7.9 – EXERCÍCIOS – pg. 333

Nos exercícios de 1 a 14, calcular a integral indefinida.

1. $\int \frac{(1+sen\ x)}{sen\ x(1+\cos x)}dx$

Fazendo:

$t = tg\ \frac{x}{2}$

Temos:

$$I = \int \frac{\left(1+\frac{2t}{1+t^2}\right)\frac{2\,dt}{1+t^2}}{\frac{2}{1+t^2}\left(1+\frac{1-t^2}{1+t^2}\right)} = \int \frac{\frac{1+t^2+2t}{1+t^2}.\frac{2\,dt}{1+t^2}}{\frac{2\,t}{1+t^2}\ \frac{1+t^2+1-t^2}{1+t^2}}$$

$$= \int \frac{\frac{2(1+t^2+2t)}{(1+t^2)^2}}{\frac{4t}{(1+t^2)^2}}dt = \int \frac{2(t+1)^2}{4t}dt = \frac{1}{2}\int \frac{(t+1)^2}{t}dt$$

$$= \frac{1}{2}\int \frac{t^2+2t+1}{t}\,dt$$

$$= \frac{1}{2}\int \left(t+2+\frac{1}{t}\right)dt$$

$$= \frac{1}{2}\left(\frac{t^2}{2}+2t+\ln|t|\right)+C$$

$$= \frac{tg^2\,\frac{x}{2}}{4}+tg\,\frac{x}{2}+\frac{1}{2}\ln\left|tg\,\frac{x}{2}\right|+C$$

2. $\int \frac{dx}{1+sen\ x+\cos x}$

Fazendo:

$t = tg\ \frac{x}{2}$

Temos:

$$= \int \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} = \int \frac{2dt}{2t+2}$$

$$= \int \frac{dt}{t+1} = \ln |t+1| + C$$

$$= \ln \left| tg\ \frac{x}{2} + 1 \right| + C$$

3. $\int \frac{2dx}{sen\ x + tg\ x}$

Fazendo:

$t = tg\ \frac{x}{2}$

Temos:

$$I = \int \frac{\frac{2 \,.\, 2\ dt}{1+t^2}}{\frac{2t}{1+t^2} + \frac{2t}{1+t^2} \cdot \frac{1+t^2}{1-t^2}}$$

$$= \int \frac{4dt}{\left(1+t^2\right) \frac{2t\left(1-t^2\right) + 2t\left(1+t^2\right)}{\left(1+t^2\right)\left(1-t^2\right)}}$$

$$= \int \frac{4dt}{\left(1+t^2\right)} \cdot \frac{\left(1+t^2\right)\left(1-t^2\right)}{2t-2t^3+2t+2t^3}$$

$$= \int \frac{4\left(1-t^2\right)}{4t}\, dt$$

$$= \int \frac{1-t^2}{t} dt$$

$$= \int \frac{dt}{t} - \int t\ dt$$

$$= \ln |t| - \frac{t^2}{2} + C$$

$$= \ln \left| tg\ \frac{x}{2} \right| - \frac{tg^2\ \frac{x}{2}}{2} + C$$

4. $\int \frac{dx}{4 + 5\cos\ x}$

Fazendo:

$$t = tg\ \frac{x}{2}$$

Temos:

$$I = \int \frac{\frac{2dt}{1+t^2}}{4+5\,.\,\frac{1-t^2}{1+t^2}} = \int \frac{2dt}{1+t^2}\cdot\frac{1}{4+\frac{5-5t^2}{1+t^2}}$$

$$= \int \frac{2dt}{1+t^2}\cdot\frac{1}{\frac{4+4t^2+5-5t^2}{1+t^2}}$$

$$= \int \frac{2dt}{1+t^2}\cdot\frac{1}{\frac{9-t^2}{1+t^2}}$$

$$= \int \frac{2dt}{1+t^2}\cdot\frac{1+t^2}{9-t^2}$$

$$= \int \frac{2dt}{9-t^2}$$

$$= 2.\frac{1}{2.3}\ln\left|\frac{t+3}{t-3}\right| + C$$

$$= \frac{1}{3}\ln\left|\frac{t+3}{t-3}\right| + C$$

$$= \frac{1}{3}\ln\left|\frac{tg\ \frac{x}{2}+3}{tg\ \frac{x}{2}-3}\right| + C$$

5. $\int \frac{dx}{3+\cos x}$

Fazendo:

$$t = tg\ \frac{x}{2}$$

Temos:

$$I = \int \frac{\frac{2dt}{1+t^2}}{3+\frac{1-t^2}{1+t^2}} = \int \frac{2dt}{1+t^2}\cdot\frac{1}{\frac{3+3t^2+1-t^2}{1+t^2}}$$

$$= \int \frac{2dt}{4+2t^2} = \frac{\sqrt{2}}{2}\, arc\ tg\ \frac{t}{\sqrt{2}} + C$$

$$= \frac{\sqrt{2}}{2}\, arc\ tg\, \frac{tg\ \frac{x}{2}}{\sqrt{2}} + C$$

6. $\int \frac{dx}{1-\cos x}$

Fazendo:

$t = tg\ \frac{x}{2}$

Temos:

$$I = \int \frac{\frac{2dt}{1+t^2}}{1-\frac{1-t^2}{1+t^2}} = \int \frac{2dt}{1+t^2} \cdot \frac{1+t^2}{1+t^2-1+t^2}$$

$$= \int \frac{2dt}{2t^2} = \int \frac{dt}{t^2}$$

$$= \frac{t^{-1}}{-1} + C$$

$$= \frac{-1}{tg\ \frac{x}{2}} + C$$

7. $\int \frac{1+\cos x}{1-sen\ x}\, dx$

Fazendo:

