7.6 – EXERCÍCIOS – pg. 325

Nos exercícios de 1 a 23, calcular a integral indefinida.

1. $\int \frac{2x^3}{x^2 + x}\, dx$

$$= 2\int \frac{x^3}{x(x+1)} dx = 2\int \frac{x^2 dx}{x+1}$$

$$= 2\int \left(x - 1 + \frac{1}{x+1} \right) dx$$

$$= 2\left(\frac{x^2}{2} - x + \ln|x+1| \right) + C$$

$$= x^2 - 2x + 2\ln|x+1| + C$$

2. $\int \frac{2x+1}{2x^2 + 3x - 2} dx$

$$= \int \frac{\frac{1}{2}(2x+1)}{x^2 + \frac{3}{2}x - 1} dx = \frac{1}{2}\int \frac{2x+1}{\left(x - \frac{1}{2}\right)(x+2)} dx$$

$$= \frac{1}{2}\int \left(\frac{A}{x - \frac{1}{2}} + \frac{B}{x+2} \right) dx$$

$$= \frac{1}{2}\int \left(\frac{4/5}{x - \frac{1}{2}} + \frac{6/5}{x+2} \right) dx$$

$$= \frac{4}{10} \ln\left| x - \frac{1}{2} \right| + \frac{3}{5} \ln|x+2| + C$$

$$= \frac{2}{5} \ln\left| x - \frac{1}{2} \right| + \frac{3}{5} \ln|x+2| + C$$

3. $\int \frac{x-1}{x^3 + x^2 - 4x - 4} dx$

$$= \int \frac{x-1}{(x-2)(x+1)(x+2)}\,dx$$

$$= \int \left(\frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x+2} \right) dx$$

Cálculo de A, B e C

$$(x-1) \equiv A(x+1)(x+2) + B(x-2)(x+2) + C(x-2)(x+1)$$

$$x = 2 \quad \rightarrow \quad A\,.\,3\,.\,4 = 1$$

$$A = \tfrac{1}{12}$$

$$x = -2 \quad \rightarrow \quad C\,.\,(-4)\,.\,(-1) = -3$$

$$C = -\tfrac{3}{4}$$

$$x = -1 \quad \rightarrow \quad B\,.\,(-3)\,.\,(1) = -2$$

$$B = \tfrac{2}{3}$$

Assim,

$$I = \int \left(\frac{\tfrac{1}{12}}{x-2} + \frac{\tfrac{2}{3}}{x+1} + \frac{-\tfrac{3}{4}}{x+2} \right) dx$$

$$= \frac{1}{12} \ln|x-2| + \frac{2}{3} \ln|x+1| - \frac{3}{4} \ln|x+2| + C$$

4. $\displaystyle\int \frac{3x^2}{2x^3 - x^2 - 2x + 1}\,dx$

$$= \int \frac{\frac{3}{2} x^2 dx}{x^3 - \frac{1}{2}x^2 - x + \frac{1}{2}}$$

$$= \frac{3}{2} \int \left(\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x - \frac{1}{2}} \right) dx$$

Cálculo de A, B e C

$$x^2 \equiv A(x+1)\left(x - \frac{1}{2}\right) + B(x-1)\left(x - \frac{1}{2}\right) + C\,(x-1)(x+1)$$

$$x = 1 \quad \rightarrow \quad A \,.\, 2 \,.\, \frac{1}{2} = 1$$
$$A = 1$$

$$x = -1 \quad \rightarrow \quad B \,.\, (-2) \,.\, \left(-\frac{3}{2}\right) = 1$$
$$B = \frac{1}{3}$$

$$x = \frac{1}{2} \quad \rightarrow \quad C \,.\, \left(-\frac{1}{2}\right) . \left(\frac{3}{2}\right) = \frac{1}{4}$$
$$C = -\frac{1}{3}$$

Assim,

$$I = \frac{3}{2} \int \left(\frac{1}{x-1} + \frac{1/3}{x+1} - \frac{1/3}{x - 1/2} \right) dx$$
$$= \frac{3}{2} \ln |x-1| + \frac{1}{2} \ln |x+1| - \frac{1}{2} \ln \left| x - \frac{1}{2} \right| + C$$

5. $\displaystyle \int \frac{x^2 + 5x + 4}{x^2 - 2x + 1}\, dx$

$$= \int dx + \int \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} \right) dx$$
$$= x + \int \left(\frac{7}{x-1} + \frac{10}{(x-1)^2} \right) dx$$
$$= x + 7 \ln |x-1| - \frac{10}{x-1} + C$$

6. $\displaystyle \int \frac{x-1}{(x-2)^2\,(x-3)^2}\, dx$

$$\int \left(\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x-3} + \frac{D}{(x-3)^2} \right) dx$$

Cálculo de A, B , C e D

$$x-1 \equiv A(x-2)(x-3)^2 + B(x-3)^2 + C(x-2)^2(x-3) + D\,(x-2)^2$$
$$x-1 \equiv A\left(x^3-8x^2+21x-18\right) + B\left(x^2-6x+9\right) + C\left(x^3-7x^2+16x-12\right) + D\left(x^2-4x+4\right)$$

$$\begin{cases} A+C=0 \\ -8A+B-7C+D=0 \\ 21A-6B+16C-4D=1 \\ -18A+9B-12C+4D=-1 \end{cases}$$

$A=3$; $B=1$; $C=-3$ e $D=2$

Assim,

$$I = \int\left(\frac{3}{x-2} + \frac{1}{(x-2)^2} + \frac{-3}{x-3} + \frac{2}{(x-3)^2}\right)dx$$
$$= 3\ln|x-2| + \frac{(x-2)^{-1}}{-1} - 3\ln|x-3| + 2\,.\,\frac{(x-3)^{-1}}{-1} + C$$
$$= 3\ln|x-2| - \frac{1}{x-2} - 3\ln|x-3| - \frac{2}{x-3} + C$$
$$= 3\ln\left|\frac{x-2}{x-3}\right| - \frac{1}{x-2} - \frac{2}{x-3} + C$$

