6. Let A and B be sets in $\mathbb{R}$ bounded from above. Let $a = \sup(A)$, $b = \sup(B)$ and let the set C be defined by $C = \{xy \mid x \in A, y \in B\}$. Show that, in general, $ab \neq \sup(C)$. If $a < 0$ and $b < 0$, then prove that $ab = \inf(C)$. If $a > 0$ and $b > 0$, and A, B have only positive elements, then also prove that $ab = \sup(C)$.

Solution: As a specific instance, let $A = \{x \in \mathbb{R} \mid -10 < x < -1\} =]-10,-1[$ and $B =]0,1/2[$, so that $a = -1$, $b = 1/2$, and $ab = -1/2$. But $C =]-5,0[$ and $\sup(C) = 0$.

Now we prove that if $a < 0$ and $b < 0$, then $ab = \inf(C)$. For this, we use the analogue of Theorem 2 for greatest lower bounds. First, let $x \in A$ and $y \in B$. We want to show $xy \geqslant ab$. But, $x \leqslant a$, $y \leqslant b$ or $-x \geqslant -a \geqslant 0$ and $-y \geqslant -b \geqslant 0$, so (using Axiom III(v) for $\mathbb{R}$), $(-x)(-y) \geqslant (-a)(-b)$ or $xy \geqslant ab$. Given $\varepsilon > 0$, we want to find $x \in A$ and $y \in B$ so that $ab > xy - \varepsilon$, or $|ab - xy| < \varepsilon$. Choose x and y so that $a < x + \varepsilon/2(|b| + 1)$, $b < y + \varepsilon/2\,|a|$, and $b < y + 1$. Then, since $|uv| = |u|\,|v|$ and $|y| < |b| + 1$, we get (using the triangle inequality) $|ab - xy| \leqslant |ab - ay| + |ay - xy| = |a|\,|b - y| + |a - x|\,|y| < |a|\,(\varepsilon/2|a|) + (\varepsilon/2(|b| + 1))(|b| + 1) = \varepsilon$.

The last assertion can be proven in an analogous way.

Exercises for Chapter 1

1. For each of the following sets S, find $\sup(S)$ and $\inf(S)$:
   (a) $\{x \in \mathbb{R} \mid x^2 < 5\}$
   (b) $\{x \in \mathbb{R} \mid x^2 > 7\}$
   (c) $\{1/n \mid n \text{ an integer}, n > 0\}$
   (d) $\{-1/n \mid n \text{ an integer}, n > 0\}$
   (e) $\{.3,.33,.333,\ldots\}$
   (f) the intervals $[a,b]$, $[a,b[$, $]a,b]$, or $]a,b[$.

2. Review the proof that $\sqrt{2}$ is irrational. [Hint: If there were a rational number m/n, where m and n have no common factor, such that $(m/n)^2 = 2$, would m be even or odd?] Generalize this to $\sqrt{k}$ for k a positive integer which is not a perfect square.

3. (a) Let $x \geqslant 0$ be a real number such that for any $\varepsilon > 0$, $x \leqslant \varepsilon$. Show that $x = 0$.
   (b) Let $S =]0,1[$. Show that for any $\varepsilon > 0$ there exists $x \in S$, such that $x < \varepsilon$, $x \neq 0$.

4. Show that $d = \inf(S)$ iff d is a lower bound for S and for any $\varepsilon > 0$ there is an $x \in S$, such that $d \geqslant x - \varepsilon$.

5. Let x_n be a monotone increasing sequence bounded above and consider the set $S = \{x_1,x_2,\ldots\}$. Using Theorem 2, show that x_n converges to $\sup(S)$. Make a similar statement for decreasing sequences.