$t = tg\ \frac{x}{2}$

Temos:

$$I=\int \frac{1+\dfrac{1-t^2}{1+t^2}}{1-\dfrac{2t}{1+t^2}}\cdot\frac{2dt}{1+t^2}$$

$$=\int \frac{\dfrac{1+t^2+1-t^2}{1+t^2}}{\dfrac{1+t^2-2t}{1+t^2}}\cdot\frac{2dt}{1+t^2}$$

$$=\int \frac{2}{t^2-2t+1}\cdot\frac{2dt}{1+t^2}$$

$$=\int \frac{4\,dt}{(t-1)^2\left(1+t^2\right)}$$

$$\frac{4}{(t-1)^2\left(1+t^2\right)}=\frac{A}{t-1}+\frac{B}{(t-1)^2}+\frac{Ct+D}{t^2+1}$$

$$4\equiv A(t-1)\left(t^2+1\right)+B\left(t^2+1\right)+(Ct+D)(t-1)^2$$

$$t=1 \quad\Rightarrow\quad 4=2B \quad\therefore\quad B=2$$

$$4\equiv A\left(t^3+t-t^2-1\right)+Bt^2+B+(Ct+D)\left(t^2-2t+1\right)$$
$$\equiv At^3+At-At^2-A+Bt^2+B+Ct^3-2Ct^2+Ct+Dt^2-2Dt+D$$

$$\begin{cases} A+C=0\\ -A+B-2C+D=0\\ A+C-2D=0\\ -A+B+D=4\end{cases}$$

$$A=-2, B=2, C=2, D=0$$

$$I=\int\left(\frac{-2}{t-1}+\frac{2}{(t-1)^2}+\frac{2t}{t^2+1}\right)dt$$
$$=-2\ln|t-1|+2\,\frac{(t-1)^{-1}}{-1}+\ln\left|t^2+1\right|+C$$
$$=-2\ln\left|tg\frac{x}{2}-1\right|-\frac{2}{tg\left(\dfrac{x}{2}\right)-1}+\ln\left|tg^2\frac{x}{2}+1\right|+C$$

8. $\int \frac{dx}{3+sen\ 2x}$

Fazendo

$u = 2x$

$du = 2dx$

e

$t = tg\ \frac{u}{2}$

Temos:

$$I = \int \frac{\frac{du}{2}}{3+sen\ u} = \frac{1}{2}\int \frac{\frac{2dt}{1+t^2}}{3+\frac{2t}{1+t^2}}$$

$$= \frac{1}{2}\int \frac{\frac{2dt}{1+t^2}}{\frac{3+3t^2+2t}{1+t^2}} = \int \frac{dt}{3t^2+2t+3}$$

$$= \frac{1}{3}\int \frac{dt}{t^2+\frac{2}{3}t+1}$$

$$= \frac{1}{3}\int \frac{dt}{\left(t+\frac{1}{3}\right)^2+\frac{8}{9}} = \frac{1}{3}.\frac{3}{\sqrt{8}}\ arc\ tg\ \frac{t+\frac{1}{3}}{\frac{\sqrt{8}}{3}}+C$$

$$= \frac{1}{\sqrt{8}}\ arc\ tg\ \frac{3\left(t+\frac{1}{3}\right)}{\sqrt{8}}+C$$

$$= \frac{1}{\sqrt{8}}\ arc\ tg\ \frac{3t+1}{\sqrt{8}}+C$$

$$= \frac{1}{\sqrt{8}}\ arc\ tg\ \frac{3tg\ \frac{u}{2}+1}{\sqrt{8}}+C$$

$$= \frac{1}{\sqrt{8}}\ arc\ tg\ \frac{3tg\ x+1}{\sqrt{8}}+C$$

9. $\int \frac{\cos\ (2t-1)}{2-\cos\ (2t-1)}dt$

Fazendo:

$u = 2t - 1$

$du = 2dt$

e

$t = tg\ \dfrac{u}{2}$

Temos:

$$I = \frac{\cos u}{2 - \cos u} \cdot \frac{du}{2}$$

$$= \frac{1}{2} \int \frac{\frac{1 - t^2}{1 + t^2} \cdot \frac{2dt}{1 + t^2}}{2 - \frac{1 - t^2}{1 + t^2}}$$

$$= \frac{1}{2} \int \frac{\frac{2(1 - t^2)}{(1 + t^2)^2} dt}{\frac{2 + 2t^2 - 1 + t^2}{1 + t^2}} = \int \frac{1 - t^2}{(1 + t^2)^2} \cdot \frac{1 + t^2}{3t^2 + 1} dt$$

$$= \int \frac{1 - t^2}{(1 + t^2)(3t^2 + 1)} dt = \frac{1}{3} \int \frac{1 - t^2}{(1 + t^2)\left(t^2 + \frac{1}{3}\right)} dt$$

Usando:

$$\frac{1 - t^2}{(1 + t^2)(t^2 + 1/3)} = \frac{At + B}{1 + t^2} + \frac{Ct + D}{t^2 + 1/3}$$

$$1 - t^2 \equiv (At + B)(t^2 + 1/3) + (Ct + D)(1 + t^2)$$

$A + C = 0$

$B + D = -1$

$\dfrac{1}{3} A + C = 0$

$\dfrac{1}{3} B + D = 1$

$A = 0, B = -3, C = 0, D = 2$

$$I=\frac{1}{3}\int\left(\frac{-3}{1+t^2}+\frac{2}{t^2+1/3}\right)dt$$

$$=-arctgt+\frac{2}{\sqrt{3}}arctg(\sqrt{3}t)+C$$

$$=-arctg\left(tg\,\frac{2t-1}{2}\right)+\frac{2}{\sqrt{3}}arctg\left(\sqrt{3}tg\,\frac{2t-1}{2}\right)+C.$$

10. $\displaystyle\int\frac{dt}{3+sent+\cos t}$

Fazendo $t=tg\,\frac{u}{2}$:

$$I=\int\frac{\dfrac{2du}{1+u^2}}{3+\dfrac{2u}{1+u^2}+\dfrac{1-u^2}{1+u^2}}$$

$$=\int\frac{2du}{1+u^2}\cdot\frac{1+u^2}{2u^2+2u+4}$$

$$=\int\frac{du}{u^2+u+2}=\int\frac{du}{\left(u+\frac{1}{2}\right)^2+\frac{7}{4}}$$

$$=\frac{1}{\frac{\sqrt{7}}{2}}arc\;tg\,\frac{u+\frac{1}{2}}{\frac{\sqrt{7}}{2}}+C$$

$$=\frac{2}{\sqrt{7}}arc\;tg\,\frac{2u+1}{\sqrt{7}}+C$$

$$=\frac{2}{\sqrt{7}}arc\;tg\left(\frac{2tg\,\frac{t}{2}+1}{\sqrt{7}}\right)+C$$

11. $\displaystyle\int\frac{e^x dx}{4sen\,e^x-3\cos e^x}$

Fazendo

$u=e^x$

$du=e^x dx$

$$I=\int\frac{du}{4sen\,u-3\cos u}$$

Fazendo $t = tg\,\frac{u}{2}$:

$$I = \int \frac{\frac{2dt}{1+t^2}}{4\,.\,\frac{2t}{1+t^2} - 3.\,\frac{1-t^2}{1+t^2}}$$

$$= \int \frac{\frac{2t}{1+t^2}}{\frac{8t-3+3t^2}{1+t^2}} = \int \frac{2dt}{3t^2+8t-3}$$

$$= \int \frac{\frac{2}{3}dt}{t^2 + \frac{8t}{3} - 1}$$

Fazendo

$$\frac{\frac{2}{3}}{t^2 + \frac{8t}{3} - 1} = \frac{A}{t+3} + \frac{B}{t - \frac{1}{3}}$$

$$\frac{2}{3} \equiv A\left(t - 1/3\right) + B(t+3)$$

$$t = 1/3 \quad \Rightarrow \quad B = 1/5$$

$$t = -3 \quad \Rightarrow \quad A = -1/5$$

Temos:

$$I = \int \left(\frac{-1/5}{t+3} + \frac{1/5}{t-1/3} \right) dt$$

$$= -\frac{1}{5}\ln\,(t+3) + \frac{1}{5}\ln\left|t - 1/3\right| + C$$

$$= -\frac{1}{5}\ln\left|tg\,\frac{u}{2} + 3\right| + \frac{1}{5}\ln\left|tg\,\frac{u}{2} - \frac{1}{3}\right| + C$$

$$= -\frac{1}{5}\ln\left|tg\,\frac{e^x}{2} + 3\right| + \frac{1}{5}\ln\left|tg\,\frac{e^x}{2} - \frac{1}{3}\right| + C$$

12. $\int \frac{\cos\theta\;d\theta}{1+\cos\theta}$

Fazendo $t = tg\,\dfrac{\theta}{2}$:

$$= \int \frac{\dfrac{1-t^2}{1+t^2} \cdot \dfrac{2dt}{1+t^2}}{1 + \dfrac{1-t^2}{1+t^2}}$$

$$= \int \frac{\dfrac{2(1-t^2)dt}{(1-t^2)^2}}{\dfrac{1+t^2+1-t^2}{1+t^2}} = \int \frac{2(1-t^2)}{(1+t^2)^2} \cdot \frac{1+t^2}{2}\,dt$$

$$= \int \frac{1-t^2}{1+t^2}\,dt$$

$$= -\int \left(1 + \frac{-2}{t^2+1}\right)dt$$

$$= -t + 2arc\ tg\ t + C$$

$$= -tg\,\frac{\theta}{2} + 2arc\ tg\left(tg\,\frac{\theta}{2}\right) + C$$

13. $\displaystyle\int \frac{dx}{sen\ x + \cos x}$

Fazendo $t = tg\,\dfrac{x}{2}$:

$$= \int \frac{\dfrac{2dt}{1+t^2}}{\dfrac{2t}{1+t^2} + \dfrac{1-t^2}{1+t^2}}$$

$$= \int \frac{\dfrac{2dt}{1+t^2}}{\dfrac{2t+1-t^2}{1+t^2}}$$

$$= -\int \frac{2dt}{t^2 - 2t - 1}$$

$$\frac{1}{t^2-2t-1} = \frac{A}{\left(t-1-\sqrt{2}\right)} + \frac{B}{\left(t-1+\sqrt{2}\right)}$$

$$1 \equiv A\left(t-1+\sqrt{2}\right) + B\left(t-1-\sqrt{2}\right)$$

$$A=\frac{1}{2\sqrt{2}}$$

$$B=\frac{-1}{2\sqrt{2}}$$

$$I=-2\left(\frac{1}{2\sqrt{2}}\int\frac{dt}{t-1-\sqrt{2}}-\frac{1}{2\sqrt{2}}\int\frac{dt}{t-1+\sqrt{2}}\right)+C$$

$$=-\frac{1}{\sqrt{2}}\ln\left|t-1-\sqrt{2}\right|+\frac{1}{\sqrt{2}}\ln\left|t-1+\sqrt{2}\right|+C$$

$$=-\frac{1}{\sqrt{2}}\ln\left|tg\,\frac{x}{2}-1-\sqrt{2}\right|+\frac{1}{\sqrt{2}}\ln\left|tg\,\frac{x}{2}-1+\sqrt{2}\right|+C$$

14. $\int\frac{d\theta}{4-sen\,\theta+\cos\theta}$

Fazendo $t=tg\,\frac{\theta}{2}$:

$$I=\int\frac{\frac{2dt}{1+t^2}}{4-\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$$

$$=\int\frac{2dt}{1+t^2}\cdot\frac{1+t^2}{4+4t^2-2t+1-t^2}$$

$$=\int\frac{2dt}{3t^2-2t+5}$$

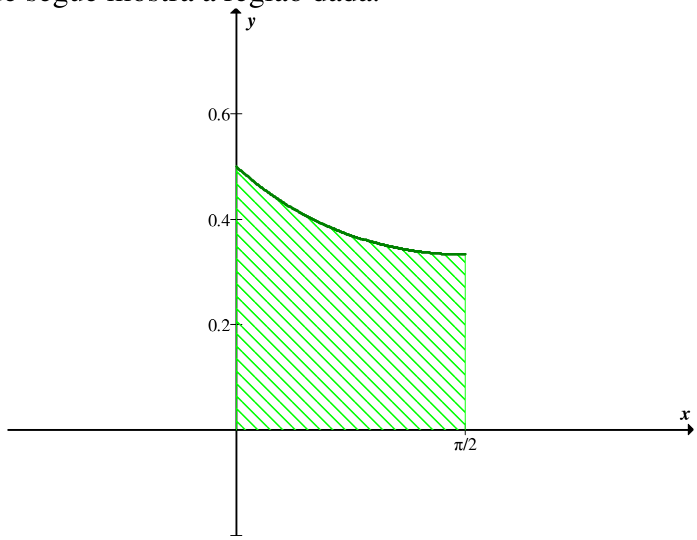
$$=\int\frac{\frac{2}{3}dt}{t^2-\frac{2}{3}t+\frac{5}{3}}=\int\frac{\frac{2}{3}dt}{\left(t-\frac{1}{3}\right)^2+\frac{14}{9}}$$

$$=\frac{2}{3}\,\frac{1}{\frac{\sqrt{14}}{3}}\,arc\,tg\,\frac{t-\frac{1}{3}}{\frac{\sqrt{14}}{3}}+C$$

$$=\frac{2}{\sqrt{14}}\,arc\,tg\,\frac{3t-1}{\sqrt{14}}+C=\frac{2}{\sqrt{14}}\,arc\,tg\,\frac{3tg\,\frac{\theta}{2}-1}{\sqrt{14}}+C$$