7. $\displaystyle\int \frac{\left(x^2+1\right)}{x^4-7x^3+18x^2-20x+8}\,dx$

$$I = \int \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}\,dx$$

Cálculo de A, B, C e D

$$x^2+1 \equiv A\,(x-2)^3 + B(x-1)(x-2)^2 + C\,(x-1)(x-2) + D\,(x-1)$$

$A=-2$; $B=2$; $C=-1$; $D=5$

Assim,

$$I=-2\ln|x-1|+2\ln|x-2|+\frac{1}{x-2}-\frac{5}{2(x-2)^2}+C$$

$$=-2\ln|x-1|+2\ln|x-2|+\frac{1}{x-2}-\frac{5}{2(x-2)^2}+C$$

$$=2\ln\frac{|x-2|}{|x-1|}+\frac{1}{x-2}-\frac{5}{2(x-2)^2}+C$$

$$=\ln\left(\frac{x-2}{x-1}\right)^2+\frac{1}{x-2}-\frac{5}{2(x-2)^2}+C$$

8. $\displaystyle\int\frac{dx}{x^3-4x^2}$

$$=\int\left(\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-4}\right)dx$$

Cálculo de A, B e C

$1\equiv Ax(x-4)+B(x-4)+Cx^2$

$x=0\quad\rightarrow\quad 1=-4B\quad\therefore\quad B={}^{-1}\!/\!_4$

$x=4\quad\rightarrow\quad 1=16C\quad\therefore\quad C={}^{1}\!/\!_{16}$

$A+C=0\quad\therefore\quad A=-{}^{1}\!/\!_{16}$

Assim,

$$I=\int\left(\frac{\frac{-1}{16}}{x}+\frac{\frac{-1}{4}}{x^2}+\frac{\frac{1}{16}}{x-4}\right)dx$$

$$=-\frac{1}{16}\ln|x|-\frac{1}{4}\frac{x^{-1}}{-1}+\frac{1}{16}\ln|x-4|+C$$

$$=-\frac{1}{16}\ln|x|+\frac{1}{4x}+\frac{1}{16}\ln|x-4|+C$$

$$=\frac{1}{16}\ln\left|\frac{x-4}{x}\right|+\frac{1}{4x}+C.$$

9. $\displaystyle\int\frac{x^3+2x^2+4}{2x^2+2}\,dx$

$$= \frac{1}{2}\int \frac{x^3 + 2x^2 + 4}{x^2 + 1}\, dx$$

$$= \frac{1}{2}\int \left(x + 2 + \frac{-x + 2}{x^2 + 1} \right) dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2} + 2x - \int \frac{xdx}{x^2 + 1} + 2\int \frac{dx}{x^2 + 1} \right)$$

$$= \frac{1}{2}\left(\frac{x^2}{2} + 2x - \frac{1}{2}\ln\left|x^2 + 1\right| + 2\, arc\ tgx \right) + C$$

$$= \frac{x^2}{4} + x - \frac{1}{4}\ln\left(x^2 + 1\right) + arc\ tg\ x + C$$

10. $\int \frac{5dx}{x^3 + 4x}$

$$= 5\int \left(\frac{A}{x} + \frac{Bx + C}{x^2 + 4} \right) dx$$

Cálculo de A, B e C

$1 \equiv A\left(x^2 + 4\right) + \left(Bx + C\right) x$

$x = 0 \quad \rightarrow \quad A = 1/4$

$$\begin{cases} A + B = 0 \quad \therefore \quad B = -1/4 \\ C = 0 \end{cases}$$

Assim,

$$I = 5\int \left(\frac{1/4}{x} + \frac{-1/4\, x}{x^2 + 4} \right) dx$$

$$= 5\left(\frac{1}{4}\ln|x| - \frac{1}{4} \cdot \frac{1}{2}\ln\left|x^2 + 4\right| \right) + C$$

$$= \frac{5}{4}\left(\ln|x| - \frac{1}{2}\ln\left(x^2 + 4\right) \right) + C$$

11. $\int \frac{3x - 1}{x^2 - x + 1}\, dx$

$$= \int \frac{3x\, dx}{x^2 - x + 1} - \int \frac{dx}{x^2 - x + 1}$$

$$= \frac{3}{2} \int \frac{2x - 1 + 1}{x^2 - x + 1}\, dx - \int \frac{dx}{x^2 - x + 1}$$

$$= \frac{3}{2} \int \frac{2x - 1}{x^2 - x + 1} dx + \frac{1}{2} \int \frac{dx}{x^2 - x + 1}$$

$$= \frac{3}{2} \ln \left| x^2 - x + 1 \right| + \frac{1}{2} \int \frac{dx}{x^2 - x + 1}$$

$$= \frac{3}{2} \ln \left| x^2 - x + 1 \right| + \frac{1}{2} \cdot \frac{1}{\frac{\sqrt{3}}{2}} arc\ tg\ \frac{x - \frac{1}{2}}{\sqrt{3/4}} + C$$

$$= \frac{3}{2} \ln \left| x^2 - x + 1 \right| + \frac{1}{\sqrt{3}} arc\ tg\ \frac{x - \frac{1}{2}}{\frac{\sqrt{3}}{2}} + C$$

$$= \frac{3}{2} \ln \left| x^2 - x + 1 \right| + \frac{1}{\sqrt{3}} arc\ tg\ \frac{2x - 1}{\sqrt{3}} + C$$