6. Let A and B be two non-empty sets of real numbers with the property that $x \leqslant y$ for all $x \in A$, $y \in B$. Show that there exists a number $c \in \mathbb{R}$ such that $x \leqslant c \leqslant y$ for all $x \in A$, $y \in B$. Give an example of this statement being false for rational numbers for all $x \in A, y \in B$. Give an example of this statement being false for rational numbers (it is, in fact, equivalent to the completeness axiom and is at the basis for another way of formulating the completeness axiom known as *Dekekind cuts*).

7. For sets $A, B \subset \mathbb{R}$, let $A + B = \{x + y \mid x \in A \text{ and } y \in B\}$. Show that $\sup(A + B) = \sup(A) + \sup(B)$. Make a similar statement for inf's.

8. For sets $A, B \subset \mathbb{R}$, determine which of the following statements are true. Prove the true statements and give a counter-example for those which are false:
   (a) $\sup(A \cap B) \leqslant \inf\{\sup(A),\sup(B)\}$
   (b) $\sup(A \cap B) = \inf\{\sup(A),\sup(B)\}$
   (c) $\sup(A \cup B) \geqslant \sup\{\sup(A),\sup(B)\}$
   (d) $\sup(A \cup B) = \sup\{\sup(A),\sup(B)\}$.

9. Demonstrate that if a subsequence of a Cauchy sequence converges to a point, then the whole sequence converges to that point. Give a counter-example if the original sequence is not a Cauchy sequence.

10. For a given sequence a_n, we define the numbers
$$\lim\sup(a_n) = \inf\{\sup\{a_n,a_{n+1},\ldots\} \mid n = 1,2,\ldots\}$$
and
$$\lim\inf(a_n) = \sup\{\inf\{a_n,a_{n+1},\ldots\} \mid n = 1,2,\ldots\}$$
Show that
   (a) $\lim\inf(a_n) \leqslant \lim\sup(a_n)$
   (b) $\lim\sup(a_n) = b$ iff for all $\varepsilon > 0$, there is an N so that $b + \varepsilon > a_n$ for all $n \geqslant N$ and $b - \varepsilon < a_n$ for some $n \geqslant N$
   (c) $a_n \to b$ iff $\lim\sup(a_n) = \lim\inf(a_n) = b$
   (d) let $a_n = (-1)^n$. Compute $\lim\inf(a_n)$, $\lim\sup(a_n)$.

Note: $\lim\sup(a_n)$ and $\lim\inf(a_n)$ always are defined (but could be $\pm\infty$) although $\lim(a_n)$ need not exist. Also, lim sup is short for *limit superior* and lim inf for *limit inferior*, and these are sometimes written as $\overline{\lim}$ and $\underline{\lim}$, respectively.

11. Show that (i), (ii), and (iii) of Theorem 3 each implies the completeness axiom for an ordered field. [Hint: (i) $\Rightarrow$ completeness axiom is almost immediate. (ii) implies (i) in much the same way as we showed in the proof of Theorem 3 that (i) implies (ii). Therefore (ii) $\Rightarrow$ completeness axiom. To show (iii) $\Rightarrow$ completeness axiom, it is sufficient to show (iii) $\Rightarrow$ (i). To do this, define the sequence x_n as in the proof of completeness axiom $\Rightarrow$ (i) and argue that x_n is a Cauchy sequence. Show that its limit is the sup of the set in question, following the proof that the completeness axiom $\Rightarrow$ (i).]

12. In $\mathbb{R}^n$ show that
   (a) $2\,\|x\|^2 + 2\,\|y\|^2 = \|x + y\|^2 + \|x - y\|^2$ (parallelogram law)
   (b) $\|x + y\|\,\|x - y\| \leqslant \|x\|^2 + \|y\|^2$
   (c) $4\langle x,y\rangle = \|x + y\|^2 - \|x - y\|^2$ (polarization identity).
   Interpret these results geometrically in terms of the parallelogram formed by x and y.

13. What is the orthogonal complement in $\mathbb{R}^4$ of the space spanned by $(1,0,1,1)$ and $(-1,2,0,0)$?