15. Calcular a área sob a curva $y = \frac{1}{2 + sen\ x}$, de $x = 0$ a $x = \frac{\pi}{2}$.

A figura que segue mostra a região dada.



$$A = \int_0^{\pi/2} \frac{dx}{2 + sen\ x}$$

$$I = \int \frac{dx}{2 + sen\ x}$$

Fazendo $t = tg\frac{x}{2}$:

$$I = \int \frac{\frac{2dt}{1+t^2}}{2 + \frac{2t}{1+t^2}} = \frac{\frac{2dt}{1+t^2}}{\frac{2+2t^2+2t}{1+t^2}}$$

$$= \int \frac{2dt}{2t^2+2t+2} = \int \frac{dt}{t^2+t+1}$$

$$= \int \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \frac{1}{\frac{\sqrt{3}}{2}} \, arc\ tg\ \frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C$$

$$= \frac{2}{\sqrt{3}} \, arc\ tg\ \frac{2t+1}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}} \, arc\ tg\ \frac{2tg\,\frac{x}{2}+1}{\sqrt{3}} + C$$

$$A = \frac{2}{\sqrt{3}} \, arc\ tg\ \frac{2tg\,\frac{x}{2}+1}{\sqrt{3}} \Bigg|_0^{\pi/2}$$

$$= \frac{2}{\sqrt{3}} \, arc\ tg\ \frac{2tg\,\frac{\pi}{4}+1}{\sqrt{3}} - \frac{2}{\sqrt{3}} \, arc\ tg\ \frac{2tg\,0+1}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} \, arc\ tg\ \frac{3}{\sqrt{3}} - \frac{2}{\sqrt{3}} \, arc\ tg\ \frac{1}{\sqrt{3}}$$

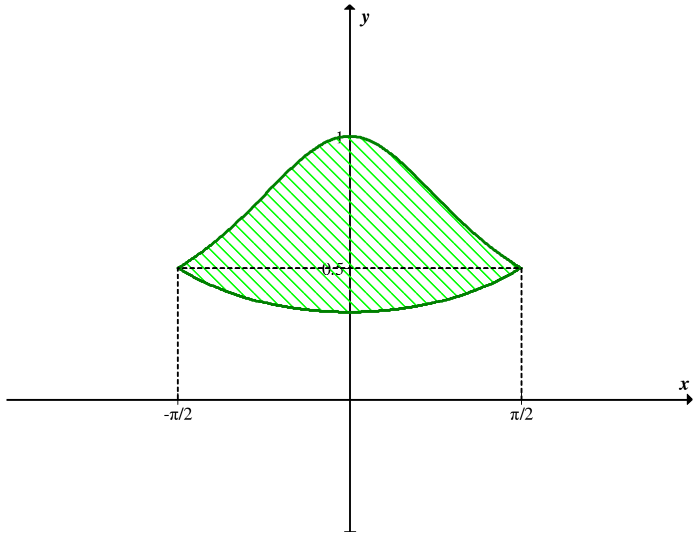
$$= \frac{2}{\sqrt{3}} \cdot \frac{\pi}{3} - \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6}$$

$$= \frac{2}{\sqrt{3}} \left(\frac{\pi}{3} - \frac{\pi}{6} \right)$$

$$= \frac{2}{\sqrt{3}} \frac{2\pi - \pi}{6} = \frac{\pi\sqrt{3}}{9} u.a.$$

16. Calcule a área limitada pelas curvas $y = \frac{1}{2+\cos x}$ e $y = \frac{1}{2-\cos x}$ entre $-\pi/2$ e $\pi/2$

A figura que segue mostra a região dada.



$$\int_{-\pi/2}^{\pi/2}\left(\frac{1}{2-\cos x}-\frac{1}{2+\cos x}\right)dx$$

$$I=\int\left(\frac{1}{2-\cos x}-\frac{1}{2+\cos x}\right)dx$$

Fazendo $t=tg\,\frac{x}{2}$:

$$I=\int\left(\frac{1}{2-\frac{1-t^2}{1+t^2}}-\frac{1}{2+\frac{1-t^2}{1+t^2}}\right)\frac{2dt}{1+t^2}$$

$$=\int\frac{2dt}{1+t^2}.\frac{1+t^2}{3t^2+1}-\int\frac{2dt}{1+t^2}.\frac{1+t^2}{t^2+3}$$

$$=\frac{2}{3}\int\frac{dt}{t^2+1/3}-2\int\frac{dt}{t^2+3}$$

$$=\frac{2}{\sqrt{3}}arc\;tg\sqrt{3}\;t+(-2)\frac{1}{\sqrt{3}}arc\;tg\,\frac{t}{\sqrt{3}}+C$$

$$=\frac{2}{\sqrt{3}}arc\;tg\left(\sqrt{3}\;tg\;\frac{x}{2}\right)-\frac{2}{\sqrt{3}}arc\;tg\left(\frac{1}{\sqrt{3}}\;tg\;\frac{x}{2}\right)+C$$

Portanto,

$$A = \frac{2}{\sqrt{3}} arc\ tg\left(\sqrt{3}\ tg\ \frac{x}{2}\right) - \frac{2}{\sqrt{3}}\ arc\ tg\left(\frac{1}{\sqrt{3}}\ tg\ \frac{x}{2}\right)\Bigg|_{-\pi/2}^{\pi/2}$$

$$= \frac{2}{\sqrt{3}} arc\ tg\ \sqrt{3} - \frac{2}{\sqrt{3}} arc\ tg\ \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}} arc\ tg\ \left(-\sqrt{3}\right) + \frac{2}{\sqrt{3}} arc\ tg\ \frac{-1}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} \cdot \frac{\pi}{3} - \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6} + \frac{2}{\sqrt{3}} \cdot \frac{\pi}{3} - \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6}$$

$$= \frac{2\,\pi}{3\sqrt{3}} = \frac{2\,\sqrt{3}}{9}\pi\ u.a.$$

Nos exercícios de 17 a 33 calcular a integral indefinida:

17. $\displaystyle\int \frac{dx}{x\sqrt{5x - x^2 - 6}}$

Fazendo

$$\sqrt{5x - x^2 - 6} = (x-2)\,.\,t$$

$$5x - x^2 - 6 = (x-2)^2\ .\,t^2$$

$$-(x-2)(x-3) = (x-2)^2\ .\,t^2$$

$$t^2 = \frac{-(x-2)(x-3)}{(x-2)^2}$$

$$t^2 = \frac{3-x}{x-2}$$

$$t = \sqrt{\frac{3-x}{x-2}}$$

temos que:

$$x = \frac{2t^2 + 3}{1 + t^2}$$

$$dx = \frac{-2t}{\left(1 + t^2\right)^2} dt$$

$$\sqrt{5x - x^2 - 6} = (x-2)\,.\,t$$

$$= \frac{t}{1 + t^2}$$

Assim,

$$I = \int \frac{\dfrac{-2t}{\left(1+t^2\right)^2}\,dt}{\dfrac{2t^2+3}{1+t^2}\cdot\dfrac{t}{1+t^2}}$$

$$= \frac{-2}{\sqrt{2}}\int \frac{\sqrt{2}\;dt}{2t^2+3}$$

$$= \frac{-2}{\sqrt{6}} arc\ tg\ \frac{\sqrt{2}}{\sqrt{3}}t + C$$

$$= -\sqrt{\frac{2}{3}} arc\ tg\ \sqrt{\frac{2\,(3-x)}{3\,(x-2)}} + C$$

18. $\displaystyle\int \frac{dx}{(x+4)\sqrt{x^2+4x+9}}$

Fazendo:

$$\sqrt{x^2+4x+9} = x+t$$

$$x = \frac{t^2-9}{4-2t}$$

$$dx = \frac{(4-2t)\,(2t)-\left(t^2-9\right)(-2)}{(4-2t)^2}\,dt$$

$$= \frac{8t-4t^2+2t^2-18}{(4-2t)^2}\,dt$$

$$= \frac{-2t^2+8t-18}{(4-2t)^2}\,dt$$

Temos:

$$I = \int \frac{\dfrac{-2t^2+8t-18}{(4-2t)^2}\,dt}{\dfrac{t^2-9+16-8t}{4-2t}\,\dfrac{t^2-9+4t-t^2}{4-2t}}$$

$$= \int \frac{-2t^2+8t-18}{(t^2-8t+7)(-t^2+4t-9)}\,dt$$

$$= 2\int \frac{dt}{t^2-8t+7}$$

$$= 2\int \left(\frac{1/6}{t-7} - \frac{1/6}{t-1} \right) dt$$

$$= 2\left(\frac{1}{6}\ln|t-7| - \frac{1}{6}\ln|t-1| \right) + C$$

$$= \frac{2}{6}\ln\left|\frac{t-7}{t-1}\right| + C$$

$$= \frac{1}{3}\ln\left|\frac{\sqrt{x^2+4x+9}-x-7}{\sqrt{x^2+4x+9-x-1}}\right| + C$$

19. $\displaystyle\int \frac{dx}{x\sqrt{4x^2+x-3}}$

Fazendo:

$$\sqrt{4x^2+x-3} = 2x+t$$

$$4x^2+x-3 = 4x^2+4xt+t^2$$

$$x = \frac{t^2+3}{1-4t}$$

$$dx = \frac{2t-4t^2+12}{(1-4t)^2}\,dt$$

Temos:

$$I = \int \frac{\dfrac{2t - 4t^2 + 12}{(1-4t^2)}}{\dfrac{t^2+3}{1-4t}\left(2 \cdot \dfrac{t^2+3}{1-4t} + t\right)} dt$$

$$= \int \frac{2}{t^2+3} dt$$

$$= \frac{2}{\sqrt{3}} arc\ tg \left(\frac{\sqrt{4x^2 + x - 3} - 2x}{\sqrt{3}} \right) + C$$

20. $\int \frac{dx}{\sqrt{1+x+x^2}}$

Fazendo:

$$\sqrt{1+x+x^2} = x + t$$

$$1 + x + x^2 = x^2 + 2xt + t^2$$

$$x - 2x\,t = t^2 - 1$$

$$x(1-2t) = t^2 - 1$$

$$x = \frac{t^2 - 1}{1-2t}$$

$$dx = \frac{(1-2t)\,2t - (t^2-1)(-2)}{(1-2t)^2} dt$$

$$= \frac{2t - 4t^2 + 2t^2 - 2}{(1-2t)^2} dt$$

$$= \frac{-2t^2 + 2t - 2}{(1-2t)^2} dt$$

$$\sqrt{1+x+x^2} = \frac{-t^2 + t - 1}{1-2t}$$

Temos:

$$I = \int \frac{-2t^2 + 2t - 2}{(1-2t)^2} \cdot \frac{1-2t}{-t^2+t-1} dt$$

$$= \int \frac{2\,dt}{1-2t} = -\ln|1-2t| + C$$

$$= -\ln\left|1 - 2\sqrt{1+x+x^2} + 2x\right| + C$$

21. $\int \frac{dx}{x\sqrt{2+x-x^2}}$

Fazendo:

$$\sqrt{2+x-x^2} = xt+\sqrt{2}$$

$$2+x-x^2 = x^2t^2+2\sqrt{2}xt+2$$

$$1-x = xt^2+2\sqrt{2}t$$

$$x = \frac{1-2\sqrt{2}\,t}{1+t^2}$$

$$dx = \frac{2\sqrt{2}t^2-2t-2\sqrt{2}}{\left(1+t^2\right)^2}$$

$$= \frac{-\sqrt{2}\,t^2+t+\sqrt{2}}{1+t^2}(-2)dt$$

Temos:

$$I = \int \frac{\frac{-2-\sqrt{2}\,t^2+t+\sqrt{2}}{\left(1+t^2\right)^2}}{\frac{1-2\sqrt{2}\,t}{1+t^2}\cdot\frac{-\sqrt{2}\,t^2+t+\sqrt{2}}{1+t^2}}\,dt$$

$$= \int \frac{-2dt}{1-2\sqrt{2}t}$$

$$= \frac{1}{\sqrt{2}}\ln\left|1-2\sqrt{2}\,t\right|+C$$

$$= \frac{1}{\sqrt{2}}\ln\left|1-2\sqrt{2}\,\frac{\left(\sqrt{2+x-x^2}-\sqrt{2}\right)}{x}\right|+C$$