12. $\int \frac{dx}{x^3 + 8}$

$$= \int \frac{dx}{(x + 2)(x^2 - 2x + 4)}$$

$$= \int \left(\frac{A}{x + 2} + \frac{Bx + C}{x^2 - 2x + 4} \right) dx$$

Cálculo de A, B e C
$1 \equiv A \left(x^2 - 2x + 4\right) + (Bx + C)(x + 2)$

$$\begin{cases} A + B = 0 \\ -2A + 2B + C = 0 \\ 4A + 2C = 1 \end{cases}$$

$A = \frac{1}{12}$; $B = \frac{-1}{12}$ e $C = \frac{1}{3}$
Assim,

$$I=\int\frac{\frac{1}{12}dx}{x+2}+\int\frac{-\frac{1}{12}x+\frac{1}{3}}{x^2-2x+4}dx$$

$$=\frac{1}{12}\ln|x+2|+\int\frac{-\frac{1}{12}x}{x^2-2x+4}dx+\int\frac{\frac{1}{3}}{x^2-2x+4}dx$$

$$=\frac{1}{12}\ln|x+2|-\frac{1}{12}\cdot\frac{1}{2}\int\frac{2x-2+2}{x^2-2x+4}\,dx+\frac{1}{3}\int\frac{dx}{x^2-2x+4}$$

$$=\frac{1}{12}\ln|x+2|-\frac{1}{24}\int\frac{(2x-2)\,dx}{x^2-2x+4}+\frac{1}{4}\int\frac{dx}{x^2-2x+4}$$

$$=\frac{1}{12}\ln|x+2|-\frac{1}{24}\ln\left|x^2-2x+4\right|+\frac{1}{4}\cdot\frac{1}{\sqrt{3}}arc\ tg\ \frac{x-1}{\sqrt{3}}+C$$

13. $\displaystyle\int\frac{x-1}{\left(x^2+2x+3\right)^2}\,dx$

$$=\frac{1}{2}\int\frac{2x+2-2}{\left(x^2+2x+3\right)^2}-\int\frac{dx}{\left(x^2+2x+3\right)^2}$$

$$=\frac{1}{2}\int\frac{2x+2-2}{\left(x^2+2x+3\right)^2}-2\int\frac{dx}{\left[(x+1)^2+2\right]^2}$$

$$=\frac{1}{2}\,\frac{\left(x^2+2x+3\right)^{-1}}{-1}-2\left[\frac{x+1}{2\,.\,2\,(2-1)\left(x^2+2x+3\right)}+\frac{1}{4(2-1)}\int\frac{dx}{x^2+2x+3}\right]$$

$$=\frac{1}{2}\,\frac{-1}{x^2+2x+3}-\frac{x+1}{2\left(x^2+2x+3\right)}-\frac{1}{2}\int\frac{dx}{x^2+2x+3}$$

$$=\frac{-x-2}{2\left(x^2+2x+3\right)}-\frac{1}{2\sqrt{2}}arc\ tg\ \frac{x+1}{\sqrt{2}}+C$$

14. $\displaystyle\int\frac{dx}{x\left(x^2-x+1\right)^2}$

$$=\int\left(\frac{A}{x}+\frac{Bx+C}{x^2-x+1}+\frac{Dx+E}{\left(x^2-x+1\right)^2}\right)dx$$

Cálculo de A, B, C, D e E.

$1\equiv A\left(x^2-x+1\right)^2+(Bx+C)x\left(x^2-x+1\right)+(Dx+E)x$

$x=0\quad\to\quad 1=A$

$1\equiv A\left(x^4-2x^3+3x^2-2x+1\right)+(Bx+C)\left(x^3-x^2+x\right)+Dx^2+Ex$

$$\begin{cases} A+B=0 \\ -2A-B+C=0 \\ 3A+B-C+D=0 \\ -2A+C+E=0 \\ A=1 \end{cases}$$

$A=1;\ B=-1,\ C=1;\ D=-1,\ E=1$

Assim,

$$I=\int\left(\frac{1}{x}+\frac{-x+1}{x^2-x+1}+\frac{-x+1}{\left(x^2-x+1\right)^2}\right)dx$$

$$=\int\frac{dx}{x}+\frac{1}{2}\int\frac{-2x-1+1}{x^2-x+1}\,dx+\int\frac{dx}{x^2-x+1}+\frac{1}{2}\int\frac{-2x+1-1}{\left(x^2-x+1\right)^2}dx+\int\frac{dx}{\left(x^2-x+1\right)^2}$$

$$=\ln|x|+\frac{1}{2}\int\frac{-2x+1}{x^2-x+1}dx+\frac{1}{2}\int\frac{dx}{x^2-x+1}+\frac{1}{2}\int\frac{(-2x+1)dx}{\left(x^2-x+1\right)^2}+\frac{1}{2}\int\frac{dx}{\left(x^2-x+1\right)^2}$$

$$=\ln|x|-\frac{1}{2}\ln\left|x^2-x+1\right|-\frac{1}{2}\frac{\left(x^2-x+1\right)^{-1}}{-1}+\frac{1}{2}\int\frac{dx}{x^2-x+1}+\frac{1}{2}\int\frac{dx}{\left(x^2-x+1\right)^2}$$

$$=\ln|x|-\frac{1}{2}\ln\left|x^2-x+1\right|+\frac{1}{2\left(x^2-x+1\right)}+\frac{1}{2}\cdot\frac{2}{\sqrt{3}}arc\ tg\ \frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}+\frac{1}{2}\int\frac{dx}{\left(x^2-x+1\right)^2}$$

$$=\ln|x|-\frac{1}{2}\ln\left|x^2-x+1\right|+\frac{5\sqrt{3}}{9}arc\ tg\ \frac{2x-1}{\sqrt{3}}+\frac{x+1}{3\left(x^2-x+1\right)}+C$$

15. $\displaystyle\int\frac{4x^4}{x^4-x^3-6x^2+4x+8}dx$

$$=\int\left(4+\frac{4x^3+24x^2-16x-32}{(x-2)^2(x+2)(x+1)}\right)dx$$

$$=\int 4dx+\int\left(\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x-2}+\frac{D}{(x-2)^2}\right)dx$$

