the Hypothesis 1 is satisfied in the region $\Omega = \{\xi \in \mathbb{R}^n \mid |\xi| \leq b\}$.

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Then, we can apply Theorem 2 to get

$$\int_S^T [E_l(s)]^{1+\beta} ds \leq C_\beta \left\{ \|u_0\|^2_{L^1} + \|u_1\|^2_{L^1} \right\}^\beta E_l(S) \leq C_\beta \left\{ \|u_0\|^2_{L^1} + \|u_1\|^2_{L^1} \right\}^\beta E_T(S) \tag{11}$$

for all $0 \leq S < T < +\infty$.

Moreover, since $[u_0, u_1] \in H^{2+\frac{1}{\beta}}(\mathbb{R}^n) \times H^{1+\frac{1}{\beta}}(\mathbb{R}^n)$ the item (i) of the Hypothesis 2 holds in the region
$\Omega = \{\xi \in \mathbb{R}^n \mid |\xi| \geq b\}$.

Moreover, since $[u_0, u_1] \in H^{2+\frac{1}{\beta}}(\mathbb{R}^n) \times H^{1+\frac{1}{\beta}}(\mathbb{R}^n)$ the item (i) of the Hypothesis 2 holds in the region $\Omega = \{\xi \in \mathbb{R}^n \mid |\xi| \geq b\}$.

Thus, we can use Theorem 3 to obtain

$$\begin{aligned} \int_S^T \left[E_h(s)\right]^{1+\beta} ds &\leq C\left\{\|u_0\|_{H^r}^2 + \|u_1\|_{H^{r-1}}^2\right\}^\beta E_h(S) \\ &\leq C\left\{\|u_0\|_{H^r}^2 + \|u_1\|_{H^{r-1}}^2\right\}^\beta E_T(S) \end{aligned} \tag{12}$$

for all $0 \leq S < T < +\infty$ with $r = 2 + \dfrac{1}{\beta}$.

Thus, it follows from (11), (12) and the Plancherel Theorem that

$$\begin{aligned}
\int_S^T [E_u(s)]^{1+\beta} ds &= \int_S^T [E_T(s)]^{1+\beta} ds \\
&\leq C \int_S^T [E_l(s)]^{1+\beta} ds + C \int_S^T [E_h(s)]^{1+\beta} ds \\
&\leq C_\beta \left\{ \|u_0\|_{L^1}^2 + \|u_1\|_{L^1}^2 \right\}^\beta E_T(S) + C [E_T(0)]^\beta E_T(S) \\
&= \left\{ C_\beta \left\{ \|u_0\|_{L^1}^2 + \|u_1\|_{L^1}^2 \right\}^\beta + C [E_u(0)]^\beta \right\} E_u(S)
\end{aligned}$$

for all $0 \leq S < T < +\infty$.

Let $T_0 > 0$ be fixed such as

$$T_0 \left[E_u(0)\right]^\beta = C_\beta \left\{\|u_0\|_{L^1}^2 + \|u_1\|_{L^1}^2\right\}^\beta + C \left[E_u(0)\right]^\beta.$$

Let $T_0 > 0$ be fixed such as

$$T_0 \, [E_u(0)]^\beta = C_\beta \left\{ \|u_0\|_{L^1}^2 + \|u_1\|_{L^1}^2 \right\}^\beta + C \, [E_u(0)]^\beta.$$

Letting $T \to +\infty$ in the previous inequality and using the Haraux-Komornik Lemma we can conclude that

$$E_u(t) \leq K_\beta \left\{ \|u_0\|_{L^1}^2 + \|u_1\|_{L^1}^2 + \|L^{1/2} u_0\|^2 + \|u_1\|^2 \right\} t^{-1/\beta},$$

for all $t \geq T_0$, where the constant $K_\beta$ depends on $\beta$.

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Thank you for your attention ! !