22. $\int \frac{x+1}{\left(2x+x^2\right)\sqrt{2x+x^2}}\,dx$

Fazendo:

$$\sqrt{2x+x^2} = x+t$$

$$x = \frac{t^2}{2-2t}$$

$$dx = \frac{-2t^2+4t}{\left(2-2t\right)^2}dt$$

$$\sqrt{2x+x^2} = \frac{-t^2+2t}{2-2t}$$

$$x+1=\frac{t^2-2t+2}{2-2t}$$

$$2x+x^2=(x+t)^2=\frac{(-t^2+2t)^2}{(2-2t)^2}$$

Temos:

$$I=\int\frac{\dfrac{t^2-2t+2}{2-2t}\cdot\dfrac{-2t^2+4t^2}{(2-2t)^2}}{\dfrac{(-t^2+2t)^2}{(2-2t)^2}\cdot\dfrac{-t^2+2t}{2-2t}}dt$$

$$=\int\frac{2(t^2-2t+2)}{(-t^2+2t)^2}\,dt$$

$$=2\int\frac{1/2}{t^2}dt+2\int\frac{1/2}{(t-2)^2}\,dt$$

$$=\frac{-1}{t}-\frac{1}{t-2}+C$$

$$=\frac{-1}{\sqrt{2x+x^2}-x}-\frac{1}{\sqrt{2x+x^2}-x-2}+C$$

23. $\displaystyle\int\frac{dx}{(x-1)\sqrt{x^2-2x-3}}$

Fazendo:

$$\sqrt{x^2-2x-3}=x+t$$

$$x=\frac{-3-t^2}{2t+2}$$

$$dx=\frac{-2t^2-4t+6}{(2t+2)^2}dt$$

$$\sqrt{x^2-2x-3}=\frac{-3-t^2}{2t+2}+t=\frac{t^2+2t-3}{2t+2}$$

$$x-1=\frac{-3-t^2}{2t+2}-1=\frac{-t^2-2t-5}{2t+2}$$

Temos:

$$I = \int \frac{\dfrac{-2t^2-4t+6}{(2t+2)^2}dt}{\dfrac{-t^2-2t-5}{2t+2}\,\dfrac{t^2+2t-3}{2t+2}}$$

$$= \int \frac{-2\,dt}{-t^2-2t-5}$$

$$= \int \frac{2\,dt}{+t^2+2t+5}$$

$$= 2\,.\frac{1}{2}\,arc\ tg\ \frac{t+1}{2} + C$$

$$= arc\ tg\ \frac{\sqrt{x^2-2x-3}-x+1}{2} + C$$

24. $\displaystyle\int \frac{1-\sqrt{1+x+x^2}}{2x^2\,\sqrt{1+x+x^2}}\,dx$

Fazendo:

$$\sqrt{1+x+x^2} = x+t$$

$$x = \frac{t^2-1}{1-2t}$$

$$dx = \frac{-2t^2+2t-2}{(1-2t)^2}dt$$

$$\sqrt{1+x+x^2} = \frac{-t^2+t-1}{1-2t}$$

$$x^2 = \frac{(t^2-1)^2}{(1-2t)^2}$$

$$1-\sqrt{1+x+x^2} = 1-x-t = \frac{t^2-3t+2}{1-2t}$$

Temos:

$$I=\frac{1}{2}\int\frac{\dfrac{t^2-3t+2}{1-2t}\cdot\dfrac{-2t^2+2t-2}{(1-2t)^2}}{\dfrac{(t^2-1)^2}{(1-2t)^2}\;\dfrac{-t^2+t-1}{1-2t}}\,dt$$

$$=\frac{1}{2}\int\frac{2(t^2-3t+2)}{(t^2-1)^2}dt$$

$$=\int\frac{(t-1)(t-2)}{(t-1)^2(t+1)^2}\,dt$$

$$=\int\frac{t-2}{(t-1)^2(t+1)^2}\,dt$$

$$=\int\left[\frac{-1/4}{t-1}+\frac{1/4}{t+1}+\frac{3/2}{(t+1)^2}\right]dt$$

$$=\frac{-1}{4}\ln(t-1)+\frac{1}{4}\ln(t+1)+\frac{3}{2}\,\frac{(t+1)^{-1}}{-1}+C$$

$$=\frac{1}{4}\ln\left|\frac{t+1}{t-1}\right|-\frac{3}{2\,(t+1)}+C$$

$$=\frac{1}{4}\ln\left|\frac{\sqrt{1+x+x^2}-x+1}{\sqrt{1+x+x^2}-x-1}\right|-\frac{3}{2\left(\sqrt{1+x+x^2}-x+1\right)}+C$$

25. $\displaystyle\int\frac{dx}{\sqrt{x^2+3x+2}}$

Fazendo:

$$\sqrt{x^2+3x+2}=(x+1)t$$

$$(x+1)(x+2)=(x+1)^2t^2$$

$$x+2=(x+1)\,t^2$$

$$x-x\,t^2=t^2-2$$

$$x(1-t^2)=t^2-2$$

$$x=\frac{t^2-2}{1-t^2}$$

$$dx = \frac{(1-t^2)2t - (t^2-2)(-2t)}{(1-t^2)^2}\, dt$$

$$= \frac{2t - 2t^3 + 2t^3 - 4t}{(1-t^2)^2}\, dt$$

$$= \frac{-2t}{(1-t^2)^2}\, dt$$

$$\sqrt{x^2+3x+2} = \left(\frac{t^2-2}{1-t^2} + 1\right)t$$

$$= \frac{(t^2 - 2 + 1 - t^2)t}{1-t^2}$$

$$= \frac{-t}{1-t^2}$$

Temos:

$$I = \int \frac{\dfrac{-2t}{(1-t^2)^2}}{\dfrac{-t}{1-t^2}} dt = \int \frac{2t}{1-t^2} \cdot \frac{1}{t} dt$$

$$= \int \frac{2}{1-t^2} dt$$

$$= \int \left(\frac{1}{1-t} + \frac{1}{1+t}\right) dt$$

$$= \ln|1-t| + \ln|1+t| + C$$

$$= \ln\left|\frac{1+t}{1-t}\right| + C$$

$$= \ln\left|\frac{1 + \dfrac{\sqrt{x^2+3x+2}}{x+1}}{1 - \dfrac{\sqrt{x^2+3x+2}}{x+1}}\right| + C$$

$$= \ln\left|\frac{x+1+\sqrt{x^2+3x+2}}{x+1-\sqrt{x^2+3x+2}}\right| + C$$

26. $\displaystyle\int \frac{dx}{\sqrt{x^2+2x-3}}$

Fazendo:

$$\sqrt{x^2+2x-3} = x+t$$
$$x^2+2x-3 = x^2+2xt+t^2$$
$$2x-2xt = t^2+3$$
$$x = \frac{t^2+3}{2-2t}$$

$$dx = \frac{(2-2t)\,2t-(t^2+3)(-2)}{(2-2t)^2}\,dt$$
$$= \frac{4t-4t^2+2t^2+6}{(2-2t)^2}\,dt$$
$$= \frac{-2t^2+4t+6}{(2-2t)^2}\,dt$$

$$\sqrt{x^2+2x-3} = \frac{t^2+3}{2-2t}+t$$
$$= \frac{t^2+3+2t-2t^2}{2-2t}$$
$$= \frac{-t^2+2t+3}{2-2t}$$