Calculando A,B,C, e D:

$$4x^3+24x^2-16x-32\equiv A(x-2)^2(x+2)+B(x-2)^2(x+1)+C\,(x-2)(x+2)(x+1)+D\,(x+2)(x+1)$$

$$\begin{cases} A+B+C=4 \\ -2A-3B+C+D=24 \\ -4A-4C+3D=-16 \\ 8A+4B-4C+2D=-32 \end{cases}$$

$A = 4/9$; $B=-4$; $C = 68/9$; $D = 16/3$

Assim,

$$I = 4x + \int \frac{4/9\, dx}{x+1} + \int \frac{-4dx}{x+2} + \int \frac{68/9\, dx}{x-2} + \int \frac{16/3 dx}{(x-2)^2}$$

$$= 4x + \frac{4}{9} \ln|x+1| - 4 \ln|x+2| + \frac{68}{9} \ln|x-2| + \frac{16}{3} \cdot \frac{(x-2)^{-1}}{-1} + C$$

16. $\displaystyle\int \frac{x^2}{3x^2 - \frac{1}{2}x - \frac{1}{2}} dx$

$$I = \int \frac{\frac{1}{3}x^2}{x^2 - \frac{1}{6}x - \frac{1}{6}} dx = \frac{1}{3} \int \frac{x^2 dx}{x^2 - \frac{1}{6}x - \frac{1}{6}}$$

$$= \frac{1}{3} \int \left(1 + \frac{\frac{1}{6}x + \frac{1}{6}}{x^2 - \frac{1}{6}x - \frac{1}{6}} \right) dx$$

$$= \frac{1}{3} \left[\int dx + \frac{1}{6} \int \left(\frac{A}{x - \frac{1}{2}} + \frac{B}{x + \frac{1}{3}} \right) dx \right]$$

$$= \frac{1}{3} \left[x + \frac{1}{6} \int \left(\frac{9/5}{x - 1/2} + \frac{-4/5}{x + 1/3} \right) dx \right] + C$$

$$= \frac{1}{3}x + \frac{1}{10} \ln\left| x - \frac{1}{2} \right| - \frac{2}{45} \ln\left| x + \frac{1}{3} \right| + C$$

17. $\displaystyle\int \frac{dx}{x^2 + 9x}$

$$= \int \left(\frac{A}{x} + \frac{Bx + C}{x^2 + 9} \right) dx$$

$$= \int \frac{1/9}{x} dx + \int \frac{-1/9x}{x^2 + 9} dx$$

$$= \frac{1}{9} \ln |x| + \frac{-1}{9} \cdot \frac{1}{2} \ln \left| x^2 + 9 \right| + C$$

$$= \frac{1}{9} \left(\ln|x| - \frac{1}{2} \ln \left(x^2 + 9 \right) \right) + C$$

18. $\int \frac{dx}{\left(x^2 + 1\right)\left(x^2 + 4\right)}$

$$\int \frac{Ax + B}{x^2 + 1} dx + \int \frac{Cx + D}{x^2 + 4} dx$$

Calculando A,B,C, e D:
$1 \equiv (Ax + B)\left(x^2 + 4\right) + (Cx + D)\left(x^2 + 1\right)$

$$\begin{cases} A + C = 0 \\ B + D = 0 \\ 4A + C = 0 \\ 4B + D' = 1 \end{cases}$$

$A = 0$; $B = 1/3$; $C = 0$; $D = -1/3$

Assim,

$$I = \int \frac{1/3}{x^2 + 1} dx + \int \frac{-1/3}{x^2 + 4} dx$$

$$= \frac{1}{3} arc\ tg\ x - \frac{1}{3} \cdot \frac{1}{2} arc\ tg\ \frac{x}{2} + C$$

$$= \frac{1}{3} arc\ tg\ x - \frac{1}{6} arc\ tg\ \frac{x}{2} + C$$

19. $\int \frac{x^3 + x^2 + 2x + 1}{x^3 - 1} dx$

$$\int \frac{x^3 + x^2 + 2x + 1}{x^3 - 1} dx = \int \left(1 + \frac{x^2 + 2x + 2}{x^3 - 1} \right) dx$$

$$= \int \left(1 + \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1} \right) dx$$

Calculando A, B e C

$$x^2+2x+2 \equiv A\left(x^2+x+1\right)+(Bx+C)(x+1)$$

$$\begin{cases} A+B=1 \\ C+A-B=2 \\ A-C=2 \end{cases}$$

$$A = 5/3\ ;\ B = -2/3\ ;\ C = -1/3$$

Assim,

$$I = \int \left(1 + \frac{5/3}{x-1} + \frac{-\frac{2}{3}x + \frac{-1}{3}}{x^2+x+1} \right) dx$$

$$= x + \frac{5}{3}\ln|x-1| - \frac{1}{3}\int \frac{2x+1}{x^2+x+1}dx$$

$$= x + \frac{5}{3}\ln|x-1| - \frac{1}{3}\ln\left|x^2+x+1\right| + C$$

20. $\displaystyle\int \frac{x^3 dx}{\left(x^2+2\right)^2}$

Fazendo

$$\frac{x^3}{\left(x^2+2\right)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{\left(x^2+2\right)^2}$$

Calculando A,B,C e D

$$x^3 \equiv (Ax+B)\left(x^2+2\right) + Cx + D$$

$$x^3 \equiv Ax^3 + 2Ax + Bx^2 + 2B + Cx + D$$

$$\begin{cases} A=1 \\ B=0 \\ 2A+C=0 \quad \Rightarrow \quad C=-2 \\ 2B+D=0 \quad \Rightarrow \quad D=0 \end{cases}$$