Temos:

$$I = \int \frac{\dfrac{-2t^2+4t+6}{(2-2t)^2}}{\dfrac{-t^2+2t+3}{2-2t}}\,dt$$
$$= \int \frac{2}{2-2t}\,dt$$
$$= -\ln|2-2t|+C$$
$$= -\ln\left|2-2\left(\sqrt{2+2x-3}-x\right)\right|+C$$
$$= -\ln\left|\sqrt{2+2x-3}-x-1\right|+C$$

27. $\displaystyle\int \frac{dx}{(2x+1)\sqrt{4x^2+4x}}$

$$I = \frac{1}{2}\int \frac{dx}{(2x+1)\sqrt{x^2+x}}$$

Fazendo:

$$\sqrt{x^2+x} = x+t$$
$$x^2+x = x^2+2xt+t^2$$
$$x-2xt = t^2$$
$$x = \frac{t^2}{1-2t}$$

$$dx = \frac{(1-2t)2t - t^2(-2)}{(1-2t)^2}\,dt$$
$$= \frac{2t-4t^2+2t^2}{(1-2t)^2}\,dt$$
$$= \frac{-2t^2+2t}{(1-2t)^2}\,dt$$

$$\sqrt{x^2+x} = \frac{t^2}{1-2t}+t$$
$$= \frac{t^2+t-2t^2}{1-2t}$$
$$= \frac{-t^2+t}{1-2t}$$

$$2x+1 = 2\,.\,\frac{t^2}{1-2t}+1$$
$$= \frac{2t^2+1-2t}{1-2t}$$

Temos:

$$I = \frac{1}{2}\int \frac{\dfrac{-2t^2+2t}{(1-2t)^2}}{\dfrac{2t^2-2t+1}{1-2t}\,.\,\dfrac{-t^2+t}{1-2t}}\,dt$$
$$= \int \frac{dt}{2t^2-2t+1}$$
$$= \int \frac{\frac{1}{2}\,dt}{t^2-t+\frac{1}{2}}$$
$$= \frac{1}{2}\int \frac{dt}{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}$$

$$= \frac{1}{2} \cdot \frac{1}{\frac{1}{2}} \, arc\ tg \, \frac{t - \frac{1}{2}}{\frac{1}{2}} + C$$

$$= arc\ tg \, \frac{2t - 1}{1} + C$$

$$= arc\ tg \left(2\left(\sqrt{x^2 + x} - x\right) - 1\right) + C$$

$$= arc\ tg \left(2\sqrt{x^2 + x} - 2x - 1\right) + C$$

28. $\displaystyle\int \frac{dx}{\sqrt{9x^2 + 12x + 5}}$

Fazendo:

$$\sqrt{9x^2 + 12x + 5} = 3x + t$$

$$9x^2 + 12x + 5 = 9x^2 + 6xt + t^2$$

$$12x - 6xt = t^2 - 5$$

$$x = \frac{t^2 - 5}{12 - 6t}$$

$$dx = \frac{(12 - 6t) \, . \, 2t - \left(t^2 - 5\right)(-6)}{(12 - 6t)^2} dt$$

$$= \frac{24t - 12t^2 + 6t^2 - 30}{(12 - 6t)^2} dt$$

$$= \frac{-6t^2 + 24t - 30}{(12 - 6t)^2} dt$$

$$\sqrt{9x^2 + 12x + 5} = 3 \, . \, \frac{t^2 - 5}{12 - 6t} + t$$

$$= \frac{3t^2 - 15 + 12t - 6t^2}{12 - 6t}$$

$$= \frac{-3t^2 + 12t - 15}{12 - 6t}$$

Temos:

$$I=\int\frac{\dfrac{-6t^2+24t-30}{(12-6t)^2}}{\dfrac{-3t^2+12t-15}{12-6t}}\,dt$$

$$=\int\frac{2}{12-6t}\,dt$$

$$=-\frac{1}{3}\int\frac{1}{t-2}\,dt$$

$$=\frac{-1}{3}\ln|t-2|+C$$

$$=-\frac{1}{3}\ln\left|2-\sqrt{9x^2+12+5}+3x\right|+C$$

29. $\displaystyle\int\frac{dx}{(2x-1)\sqrt{x^2-x+5/4}}$

Fazendo:

$$\sqrt{x^2-x+5/4}=x+t$$

$$x^2-x+5/4=x^2+2xt+t^2$$

$$2xt+x=\frac{5}{4}-t^2$$

$$x=\frac{\frac{5}{4}-t^2}{2t+1}$$

$$dx=\frac{(2t+1)(-2t)-\left(\frac{5}{4}-t^2\right)^2.2}{(2t+1)^2}\,dt$$

$$=\frac{-4t^2-2t-\frac{5}{2}+2t^2}{(2t+1)^2}\,dt$$

$$=\frac{-2t^2-2t-5/2}{(2t+1)^2}\,dt$$

$$2x-1=2\,.\,\frac{5/4-t^2}{2t+1}-1$$

$$=\frac{5/2-2t^2-2t-1}{2t-1}=\frac{-2t^2-2t+3/2}{2t-1}$$

Temos:

$$I=\int\frac{\dfrac{-2t^2-2t-5/2}{(2t+1)^2}}{\dfrac{-2t^2-2t+3/2}{2t-1}\,.\,\dfrac{5/4-t^2+2t^2+t}{2t+1}}\,dt$$

$$=\int\frac{-2t^2-2t-5/2}{\left(-2t^2-2t+3/2\right)\left(t^2+t+5/4\right)}dt$$

$$=\int\frac{-2dt}{-2t^2-2t+3/2}$$

$$=\int\frac{dt}{t^2+t-3/4}=\int\left(\frac{A}{t-1/2}+\frac{B}{t+3/2}\right)dt$$

$$=\int\frac{1/2}{t-1/2}\,dt+\int\frac{-1/2}{t+3/2}\,dt$$

$$=\frac{1}{2}\ln\left|t-\frac{1}{2}\right|-\frac{1}{2}\ln\left|t+\frac{3}{2}\right|+C$$

$$=\frac{1}{2}\ln\left|\frac{t-1/2}{t+3/2}\right|+C=\frac{1}{2}\ln\left|\frac{2t-1}{2t+3}\right|+C$$

$$=\frac{1}{2}\ln\left|\frac{2\sqrt{x^2-x+5/4}-2x-1}{2\sqrt{x^2-x+5/4}-2x+3}\right|+C$$