Assim,

$$I = \int \frac{x}{x^2+2}\, dx + \int \frac{-2x}{\left(x^2+2\right)^2}\, dx$$

$$= \frac{1}{2} \ln \left|x^2+2\right| - \frac{\left(x^2+2\right)^{-1}}{-1} + C$$

$$= \frac{1}{2} \ln \left|x^2+2\right| + \frac{1}{x^2+2} + C$$

21. $\displaystyle\int \frac{dx}{x^4 - 3x^3 + 3x^2 - x}$

$$\frac{1}{x^4 - 3x^3 + 3x^2 - x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

Calculando A,B,C e D

$1 \equiv A(x-1)^3 + Bx\,(x-1)^2 + Cx\,(x-1) + Dx$

$x = 0 \quad \rightarrow \quad 1 = -A \quad \therefore \quad A = -1$

$x = 1 \quad \rightarrow \quad 1 = D \quad \therefore \quad D = 1$

$A + B = 0$

$B = -A \quad \therefore \quad B = 1$

$3A + B - C + D = 0 \;\; \therefore \; C = -1$

Assim,

$$I = \int \left(\frac{-1}{x} + \frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3} \right) dx$$

$$= -\ln|x| + \ln|x-1| - \frac{(x-1)^{-1}}{-1} + \frac{(x-1)^{-2}}{-2} + C$$

$$= \ln\left|\frac{x-1}{x}\right| + \frac{1}{x-1} - \frac{1}{2(x-1)^2} + C$$

22. $\displaystyle\int \frac{x\, dx}{(x-1)^2\,(x+1)^2}$

$$\frac{x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

Calculando A,B,C e D

$$x \equiv A\,(x-1)\,(x+1)^2 + B(x+1)^2 + C(x-1)^2\,(x+1) + D(x-1)^2$$
$$x \equiv A\,\left(x^3 + 2x^2 + x - x^2 - 2x - 1\right) + Bx^2 + 2Bx + B + C\left(x^3 + x^2 - 2x^2 - 2x + x + 1\right) + Dx^2 - 2Dx + D$$

$$\begin{cases} A + C = 0 \\ A + B - C + D = 0 \\ -A + 2B - C - 2D = 1 \\ -A + B + C + D = 0 \end{cases}$$

$A = 0, B = 1/4, C = 0, D = -1/4$

Assim,

$$\begin{aligned} I &= \int \left(\frac{1/4}{(x-1)^2} + \frac{-1/4}{(x+1)^2} \right) dx \\ &= \frac{1}{4}\,\frac{(x-1)^{-1}}{-1} - \frac{1}{4}\,\frac{(x+1)^{-1}}{-1} + C \\ &= \frac{1}{4}\,\frac{1}{x-1} + \frac{1}{4}\,\frac{1}{x+1} + C \end{aligned}$$

23. $\displaystyle\int \frac{x^2 + 2x - 1}{(x-1)^2\left(x^2+1\right)}\,dx$

$$\frac{x^2 + 2x - 1}{(x-1)^2\,\left(x^2+1\right)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx + D}{x^2 + 1}$$

Calculando A,B,C e D

$$x^2 + 2x - 1 \equiv A(x-1)\left(x^2+1\right) + B\left(x^2+1\right) + (Cx + D)\,(x-1)^2$$

$x = 1 \quad \rightarrow \quad 2 = 2B \quad \therefore \quad B = 1$

$$x^2 + 2x - 1 \equiv A\left(x^3 + x - x^2 - 1\right) + Bx^2 + B + Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D$$

$$\begin{cases} A + C = 0 \\ -A + B - 2C + D = 1 \\ A + C - 2D = 2 \\ -A + B + D = -1 \end{cases}$$

$A = 1, B = 1, C = -1, D = -1$

Assim,

$$I = \int \left(\frac{1}{x-1} + \frac{1}{(x-1)^2} + \frac{-x-1}{x^2+1} \right) dx$$

$$= \ln|x-1| + \frac{(x-1)^{-1}}{-1} + \int \frac{-x\,dx}{x^2+1} - \int \frac{dx}{x^2+1}$$

$$= \ln|x-1| - \frac{1}{x-1} - \frac{1}{2}\ln|x^2+1| - arc\ tg\ x + C$$

24. Verificar a formula

$$\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$$

$$\frac{1}{u^2-a^2} = \frac{A}{u-a} + \frac{B}{u+a}$$

Calculando A eB

$1 \equiv A\,(u+a) + B\,(u-a)$

$u = a \quad \to \quad 1 = 2a\,A \quad \therefore \quad A = 1/2a$

$u = -a \quad \to \quad 1 = -2a\,B \quad \therefore \quad B = -1/2a$

Assim,

$$I = \int \frac{du}{a^2-u^2} = -\int \frac{du}{u^2-a^2}$$

$$= -\left[\int \frac{1/2a}{u-a}du + \int \frac{-1/2a}{u+a}du \right]$$

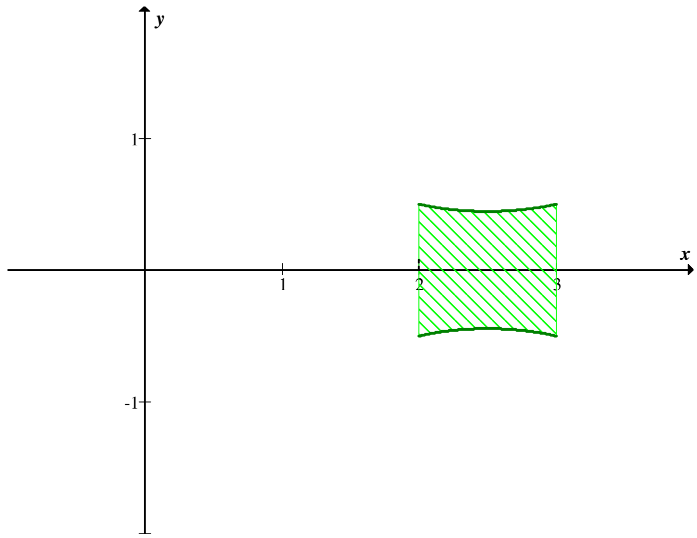
$$= \frac{-1}{2a}\ln|u-a| + \frac{1}{2a}\ln|u+a| + C$$

$$= \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$$

25. Calcular a área da região limitada pelas curvas $y = \dfrac{1}{(x-1)(x-4)}$, $y = \dfrac{1}{(1-x)(x-4)}$,

$x = 2$ e $x = 3$

A Figura que segue mostra a área.



$$A=\int_{2}^{3}\left(\frac{1}{(1-x)(x-4)}-\frac{1}{(x-1)(x-4)}\right)dx$$

$$=\int_{2}^{3}\frac{-2}{(x-1)(x-4)}dx$$

Fazendo:

$$\frac{-2}{(x-1)(x-4)}=\frac{A}{x-1}+\frac{B}{x-4}$$

Calculando A e B:

$$-2\equiv A(x-4)+B(x-1)$$

$$x=4 \quad\rightarrow\quad -2=3B \quad\therefore\quad B=-2/3$$

$$x=1 \quad\rightarrow\quad -2=-3A \quad\therefore\quad A=2/3$$

Assim,

$$\int\frac{-2dx}{(x-1)(x-4)}=\int\left(\frac{2/3}{x-1}-\frac{2/3}{x-4}\right)dx$$

$$=\frac{2}{3}\ln|x-1|-\frac{2}{3}\ln|x-4|+C$$

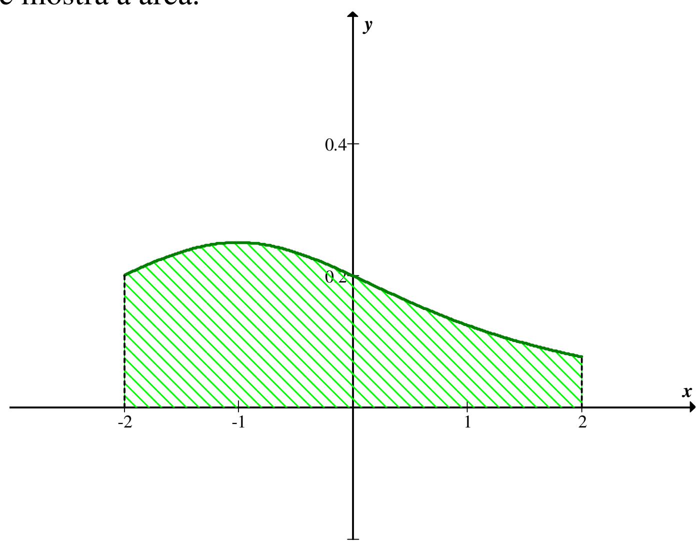
$$A=\left[\left(\frac{2}{3}\ln|x-1|-\frac{2}{3}\ln|x-4|\right)\Big|_{2}^{3}\right]$$

$$=\frac{2}{3}(\ln 2-\ln 1)-\frac{2}{3}(\ln 1-\ln 2)$$

$$=\frac{2}{3}\ln 2+\frac{2}{3}\ln 2=\frac{4}{3}\ln 2 u.a.$$

26. Calcular a área da região sob o gráfico de $y = \dfrac{1}{x^2 + 2x + 5}$ de $x = -2$ ate $x = 2$

A Figura que segue mostra a área.



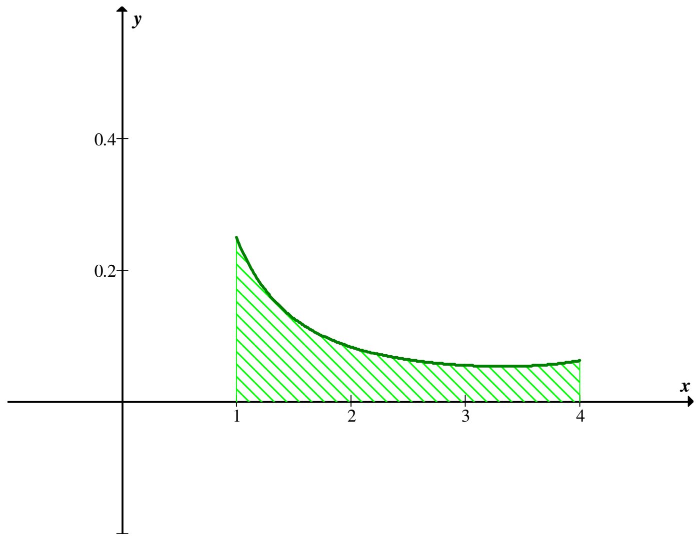
$$\int_{-2}^{2} \frac{dx}{x^2 + 2x + 5} = \int_{-2}^{2} \frac{dx}{(x+1)^2 + 4}$$

$$= \frac{1}{2}\, arc\ tg\ \frac{x+1}{2}\bigg|_{-2}^{2}$$

$$= \frac{1}{2}\, arc\ tg\ \frac{3}{2} - \frac{1}{2} arc\ tg\ \left(-\frac{1}{2}\right) u.a.$$

27. Calcular a área da região sob o gráfico $y = \dfrac{-1}{x^2(x-5)}$ de $x = 1$ ate $x = 4$

A Figura que segue mostra a área.