30. $\displaystyle\int\frac{dx}{x\sqrt{x^2+x-3}}$

Fazendo:

$$\sqrt{x^2+x-3}=x+t$$

$$x^2+x-3=x^2+2xt+t^2$$

$$x-2xt=t^2+3$$

$$x=\frac{t^2+3}{1-2t}$$

$$dx = \frac{(1-2t)\,2t - (t^2+3)(-2)}{(1-2t)^2}\,dt$$
$$= \frac{2t - 4t^2 + 2t^2 + 6}{(1-2t)^2}\,dt$$
$$= \frac{-2t^2 + 2t + 6}{(1-2t)^2}\,dt$$

$$\sqrt{x^2+x-3} = \frac{t^2+3}{1-2t} + t$$
$$= \frac{t^2+3+t-2t^2}{1-2t}$$
$$= \frac{-t^2+t+3}{1-2t}$$

Temos:

$$I = \int \frac{\dfrac{-2t^2+2t+6}{(1-2t)^2}}{\dfrac{t^2+3}{1-2t} \cdot \dfrac{-t^2+t+3}{1-2t}}dt$$
$$= \int \frac{-2t^2+2t+6}{(t^2+3)(-t^2+t+3)}dt$$
$$= \int \frac{2dt}{t^2+3}$$
$$= 2.\frac{1}{\sqrt{3}}\,arc\ tg\ \frac{t}{\sqrt{3}} + C$$
$$= \frac{2}{\sqrt{3}}\,arc\ tg\ \frac{\sqrt{x^2+x-3}-x}{\sqrt{3}} + C$$

31. $\displaystyle\int \frac{dx}{x\sqrt{x^2-4x-4}}$

Fazendo:

$$\sqrt{x^2-4x-4} = x + t$$
$$x^2-4x-4 = x^2+2xt+t^2$$
$$2xt+4x = -4-t^2$$
$$x = \frac{-4-t^2}{2t+4}$$

$$dx = \frac{(2t+4)(-2t) - (-4-t^2)\,2}{(2t+4)^2}\,dt$$

$$= \frac{-4t^2 - 8t + 8 + 2t^2}{(2t+4)^2}\,dt$$

$$= \frac{-2t^2 - 8t + 8}{(2t+4)^2}\,dt$$

$$\sqrt{x^2 - 4x - 4} = \frac{-4-t^2}{2t+4} + t$$

$$= \frac{-4 - t^2 + 2t^2 + 4t}{2t+4}$$

$$= \frac{t^2 + 4t - 4}{2t+4}$$

Temos:

$$I = \int \frac{\dfrac{-2t^2 - 8t + 8}{(2t+4)^2}}{\dfrac{-4-t^2}{2t+4} \cdot \dfrac{t^2+4t-4}{2t+4}}\,dt$$

$$= \int \frac{-2dt}{-4-t^2}$$

$$= \frac{2}{2}\, arc\ tg\ \frac{t}{2} + C$$

$$= arc\ tg\ \frac{\sqrt{x^2 - 4x - 4} - x}{2} + C$$

32. $\displaystyle\int \frac{x+3}{\sqrt{x^2+2x}}\,dx$

Fazendo:

$$\sqrt{x^2+2x} = x + t$$

$$x^2 + 2x = x^2 + 2xt + t^2$$

$$2x - 2xt = t^2$$

$$x = \frac{t^2}{2-2t}$$

$$dx = \frac{-2t^2 + 4t}{(2-2t)^2}\,dt$$

$$\sqrt{x^2+2x}=\frac{-t^2+2t}{2-2t}$$

$$x+3=\frac{t^2+6-6t}{2-2t}$$

Temos

$$I=\int\frac{\frac{t^2-6t+6}{2-2t}\cdot\frac{-2t^2+4t}{(2-2t)^2}}{\frac{-t^2+2t}{2-2t}}dt$$

$$=\int\frac{2(t^2-6t+6)}{(2-2t)^2}\,dt$$

$$=2\int\left(\frac{1}{4}+\frac{-4t+5}{4t^2-8t+4}\right)dt$$

$$=2\int\left(\frac{1}{4}+\frac{-4t+5}{4(t-1)^2}\right)dt$$

$$=\frac{1}{2}t+\frac{1}{2}\int\frac{-4t+5}{(t-1)^2}dt$$

$$=\frac{1}{2}t-2\ln|t-1|+\frac{1}{2}\,\frac{(t-1)^{-1}}{-1}+C$$

$$=\frac{1}{2}\left(\sqrt{x^2+2x}-x\right)-2\ln\left|\sqrt{x^2+2x}-x-1\right|-\frac{1}{2}\,\frac{1}{\sqrt{x^2+2x}-x-1}+C$$

33. $\displaystyle\int\frac{dx}{\sqrt{3-2x-x^2}}$

Fazendo:

$$\sqrt{3-2x-x^2}=xt+\sqrt{3}$$

$$x=\frac{-2-2\sqrt{3}\,t}{t^2+1}$$

$$dx=\frac{2\sqrt{3}\,t^2+4t-2\sqrt{3}}{(t^2+1)^2}\,dt$$

$$\sqrt{3-2x-x^2}=\frac{-\sqrt{3}t^2-2t+\sqrt{3}}{t^2+1}$$

Temos:

$$I = \int \frac{\dfrac{2\sqrt{3}\, t^2 + 4t - 2\sqrt{3}}{\left(t^2+1\right)^2}}{\dfrac{-\sqrt{3}t^2 - 2t + \sqrt{3}}{t^2+1}}\, dt$$

$$= \int \frac{-2}{t^2+1} dt$$

$$= -2 arc\ tg\ t + C$$

$$= -2 arc\ tg\ \frac{\sqrt{3-2x-x^2} - \sqrt{3}}{x} + C$$