$$I = \int \frac{-\,dx}{x^2(x-5)}$$

$$\frac{1}{x^2(x-5)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-5}$$

$$1 \equiv A(x-5)x + B(x-5) + Cx^2$$

$$x = 0 \quad \rightarrow \quad 1 = -5B \quad \therefore \quad B = -\tfrac{1}{5}$$

$$x = 5 \quad \rightarrow \quad 1 = 25C \quad \therefore \quad C = \tfrac{1}{25}$$

$$A + C = 0 \quad \therefore \quad A = {-1}/{25}$$

$$I = -\int \left(\frac{-1/25}{x} + \frac{-1/5}{x^2} + \frac{1/25}{x-5} \right) dx$$

$$= \frac{1}{25} \ln |x| + \frac{1}{5} . \frac{1}{x} - \frac{1}{25} \ln |x-5| + C$$

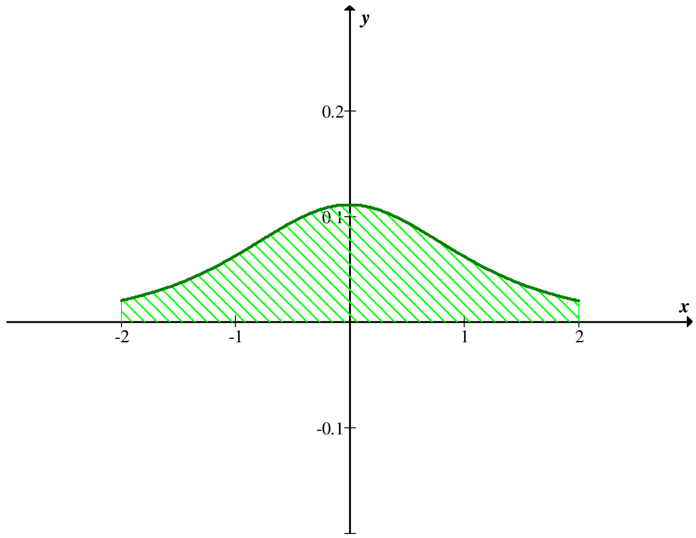
$$A = \left(\frac{1}{25} \ln |x| + \frac{1}{5} . \frac{1}{x} - \frac{1}{25} \ln |x-5| \right) \Bigg|_1^4$$

$$= \frac{1}{25} \ln 4 + \frac{-1}{20} + \frac{1}{5} + \frac{1}{25} \ln 4$$

$$= \left(\frac{2}{25} \ln 4 + \frac{3}{20} \right) u.\, a.$$

28. Calcular a área da região sob o gráfico de $y = \dfrac{1}{(x^2+3)^2}$ de $x = -2$ ate $x = 2$

A Figura que segue mostra a área.



$$I = \int \frac{dx}{(x^2+3)^2}$$

$$\begin{aligned}
x &= \sqrt{3}\ tg\ \theta \\
dx &= \sqrt{3}\ \sec^2\theta\ d\theta \\
x^2 + 3 &= 3\ tg^2\theta + 3 \\
&= 3\left(tg^2\theta + 1\right) \\
&= 3\sec^2\theta
\end{aligned}$$

$$\begin{aligned}
I &= \int \frac{\sqrt{3}\sec^2\theta}{9\sec^4\theta}\,d\theta = \frac{\sqrt{3}}{9}\int\frac{d\theta}{\sec^2\theta} \\
&= \frac{\sqrt{3}}{9}\int\cos^2\theta\ d\theta \\
&= \frac{\sqrt{3}}{9}\left(\frac{1}{2}\theta + \frac{1}{4}sen\ 2\theta\right) + C \\
&= \frac{\sqrt{3}}{9}\left(\frac{1}{2}\,arc\ tg\ \frac{x}{\sqrt{3}} + \frac{1}{4}\,.\,2\,.\,\frac{x}{\sqrt{3+x^2}}\,.\,\frac{\sqrt{3}}{\sqrt{3+x^2}}\right)
\end{aligned}$$

$$A = \int_{-2}^{2} \frac{dx}{\left(x^2+3\right)^2} = \left(\frac{\sqrt{3}}{18} arc\ tg \frac{x}{\sqrt{3}} + \frac{\sqrt{3}}{9} \cdot \frac{\sqrt{3}x}{2 \cdot \left(3+x^2\right)} \right) \Bigg|_{-2}^{2}$$

$$= \frac{\sqrt{3}}{9} arc\ tg \frac{2}{\sqrt{3}} + \frac{2}{21} u.a.$$

29. Investigar as integrais impróprias

(a) $I = \int_{10}^{+\infty} \frac{dx}{x^2 (x-5)}$

$$I_1 = \int \frac{dx}{x^2 (x-5)}$$

$$\frac{1}{x^2(x-5)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-5}$$

$$= \frac{A(x-5)x + B(x-5) + C\left(x^2\right)}{x^2(x-5)}$$

$$1 = A(x-5)x + B(x-5) + C\left(x^2\right)$$

$$x = 5 \quad \Rightarrow \quad 1 = C \,.\, 25 \quad \Rightarrow \quad C = \frac{1}{25}$$

$$x = 0 \quad \Rightarrow \quad 1 = B \,.\, (-5) \quad \Rightarrow \quad B = -\frac{1}{5}$$

$$x = 1 \quad \Rightarrow \quad 1 = -4A - 4B + C$$

$$1 = -4A - \frac{4}{5} + \frac{1}{25}$$

$$4A = \frac{4}{5} + \frac{1}{25} - 1 = \frac{20+1-25}{25}$$

$$4A = \frac{-4}{25}$$

$$A = -\frac{1}{25}$$

$$I_1 = \int -\frac{1}{25}\,.\frac{dx}{x} + \int -\frac{1}{5}\frac{dx}{x^2} + \int \frac{1}{25}\,\frac{dx}{x-5}$$

$$= -\frac{1}{25}\int \frac{dx}{x} - \frac{1}{5}\int x^{-2}dx - \frac{1}{25}\int \frac{dx}{x-5}$$

$$= -\frac{1}{25}\ln|x| + \frac{1}{5x} + \frac{1}{25}\ln|x-5| + c$$

$$= \frac{1}{5x} + \frac{1}{25}\ln\left|\frac{x-5}{x}\right| + c$$

$$I = \lim_{b\to+\infty}\int_{10}^{b}\frac{dx}{x^2\,(x-5)}$$

$$= \lim_{b\to+\infty}\left(\frac{1}{5x} + \frac{1}{25}\ln\left|\frac{x-5}{x}\right|\right)\Bigg|_{10}^{b}$$

$$= \lim_{b\to+\infty}\left[\frac{1}{5b} + \frac{1}{25}\ln\left|\frac{b-5}{b}\right|\right] - \frac{1}{5\,.\,10} - \frac{1}{25}\ln\left|\frac{10-5}{10}\right|$$

$$= -\frac{1}{50} - \frac{1}{25}\ln\frac{1}{2} = \frac{\ln 2}{25} - \frac{1}{50}$$

A integral converse e tem como resultado $\frac{\ln 2}{25} - \frac{1}{50}$.

(b) $I = \int_0^2 \frac{dx}{x^2(x-5)}$

$$I = \int_0^2 \frac{dx}{x^2(x-5)} = \lim_{r\to 0^+}\int_r^2 \frac{dx}{x^2(x-5)}$$

$$= \lim_{r\to 0^+}\left(\frac{1}{5x} + \frac{1}{25}\ln\left|\frac{x-5}{x}\right|\right)\Bigg|_r^2$$

$$= \frac{1}{10} + \frac{1}{25}\ln\frac{3}{2} - \lim_{r\to 0^+}\left(\frac{1}{5r} + \frac{1}{25}\ln\left|\frac{r-5}{r}\right|\right)$$

$$= +\infty$$

A integral diverge.

c) $I = \int_5^{+\infty}\frac{dx}{x^2(x-5)}$

$$I = \int_{5}^{+\infty} \frac{dx}{x^2(x-5)}$$

$$= \int_{5}^{10} \frac{dx}{x^2(x-5)} + \int_{10}^{+\infty} \frac{dx}{x^2(x-5)}$$

$$I_1 = \int_{5}^{10} \frac{dx}{x^2(x-5)} = \lim_{r \to 5^+} \int_{r}^{10} \frac{dx}{x^2(x-5)}$$

$$= \lim_{r \to 5^+} \left(\frac{1}{5x} + \frac{1}{25} \ln \left| \frac{x-5}{x} \right| \right) \Bigg|_{r}^{10}$$

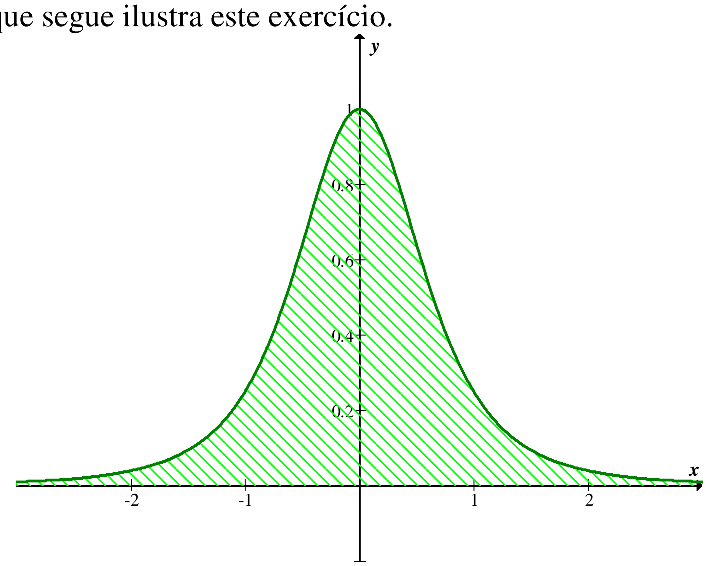
$$= \lim_{r \to 5^+} \left(\frac{1}{50} + \frac{1}{25} \ln \left| \frac{10-5}{10} \right| \right) - \left(\frac{1}{5r} + \frac{1}{25} \ln \left| \frac{r-5}{r} \right| \right)$$

$$= +\infty$$

A integral diverge

30. Determinar, se possível, a área da região sob o gráfico da função $y = \frac{1}{\left(x^2+1\right)^2}$ de $-\infty$ a $+\infty$.

A Figura que segue ilustra este exercício.



$$I = \int_{-\infty}^{+\infty} \frac{dx}{\left(x^2+1\right)^2}$$

$$I_1 = \int \frac{dx}{\left(x^2+1\right)^2} = \frac{x\left(x^2+1\right)^{1-2}}{2\,.\,(2-1)} + \frac{1}{2\,.\,1}\int \frac{dx}{\left(x^2+1\right)^{2-1}}$$

$$= \frac{x}{2\left(x^2+1\right)} + \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{x}{2\left(x^2+1\right)} + \frac{1}{2}\, arc\ tg\ x + c$$

$$I = \int_{-\infty}^{+\infty} \frac{dx}{\left(x^2+1\right)^2} = \int_{-\infty}^{0} \frac{dx}{\left(x^2+1\right)^2} + \int_{0}^{+\infty} \frac{dx}{\left(x^2+1\right)^2}$$

Como o integrando é uma função par, basta investigar a integral $\int_0^{+\infty} \frac{dx}{\left(x^2+1\right)^2}$.

Temos,

$$I_1 = \lim_{b \to +\infty} \int_0^b \frac{dx}{\left(x^2+1\right)^2}$$

$$= \lim_{b \to +\infty} \left(\frac{x}{2\left(x^2+1\right)} + \frac{1}{2}\, arc\ tg\ x \right)\Bigg|_0^b$$

$$= \frac{1}{2}\, arc\ tg\left(+\infty\right)$$

$$= \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

Logo $I = 2\,.\,\frac{\pi}{4} = \frac{\pi}{2}$